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CONVERGENCE BALL OF ITERATIONS WITH ONE PARAMETER

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Abstract. Under the weak Lipschitz condition about the solution of the equation, convergence theorems for a family of iterations with one parameter are obtained. An estimation of the radius of the attraction ball is shown. At last two examples are given.

§1 Introduction

Of the various iteration methods used to solve the equation $F(x)=0$, the family of iterative methods with one parameter

$$\begin{cases} x_{n+1} = x_n - \left[I + \frac{1}{2}P_F(I-\alpha P_F)^{-1}\right]F'(x_n)^{-1}F(x_n), \\ P_F(x) = F'(x)^{-1}F''(x)F'(x)^{-1}F(x), \end{cases} \quad \alpha \in [0,1], \tag{1}$$

is remarkable because it includes the super-Halley method, the Chebyshev method and Halley's method as its special cases when α is equal to 1, 0 and 1/2, respectively.

There are a number of papers concerning the above three iterations. In [1-7], Halley's method is studied. Among these references, [1] gives the convergence of Halley's method under the conditions of point estimation. The convergence and error estimations are obtained by means of cubic majorant function (see [2-5]). The convergence of the Chebyshev method by using recurrence relations is given in [8] and [9]. About the super-Halley method, cubic majorizing function is used in [10] and recurrence relations are used in [11]. The conditions the above papers need are usually added to the initial point x_0. Now our conditions are added to the equation solution. The aim of this paper is to compare the radius of the attraction ball in the family of iterative methods (1).

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§ 2 Convergence ball of iterative family(1)

Define $\rho=\rho(x)=\|x-x^*\|$, $L_\rho=L(\rho)=L(\rho)\rho$, $a_\rho=a(\rho)=1-\int_0^\rho L(t)\mathrm{d}t$, $b_\rho=b(\rho)=\frac{1}{\rho}\int_0^\rho L(t)t\mathrm{d}t$, in which $L(t)$ is a nondecreasing positive integral function.

For $r>0$, $B(x^*,r)$ denotes the open ball with the radius r and the center x^*. then we have

Theorem 1. Suppose that F has first and second order continuous Frechet derivatives in some neighbourhood of the point x^*, $F'(x^*)^{-1}$ exists and

$$\|F'(x^*)^{-1}F''(x^*)\|\leqslant L(0),$$

$$\|F'(x^*)^{-1}[F''(x)-F''(x')]\|\leqslant\left|\int_{\rho(x')}^{\rho(x)}L'(u)\mathrm{d}u\right|,\ \forall x\in B(x^*,r).$$

If r satisfies

$$h(r):=\frac{|a_r|}{|a_r^2-\alpha L_r|}\left[\frac{L_r}{2}+\frac{L_rb_r}{2|a_r|}+\left(\frac{1}{2}-\alpha\right)\frac{L_rb_r}{a_r^2}\right]\leqslant 1,$$

then the family (1) is convergent for all $x_0\in B(x^*,r)$ and

$$\|x_n-x^*\|\leqslant q^{2^n-1}\|x_0-x^*\|,\ n=1,2,\ldots,$$

where

$$q=\frac{a_{\rho_0}}{a_{\rho_0}^2-\alpha L_{\rho_0}}\left[\frac{L_{\rho_0}}{2}+\frac{L_{\rho_0}b_{\rho_0}}{2a_{\rho_0}}+\left(\frac{1}{2}-\alpha\right)\frac{L_{\rho_0}b_{\rho_0}}{a_{\rho_0}^2}\right]<1.$$

In order to prove this theorem, we need

Lemma 1[12]. $\frac{1}{t}\int_0^t L(u)u\mathrm{d}u$ is nondecreasing with respect to t.

Lemma 2. Suppose that F satisfies the condition of Theorem 1, $\int_0^r L(t)\,\mathrm{d}t<1$ and $\alpha\frac{L_r}{a_r^2}<1$, then we have

(1) $\|F'(x^*)^{-1}F''(x)\|\leqslant L(\rho)$;

(2) $\|F'(x)^{-1}F'(x^*)\|\leqslant\dfrac{1}{1-\int_0^\rho L(t)\mathrm{d}t}=\dfrac{1}{a_\rho}$;

(3) $\|F'(x)^{-1}F(x)\|\leqslant\dfrac{\rho}{a_\rho}$;

(4) $\|L_F(x)\|\leqslant\dfrac{L_\rho}{a_\rho^2}$;

(5) $\|(I-\alpha L_F)^{-1}\|\leqslant\dfrac{a_\rho}{a_\rho^2-\alpha L_\rho}$;

(6) $\left\|\int_0^1F'(x^*)^{-1}[F''(x+\tau(x^*-x))-F''(x)](1-\tau)(x^*-x)^2\mathrm{d}\tau\right\|\leqslant\dfrac{L_\rho\rho}{2}$.

