Classification of simple amenable $C^*$-algebras

Huaxin Lin
East China Normal University
and University of Oregon

References


★. A $C^*$-algebra is a complete normed algebra over $\mathbb{C}$ with an involution ($\ast$) for which

$$\|a^*\| = \|a\| \quad \text{and} \quad \|a^*a\| = \|a\|^2.$$ 

★. Every $C^*$-algebra is closed and adjoint closed sub-algebra of $B(H)$, where $B(H)$ is the $C^*$-algebra of all bounded operators on a Hilbert space $H$.

★. Examples: $\mathbb{C}$, $M_n$, $C_0(X)$, where $X$ is a locally compact Hausdorff space, $\mathcal{K}$, the algebra of all compact operators on $l^2$.

★. Gelfand: Every (unital) commutative $C^*$-algebra is isomorphic (as $C^*$-algebra) to $C(X)$ for some compact Hausdorff space $X$.

★. $C^*$-algebra are viewed as non-commutative topology.
Minimal dynamical systems.

Let \( X \) be a compact metric space and \( \alpha : X \to X \) be a minimal homeomorphism. There is an \( \alpha \)-invariant normalized Borel measure \( \mu \). Consider the Hilbert space \( H = L^2(X, \mu) \) and homomorphism \( \pi : C(X) \to B(H) \) defined by

\[
\pi(g)(f) = gf \quad \text{for} \quad f \in L^2(X, \mu)
\]

for all \( g \in C(X) \). Define

\[
U(f) = f \circ \alpha^{-1} \quad f \in L^2(X, \mu).
\]

Then \( U \) gives a homomorphism from \( \mathbb{Z} \) into the unitary group of \( B(H) \). The \( C^* \)-algebra generated by \( \pi(C(X)) \) and \( \alpha \) is denoted by \( A_\alpha = C(X) \rtimes_\alpha \mathbb{Z} \) and is called the crossed product of \( C(X) \) by \( \mathbb{Z} \) via \( \alpha \).

In this special case \( A_\alpha \) is a unital separable simple \( C^* \)-algebra. (no proper ideal).

Irrational rotations.

Let \( S^1 \) be the unit circle and \( \theta \) be an irrational number. Define \( \alpha : S^1 \to S^1 \) by \( \alpha(e^{2\pi i t}) = e^{2\pi i (t+\theta)} \). This is an irrational rotation.
One can show that $A_\alpha$ is the universal $C^*$-algebra generated by two uniatres $u$ and $v$ with relation:

$$uv = e^{2\pi i \theta} vu.$$  

(non-commutative torus).

★. Question: when two $C^*$-algebras are isomorphic?

★.

Two unital commutative $C^*$-algebras $A = C(X)$ and $B = C(Y)$ are isomorphic if and only if $X$ and $Y$ are homeomorphic.

★. We are NOT going to classify commutative $C^*$-algebras.
★. We consider simple separable amenable $C^*$-algebras with lower rank (for this talk zero rank).

★. AF-algebras can be classified by (scaled) dimension groups (G. A. Elliott—1978).

AF==== approximately finite dimensional. $C(X)$ is AF if and only if $\dim X = 0$.

★. A $C^*$-algebra has real rank zero if the set of invertible self-adjoint elements is dense in $A_{s.a.}$.

Every AF-algebra has real rank zero.

$C(X)$ has real rank zero if and only if $\dim X = 0$.

Every Von-Neumann algebras has real rank zero.
★. AH-algebras:

\[ A = \lim_{n \to \infty} (A_n, \phi_n), \text{ where} \]

\[ A_n = P(C(X_n) \otimes F_n)P = P(C(X, F_n))P, \]

where \( X_n \) is a finite CW-complex, \( P \) is a projection in \( C(X, F_n) \).

\( A \) is said to be AT-algebra if each \( X_n \) can be taken as the unit circle.

★. Theorem (Elliott-Gong) (On the classification of \( C^* \)-algebras of real rank zero. II. Ann. of Math. 144 (1996), 497–610.)

Let \( A \) and \( B \) be two unital AH-algebras with no dimension growth and with real rank zero. Then \( A \cong B \) if and only if

\[ (K_0(A), K_0(A)_+, [1_A], K_1(A)) \]

\[ \cong (K_0(B), K_0(B)_+, [1_B], K_1(B)). \]
Moreover, for any countable abelian group $G_1$ and any countable weakly unperforated (partial) ordered group $G_0$ with order unit $u \in G_0$ with the Riesz interpolation, there is a unital simple AH-algebra with no dimension growth and with real rank zero such that

$$(K_0(A), K_0(A)_+, [1_A], K_1(A)) = (G_0, (G_0)_+, u, G_1).$$

**Theorem** (Elliott and Evans)

Let $\theta$ be an irrational number and $\alpha : S^1 \to S^1$ is defined by $\alpha(z) = e^{2\pi i \theta} z$. Then $C(X) \rtimes_{\alpha} \mathbb{Z}$ is a unital simple $AT$-algebra with real rank zero.

