Floquet Theory

Consider the linear periodic system as follows.

\[ \dot{x} = A(t)x, \quad A(t + p) = A(t), \quad p > 0, \]

where \( A(t) \in C(R) \).

**Lemma 8.4** If \( C \) is a \( n \times n \) matrix with \( \det C \neq 0 \), then, there exists a \( n \times n \) (complex) matrix \( B \) such that \( e^B = C \).

**Proof:** For any matrix \( C \), there exists an invertible matrix \( P \), s.t. \( P^{-1}CP = J \), where \( J \) is a Jordan matrix.

If \( e^B = C \), then, \( e^{P^{-1}BP} = P^{-1}e^B P = P^{-1}CP = J \). Therefore, it is suffice to prove the result when \( C \) is in a canonical form.

Suppose that \( C = \text{diag}(C_1, \cdots, C_s) \), \( C_j = \lambda_j I_j + N_j \), where \( N_j \) is nilpotent, that is,

\[
N_j = \begin{pmatrix}
0 & 1 & \cdots & 0 \\
0 & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 \\
0 & \cdots & 0 & 0
\end{pmatrix} \quad \text{with} \quad N_j^{n_j} = O.
\]

Since \( C \) is invertible for each \( \lambda_j \neq 0 \).

If we can show that for each \( C_j \), there exists \( B_j \) s.t. \( C_j = e^{B_j} \Rightarrow C = e^B \).

Since \( C_j = \lambda_j (I_j + \frac{N_j}{\lambda_j}) \), using the expansion of \( \ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k \), \( |x|<1 \), we have

\[
B_j = \ln C_j = \ln \left\{ \lambda_j \left( I_j + \frac{N_j}{\lambda_j} \right) \right\} = I_j \ln \lambda_j + \ln \left( I_j + \frac{N_j}{\lambda_j} \right) \\
= I_j \ln \lambda_j + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left( \frac{N_j}{\lambda_j} \right)^k.
\]

Since \( N_j^{n_j} = O \), we actually have

\[
B_j = \ln C_j = I_j \ln \lambda_j + \sum_{k=1}^{n_j-1} \frac{(-1)^{k+1}}{k} \left( \frac{N_j}{\lambda_j} \right)^k = I_j \ln \lambda_j + M_j, \quad j = 1, 2, \cdots, s,
\]
where \( M_j = \sum_{k=1}^{n-1} \frac{(-1)^{k-1} \lambda^k}{k} \). Therefore, we have
\[
e^{B_j} = \exp \{ I_j \ln \lambda + M_j \} = \exp \{ \ln C_j \} = C_j, \quad j = 1, 2, \ldots, s.
\]

Let \( B = \text{diag}(B_1, \cdots, B_s) \), where \( B_j \) is defined above. We have the desired result given by
\[
e^B = \text{diag}(e^{B_1}, e^{B_2}, \cdots, e^{B_s}) = \text{diag}(C_1, C_2, \cdots, C_s) = C. \quad \Box
\]

**Remark 8.16** Clearly, \( B \) is not unique since \( e^{B + 2\pi k I_s} = e^B e^{2\pi k I_s} = e^B e^{2\pi k I_n} = e^B e^{2\pi ki} = e^B \) for any integer \( k \).

**Theorem 8.6 (Floquet Theorem)** If \( \Phi(t) \) is a fundamental matrix solution of the periodic system \( \dot{x} = Ax \), then so is \( \Phi(t + p) \). Moreover, there exists an invertible matrix \( P(t) \) with \( p \)-period such that
\[
\Phi(t) = P(t)e^{Bt}.
\]

**Proof.** Let \( \Psi(t) = \Phi(t + p) \). Since \( \Phi'(t) = A(t)\Phi(t) \), it follows that
\[
\Psi'(t) = \Phi'(t + p) = A(t + p)\Phi(t + p) = A(t)\Psi(t),
\]
Hence, \( \Psi(t) \) is also a matrix solution. Since \( \Phi(t) \) is invertible for all \( t \in \mathbb{R} \), so is \( \Phi(t + p) \Rightarrow \Psi(t) \) is also a fundamental matrix solution. Therefore, there exists an invertible matrix \( C \) (for example, if \( \Phi(t) \) satisfies \( \Phi(0) = I_n \), then \( C = \Phi(p) \))!

Depends on solutions. It is a point of difficulty for computation) s.t.
\[
\Phi(t + p) = \Phi(t)C \quad \text{for all} \quad t \in \mathbb{R}.
\]

By Lemma 8.4, there exists a matrix \( B \) such that \( e^{Bp} = C \). For such a matrix \( B \), we take \( P(t) := \Phi(t)e^{-Bt} \), that is, \( \Phi(t) = P(t)e^{Bt} \). Then
\[
P(t + p) = \Phi(t + p)e^{-B(t+p)} = \Phi(t)Ce^{-B(t+p)} = \Phi(t)e^{-Bt} = P(t).
\]

Therefore \( P(t) \) is invertible for all \( t \in \mathbb{R} \) and \( p \)-periodic. This concludes the proof.
Remark 8.17

1) If we know $\Phi(t)$ over $[t_0, t_0 + p]$, then we will know $\Phi(t)$ for all $t \in R$ by Floquet Theorem. This means that $\Phi(t)$ on $[t_0, t_0 + p]$ determines $\Phi(t)$ for all $t \in R$.