(7) $\left\|\int_0^1F'(x^*)^{-1}F''(x+\tau(x^*-x))(1-\tau)(x^*-x)^2\mathrm{d}\tau\right\|\leqslant\rho b_\rho$.

Proof. (1) From the conditions on F, it is easy to get

$$(1)\quad \| F'(x^*)^{-1}F''(x) \| = \| F'(x^*)^{-1}[F''(x) - F''(x^*)] + F'(x^*)^{-1}F''(x^*) \| \leqslant \int_0^{\rho} L'(u)du + L(0) = L(\rho).$$

(2) From (1) we have

$$\| I - F'(x^*)^{-1}F'(x) \| = \| F'(x^*)^{-1}[F'(x^*) - F'(x)] \| = \| F'(x^*)^{-1}\int_0^1 F''(x + \tau(x^* - x))(x^* - x)\mathrm{d}\tau \| \leqslant \int_0^1 L((1-\tau)\rho)\rho \mathrm{d}\tau = \int_0^{\rho} L(t)\mathrm{d}t,$$

so the inequality

$$\| F'(x)^{-1}F'(x^*) \| \leqslant \frac{1}{1 - \int_0^{\rho} L(t)\mathrm{d}t} = \frac{1}{a_\rho}$$

holds from the Banach Lemma and $\int_0^{\rho} L(t)\mathrm{d}t < 1$.

(3) $F'(x)^{-1}F(x) = F'(x)^{-1}[F(x) - F(x^*)] =$

$$- \int_0^1 F'(x)^{-1}F'(x + \tau(x^* - x))(x^* - x)\mathrm{d}\tau = - \int_0^1 F'(x)^{-1}F'(x_\tau)(x^* - x)\mathrm{d}\tau (\text{let} x_\tau = x + \tau(x^* - x)).$$

On the other hand, by equalities

$$\rho(x_\tau) = \| x_\tau - x^* \| = \| x - x^* + \tau(x^* - x) \| = (1-\tau)\rho(x),$$
$$\rho(x + \sigma(x_\tau - x)) = \| x - x^* + \sigma\tau(x^* - x) \| = (1 - \sigma\tau)\rho(x),$$

we have

$$\| I - F'(x_\tau)^{-1}F'(x) \| = \| F'(x_\tau)^{-1}[F'(x_\tau) - F'(x)] \| = \| F'(x_\tau)^{-1}\int_0^1 F''(x + \sigma(x_\tau - x))(x_\tau - x)\mathrm{d}\sigma \| \leqslant$$
$$\| F'(x_\tau)^{-1}F'(x^*) \| \cdot \| \int_0^1 F'(x^*)^{-1}F''(x + \sigma(x_\tau - x))(x_\tau - x)\mathrm{d}\sigma \| \leqslant \frac{1}{1 - \int_0^{\rho(x_\tau)} L(t)\mathrm{d}t}\int_0^1 L(\rho(x + \sigma(x_\tau - x))) \| x_\tau - x \| \mathrm{d}\sigma \leqslant$$
$$\frac{1}{1 - \int_0^{(1-\tau)\rho} L(t)\mathrm{d}t}\int_0^1 L((1 - \sigma\tau)\rho)\tau\rho \mathrm{d}\sigma = \frac{1}{1 - \int_0^{(1-\tau)\rho} L(t)\mathrm{d}t}\int_{(1-\tau)\rho}^{\rho} L(t)\mathrm{d}t.$$

It is clear that the above inequality is bounded by 1. Thus, from the Banach lemma, the following inequality is valid:

$$\| F'(x)^{-1}F'(x_\tau) \| \leqslant \frac{1-\int_0^{(1-\tau)\rho} L(t)\mathrm{d}t}{1-\int_0^{\rho} L(t)\mathrm{d}t},$$

$$\| F'(x)^{-1}F(x) \| \leqslant \int_0^1 \| F'(x)^{-1}F'(x_\tau)(x^* - x) \| \mathrm{d}\tau \leqslant$$

$$\int_0^1 \frac{1-\int_0^{(1-\tau)\rho} L(t)\mathrm{d}t}{1-\int_0^{\rho} L(t)\mathrm{d}t}\rho \mathrm{d}\tau \leqslant \frac{\rho}{1-\int_0^{\rho} L(t)\mathrm{d}t} = \frac{\rho}{a_\rho}.$$

(4) $\| L_F(x) \| = \| F'(x)^{-1}F''(x)F'(x)^{-1}F(x) \| \leqslant$

$$\| F'(x)^{-1}F'(x^*) \| \; \| F'(x^*)^{-1}F''(x) \| \; \| F'(x)^{-1}F(x) \| \leqslant$$

$$\frac{1}{a_\rho}L(\rho)\frac{\rho}{a_\rho} = \frac{L_\rho}{a_\rho^2}.$$

(5) Since $\alpha \frac{L_r}{a_r^2} < 1$, we have

$$\| (I-\alpha L_F)^{-1} \| \leqslant \frac{1}{1-\alpha \| L_F \|} \leqslant \frac{a_\rho}{a_\rho^2 - \alpha L_\rho}.$$