★. All purely infinite simple $C^*$-algebras have real rank zero. (S. Zhang)

★. Tracial (topological) rank was introduced in 1998. If $A$ has tracial rank zero, it will written $TR(A) = 0$.

★. Every unital simple $C^*$-algebra with $TR(A) = 0$ is quasidiagonal, has real rank zero, stable rank and weakly unperforated $K_0$.  

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★. Every unital simple AH-algebra with no dimension growth and with real rank zero has tracial rank zero.

★. $\mathcal{N}$: the so-called Bootstrap class of $C^*$-algebras.

It contains most interesting separable $C^*$-algebras. It contains all commutative $C^*$-algebras, type $III$ $C^*$-algebras, closed under inductive limit, quotient, ideal, tensor product with AF-algebras, crossed products by $\mathbb{Z}$,...

We are only interested in $C^*$-algebras in $\mathcal{N}$.

★. **Theorem A (L—)**

Let $A$ and $B$ be two unital separable simple $C^*$-algebras in $\mathcal{N}$. Suppose that $TR(A) = TR(B) = 0$. Then

\[ A \cong B \]

if and only if

\[(K_0(A), K_0(A)_+, [1_A], K_1(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B)).\]
Consider

\[ \alpha : K_*(A) \longrightarrow K_*(B). \]

We hope to establish homomorphisms \( \phi : A \rightarrow B \) so that \([\phi] = \alpha\).

One can settle for “approximately multiplicative maps”.

★ A sequence of positive linear maps \( \phi_n : A \rightarrow B \) is said to be asymptotically multiplicative if

\[ \lim_{n \to \infty} \| \phi_n(a)\phi_n(b) - \phi_n(ab) \| = 0 \]

for all \( a, b \in A \).

In general, asymptotically multiplicative completely positive linear maps cannot be “close” to any homomorphisms
For unital separable amenable simple $C^*$-algebras $A$ and $B$ with $TR(A) = TR(B) = 0$, given any
\[
\alpha : (K_0(A), K_0(A)_+, [1_A], K_1(A)) \rightarrow (K_0(B), K_0(B)_+, [1_B], K_1(B)).
\]
There existence a (sequence) of asymptotically multiplicative contractive completely positive linear maps $\{\phi_n\} : A \rightarrow B$ such that “locally” $\{\phi_n\}$ gives $\alpha$.

★ (A non-commutative diagram)
\[
\begin{array}{ccc}
A & \overset{id}{\rightarrow} & A & \overset{id}{\rightarrow} & A \\
\downarrow \phi_1 & \nearrow \psi_1 & \downarrow \phi_2 \\
B & \overset{id}{\rightarrow} & B & \overset{id}{\rightarrow} & B
\end{array}
\]

★ Question:
Given two maps from $L_1, L_2 : \rightarrow B$.
1) When are they the “same”?
2) When are they unitarily equivalent?
Consider a spacial case:

Let $X$ be a compact metric space and let $A$ be a unital separable simple $C^*$-algebra with $TR(A) = 0$. Suppose $h_1, h_2 : C(X) \to A$ are two unital monomorphisms. When they are “equivalent?”

★. Theorem B (L—)

Let $A$ be a unital simple $C^*$-algebra with tracial rank zero and $X$ be a compact metric space. Suppose that $h_1, h_2 : C(X) \to A$ are two unital monomorphisms. Then $h_1$ and $h_2$ are approximately unitarily equivalent if and only if

$$[h_1] = [h_2] \text{ in } KL(C(X), A) \text{ and } \tau(h_1(f)) = \tau(h_2(f))$$

for every $f \in C(X)$ and every trace $\tau$ of $A$.

★. Approximately unitarily equivalent:

There exists a sequence of unitaries $u_n \in A$ such that

$$\lim_{n \to \infty} \|u_n^* h_1(a) u_n - h_2(a)\| = 0$$

for all $f \in C(X)$. 

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Let $A$ be the Calkin algebra. Suppose that $h_1, h_2 : C(X) \to A$ are two unital monomorphisms. Then $h_1$ and $h_2$ are unitarily equivalent if and only if
$$[h_1] = [h_2] \text{ in } KK(C(X), A).$$

If $K_*(C(X))$ is torsion free, in Theorem A, condition about $KL$ can be replaced by
$$(h_1)_{*i} = (h_2)_{*i}, \quad i = 0, 1,$$
where $(h_j)_{*i} : K_i(C(X)) \to K_i(A)$ ($i = 0, 1$) is the induced homomorphism on $K_i$.

If $A$ has a trace $\tau$ and $h_1$ and $h_2$ are approximately unitarily equivalent, then
$$\tau(h_1(f)) = \tau(h_2(f))$$
for all $f \in C(X)$.

Calkin algebra is purely infinite and simple—no trace.
★. **Theorem C** Let $A$ and $B$ be two unital separable simple $C^*$-algebras with $TR(A) = TR(B) = 0$. Suppose that $\{\phi_n\}, \{\psi_n\} : A \to B$ are two sequences of completely positive linear maps which are asymptotically multiplicative such that

$$[[\phi_n]] = [[\psi_n]] \text{ in } KL(A, B)$$

. Then there exists a sequence of unitaries $\{u_k\} \subset B$ such that

$$\lim_{k \to \infty} \|u_k^* \phi_{n_k}(a) u_k - \psi_{n_k}(a)\| = 0$$

for all $a \in A$.