Reasoning:

Suppose $\Phi(t)$ is known on $[t_0, t_0 + p]$. Since $\Phi(t + p) = \Phi(t)C$, we take $C = \Phi^{-1}(t_0)\Phi(t_0 + p)$ and $B = p^{-1}\ln C$. $P(t) = \Phi(t)e^{-Bt}$ is known on $[t_0, t_0 + p]$. Since $P(t)$ is periodic for $t \in R$, $\Phi(t)$ is given over $t \in R$ by $\Phi(t) = P(t)e^{Bt}$.

2) If $\Phi(t)$ determines $e^{Bt}$ (or $B$), then any fundamental matrix solution $\Psi(t)$ determines a similar matrix $Se^{Bp}S^{-1}$ (or $SBS^{-1}$).

Reasoning:

For any fundamental matrix solution $\Psi(t)$, there exists $S$ with $\det S \neq 0$ s.t. $\Phi(t) = \Psi(t)S$. Since $\Phi(t + p) = \Phi(t)e^{Bp}$, we have

$$\Psi(t + p)S = \Psi(t)Se^{Bp} \Rightarrow \Psi(t + p) = \Psi(t)S e^{Bp}S^{-1} = \Psi(t)e^{SB^{-1}p}.$$

3) For the linear periodic system, its solutions are not necessarily periodic. That is, $\Phi(t) \neq \Phi(t + p)$ in general!!! Give counter-example by yourselves.

Corollary 8.1 Under the transformation $x = P(t)y$, which is invertible and periodic, the periodic system $\dot{x} = A(t)x \Rightarrow$ a time-invariant system.

Proof. Suppose $P(t)$ and $B$ defined by before and let $x = P(t)y$. Then

$$x' = P'(t)y + P(t)y' \quad \text{and} \quad x' = A(t)x = A(t)P(t)y \Rightarrow P'(t)y + P(t)y' = A(t)P(t)y,$$

$$\Rightarrow y' = P^{-1}(t)[A(t)P(t) - P'(t)]y.$$

By Floquet Theorem with $P(t) = \Phi(t)e^{-Bt}$, we have

$$P'(t) = A(t)\Phi(t)e^{-Bt} + \Phi(t)e^{-Bt}(-B) = A(t)P(t) - P(t)B.$$
It follows that
\[ y' = P^{-1}(t) [A(t)P(t) - P'(t)] y = P^{-1}(t) P(t) B y = B y. \]

This completes the proof. □

**Remark 8.18**

1) \( x = P(t)y \) is called Lyapunov transformation. \( P(t) \), which plays an important role. But it is difficult to be found explicitly since the computation of \( P(t) = \Phi(t) e^{-B t} \) depends on a fundamental matrix solution \( \Phi(t) \).

2) Since \( \Phi(t + p) = \Phi(t) C \) with \( \det C \neq 0 \), \( e^B = C \), the eigenvalues \( \rho \) of \( C \) are called the characteristic multipliers of the periodic linear system. The eigenvalues \( \lambda \) of \( B \) are called characteristic exponents of the periodic linear system. \( \rho = e^{\lambda_p} \).

3) Since \( B \) is not unique, the characteristic exponents are not uniquely defined, but the multipliers \( \{ \rho \} \) are uniquely defined (Why?) We always choose the exponents \( \{ \lambda \} \) as the eigenvalues of \( B \), where \( B \) is any matrix such that \( e^{B_p} = C \).

4) Since \( B \) is not unique and satisfies \( e^{B_p} = C \), so \( B \) is not necessarily real.

5) \( B \) may be complex, even if \( C \) is real. However, if \( A(t) \) is real (so that \( C \) is real), then, there exists a real \( S \) such that \( e^{2S_p} = C^2 \).

**Reasoning:**

Suppose \( \Phi(t) \) with \( \Phi(0) = I_n \), then \( C = \Phi(p) = e^{B_p} \), so
\[
\Phi^2(p) = e^{B_p} e^{B_p} = e^{(B+B)p}.
\]

Let \( S = \frac{B + \bar{B}}{2} \), then \( S \) is real s.t. \( e^{2S_p} = \Phi^2(p) = C^2 \).

6) Let \( S(t) = \Phi(t)e^{-St} \). Then \( S(t) \) is real, \( 2p \)-periodic.

Moreover, \( x = S(t)z \) reduces the periodic system \( \dot{x} = A(t)x \) into \( z' = S z \).

**Reasoning:**

Clearly, \( S(t) \) is real since \( S \) is real, and
\[
S(t + 2p) = \Phi(t + 2p)e^{-S(t+2p)} = \Phi(t) C^2 e^{-2S_p} e^{-St} = \Phi(t) e^{-St} = S(t);\]
It is similar to obtain $\dot{z} = Sz$ under the transformation $x = S(t)z$.

- Floquet theory gives a theoretical result which reduces it into linear systems with constant coefficients. However, The Lyapunov transformation can not be computed.

- Floquet theory is very useful to study stability of a given periodic solution, noted that not equilibrium here. This is a topic of research for dynamic systems, or it is also named as geometric theory of differential equations. It is noted that this type of stability is not in Lyapunov sense.