(6) $\| \int_0^1 F'(x^*)^{-1}[F''(x+\tau(x^*-x)) - F''(x)](1-\tau)(x^*-x)^2 \mathrm{d}\tau \| \leqslant$

$$\int_0^1\int_{(1-\tau)\rho}^{\rho} L'(u)(1-\tau)\rho^2 \mathrm{d}u\mathrm{d}\tau = \int_0^1 [L(\rho) - L((1-\tau)\rho)](1-\tau)\rho^2 \mathrm{d}\tau =$$

$$\frac{L(\rho)\rho^2}{2} - \int_0^1 L((1-\tau)\rho)(1-\tau)\rho^2 \mathrm{d}\tau \;(\text{let}\,(1-\tau)\rho = t) =$$

$$\frac{L_\rho \rho}{2} - \int_0^\rho L(t)t\mathrm{d}t = \frac{L_\rho \rho}{2} - \rho b_\rho \leqslant \frac{L_\rho \rho}{2}.$$

(7) $\| \int_0^1 F'(x^*)^{-1}F''(x+\tau(x^*-x))(1-\tau)(x^*-x)^2 \mathrm{d}\tau \| \leqslant$

$$\int_0^1 L(\rho(x_\tau))(1-\tau)\rho^2 \mathrm{d}\tau = \int_0^1 L((1-\tau)\rho)(1-\tau)\rho^2 \mathrm{d}\tau = \int_0^\rho L(t)t\mathrm{d}t = \rho b_\rho.$$

Proof of Theorem 1. The method is the same as that of [13].

When $0 \leqslant \alpha \leqslant \frac{1}{2}$,

$$x^* - x_{n+1} \triangleq (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) =$$

$$-Q_F F'(x_n)^{-1}\int_0^1 [F''(x_n+\tau(x^*-x_n)) - F''(x_n)](1-\tau)(x^*-x_n)^2 \mathrm{d}\tau -$$

$$\left(\frac{1}{2}-\alpha\right) Q_F L_F F'(x_n)^{-1}\int_0^1 F''(x_n+\tau(x^*-x_n))(1-\tau)(x^*-x_n)^2 \mathrm{d}\tau +$$

$$\frac{1}{2} Q_F F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}\int_0^1 F''(x_n+\tau(x^*-x_n))(1-\tau)(x^*-x_n)^2 \mathrm{d}\tau (x^*-x_n),$$

where $Q_F = (I-\alpha L_F)^{-1}$. If we set $x_n = x$, then

$$\| (\mathrm{I}) \| + \| (\mathrm{II}) \| + \| (\mathrm{III}) \| \leqslant \frac{a_\rho^2}{a_\rho^2 - \alpha L_\rho} \cdot \frac{1}{a_\rho}\left[\frac{L_\rho \cdot \rho}{2} + \frac{L_\rho b_\rho \cdot \rho}{2a_\rho} + \left(\frac{1}{2}-\alpha\right) \cdot\right.$$

$$\left.\frac{L_\rho b_\rho\cdot\rho}{a_\rho^2}\right]=\frac{a_\rho}{a_\rho^2-\alpha L_\rho}\left[\frac{L_\rho}{2}+\frac{L_\rho b_\rho}{2a_\rho}+\left(\frac{1}{2}-\alpha\right)\cdot\frac{L_\rho b_\rho}{a_\rho^2}\right]\cdot\frac{1}{\rho}\cdot\rho^2.$$

Therefore

$$\| x^* - x_{n+1} \| \leqslant \frac{q}{\rho(x_0)}\rho(x_n)^2 \leqslant \ldots \leqslant q^{2^{n+1}-1}\rho(x_0),$$

i.e.,

$$\| x^* - x_n \| \leqslant q^{2^n-1}\rho(x_0),\ n = 1,2,\ldots.$$

When $\frac{1}{2}\leqslant\alpha\leqslant 1$,

$$\begin{aligned}&\| x_{n+1} - x^* \| \leqslant \\ &\| Q_F F'(x_n)^{-1}\int_0^1[F''(x_n+\tau(x^*-x_n))-F''(x_n)](1-\tau)(x^*-x_n)^2 d\tau \| + \\ &\| \left(\alpha-\frac{1}{2}\right)Q_F F'(x_n)^{-1}F''(x_n)[F'(x_n)^{-1}\int_0^1 F''(x_n+\tau(x^*-x_n))\cdot \\ &(1-\tau)(x^*-x_n)^2 d\tau]^2 \| + \| (\alpha-1)Q_F F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}\cdot \\ &\int_0^1 F''(x_n+\tau(x^*-x_n))(1-\tau)(x^*-x_n)^2 d\tau(x^*-x_n) \| \leqslant \frac{a_{\rho_n}^2}{a_{\rho_n}^2-\alpha L_{\rho_n}}\cdot\frac{1}{a_\rho}\cdot \\ &\left[\frac{L_{\rho_n}\cdot\rho}{2}+(1-\alpha)\frac{L_{\rho_n}b_{\rho_n}\cdot\rho}{a_{\rho_n}}+\left(\alpha-\frac{1}{2}\right)L_{\rho_n}\left(\frac{b_{\rho_n}\cdot\rho}{a_{\rho_n}}\right)^2\right]= \\ &\frac{a_{\rho_n}}{a_{\rho_n}^2-\alpha L_{\rho_n}}\left[\frac{L_{\rho_n}}{2}+(1-\alpha)\frac{L_{\rho_n}b_{\rho_n}}{a_{\rho_n}}+\left(\alpha-\frac{1}{2}\right)\frac{L_{\rho_n}b_{\rho_n}^2}{a_{\rho_n}^2}\right]\cdot\frac{1}{\rho}\cdot\rho^2.\end{aligned}$$

Considering the expressions of a_ρ and b_ρ, and then estimating the above formula, we have the following expression which is also ture for $\frac{1}{2}\leqslant\alpha\leqslant 1$:

$$\| x_{n+1} - x^* \| \leqslant \frac{a_{\rho_n}}{a_{\rho_n}^2-\alpha L_{\rho_n}}\left[\frac{L_{\rho_n}}{2}+\frac{L_{\rho_n}b_{\rho_n}}{2a_{\rho_n}}+\left(\frac{1}{2}-\alpha\right)\frac{L_{\rho_n}b_{\rho_n}}{a_{\rho_n}^2}\right]\cdot\frac{1}{\rho}\cdot\rho^2.$$

Noticing the conditions $a_r>0$ and $\alpha\frac{L_r}{a_r^2}<1$, we have the conclusion as required.

§ 3 Example

If we set $L(u)=\gamma+Lu$ and $L(u)=\frac{2\gamma}{(1-\gamma u)^3}$, the corresponding values are shown in Tables 1 and 2. The numbers beginning from line 2 column 2 mean the suprema of the radius of attraction ball r. The computation is exact to four significant figures.

Table 1 $L(u)=\gamma+Lu$

α	$\gamma=1$ $L=1$	$\gamma=2$ $L=2$	$\gamma=3$ $L=3$	$\gamma=0$ $L=1$	$\gamma=0$ $L=2$	$\gamma=0$ $L=3$	$\gamma=1$ $L=0$	$\gamma=2$ $L=0$	$\gamma=3$ $L=0$
1	1.275	0.7510	0.5391	2.081	1.4718	1.202	2.000	1.000	0.6667
$\frac{1}{2}$	0.3234	0.5000	0.1224	0.7420	0.5246	0.4284	0.4000	0.6306	0.4204
0	0.3735	0.2055	0.1423	0.8317	0.5881	0.4802	0.4684	0.2342	0.1561

From Table 1 we can conclude that the range of r decreases as the values of γ and L increase. Furthermore, when $\gamma=0$ and the value of L increases, the range of r also seems decreasing. This also happens when $L=0$ and γ increases. On the other hand, the value of r at $\alpha=1$ seems larger than that at $\alpha=\frac{1}{2}$ and $\alpha=0$, but no conclusion can be drawn from $\alpha=\frac{1}{2}$ and $\alpha=0$.

Table 2 $L(u)=\frac{2\gamma}{(1-\gamma u)^3}$

α	$\gamma=1$	$\gamma=2$	$\gamma=3$	$\gamma=4$	$\gamma=5$	$\gamma=6$	$\gamma=10$	$\gamma=20$	$\gamma=30$
1	0.4408	0.06000	0.04138	0.1102	0.02483	0.07347	0.04408	0.02204	0.01469
$\frac{1}{2}$	0.3293	0.07095	0.04730	0.03548	0.02838	0.05488	0.03293	0.007095	0.01098
0	0.1619	0.08097	0.05398	0.04048	0.03239	0.02699	0.01619	0.008097	0.005398

Table 2 gives us the impression that the decreasing tendency of r seems to be ture.

§ 4 Open problem

It's well-known that the super-Halley method, the Chebyshev method and Halley's method all have the third-order convergence. It is to be proven that the error estimates of the iterative family (1) are also of third order. On the other hand, the message that the super-Halley method seems superior to the other two iterations can be concluded from the tables, do we have the same conclusion from the expression $h(r)\leqslant 1$?

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