★ If $K_*(A)$ is torsion free,

$$KL(A, B) = Hom(K_*(A), K_*(B)).$$

By using the "existence theorem" and the "uniqueness theorem" one can construct an approximate intertwining:

★

$$\begin{align*}
A & \xrightarrow{\id} A \\
\downarrow \phi_1 & \xleftarrow{\text{ad } u_1 \circ \psi_1} \uparrow \text{ad } v_2 \circ \phi_2 \\
B & \xrightarrow{\id} B \\
\end{align*}$$

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Minimal dynamical systems

★. Let $X$ be a compact metric space and $\alpha$ be a homeomorphism on $X$. Set $A_\alpha = C(X) \rtimes_\alpha \mathbb{Z}$ and $j_\alpha : C(X) \to A_\alpha$ the obvious embedding map.

★. Let $X$ be a compact metric space and $\alpha, \beta : X \to X$ be minimal homeomorphisms. We say $\alpha$ and $\beta$ are conjugate if there exists homeomorphism $\sigma : X \to X$ such that

$$\sigma^{-1} \circ \beta \circ \sigma = \alpha.$$ 

We say $\alpha$ and $\beta$ are flip conjugate if either $\alpha$ and $\beta^{-1}$ (or $\alpha^{-1}$ and $\beta$ or $\alpha$ and $\beta$) are conjugate.
★. Theorem T (J. Tomiyama)

Let $X$ be a compact metric space and $\alpha, \beta : X \to X$ be homeomorphisms. Suppose that $(X, \alpha)$ and $(X, \beta)$ are topologically transitive. Then $\alpha$ and $\beta$ are flip conjugate if and only if there is an isomorphism $\phi : C(X) \rtimes_\alpha \mathbb{Z} \to C(X) \rtimes_\beta \mathbb{Z}$ such that $\phi \circ j_\alpha = j_\beta \circ \chi$ for some isomorphism $\chi : C(X) \to C(X)$.

It should be noted that all minimal dynamical systems are transitive.

★. Definition

Let $(X, \alpha)$ and $(X, \beta)$ be two topological transitive systems. $(X, \alpha)$ and $(X, \beta)$ are $C^*$-strongly approximately flip conjugate if there exists an $\phi : A_\alpha \to A_\beta$, a sequence of unitaries $u_n \in C(X) \rtimes_\alpha \mathbb{Z}$ and an isomorphism $\chi : C(X) \to C(X)$ such that

$$\lim_{n \to \infty} \| \text{ad} \ u_n \circ \phi \circ j_\alpha(f) - j_\beta \circ \chi(f) \| = 0 \text{ for } f \in C(X).$$
In Theorem T, let $\theta = [\phi]$ in $KK(A_\alpha, A_\beta)$. Let $\Gamma(\theta)$ be the induced element in $\text{Hom}(K_*(A_\alpha), K_*(A_\beta))$ which preserves the order and the unit. Then one has

$$[j_\alpha] \times \theta = [j_\beta \circ \chi]$$

Let $A$ be a stably finite $C^*$-algebra and $T(A)$ be the space of tracial states on $A$. There is a positive homomorphism $\rho_A : K_0(A) \to \text{Aff}(T(A))$, where $\text{Aff}(T(A))$ is the set of all real affine continuous functions on $T(A)$.

Suppose that $TR(A_\alpha) = TR(A_\beta) = 0$. Then $\rho_{A_\alpha}(K_0(A_\alpha))$ and $\rho_{A_\beta}(K_0(A_\beta))$ are dense in $\text{Aff}(T(A_\alpha))$ and $\text{Aff}(T(A_\beta))$ respectively. Thus $\Gamma(\theta)$ induces an order and unit preserving affine isomorphism $\theta_\rho : \text{Aff}(T(A_\alpha)) \to \text{Aff}(T(A_\beta))$.

For each $a \in A_{s.a.}$, one defines an element $\hat{a} \in \text{Aff}(T(A_\alpha))$ by $\hat{a}(\tau) = \tau(a)$. In particular, each element in $j_\alpha(C(X)_{s.a})$ gives an element in $\text{Aff}(T(A_\alpha))$. Therefore, in terms of $K$-theory and $KK$-theory, one has the following: If $\alpha$ and $\beta$ are flip conjugate, then there is an isomorphism.
\[ \chi : C(X) \to C(X) \text{ such that} \]
\[ [j_\alpha] \times \theta = [j_\beta \circ \chi] \text{ in } KK(C(X), A_\beta) \quad \text{and} \]
\[ \theta_\rho \circ \rho_{A_\alpha} \circ j_\alpha = \rho_{A_\beta} \circ j_\beta \circ \chi. \]

\[ \star. \textbf{Theorem D (L—2004)} \]

Let \((X, \alpha)\) and \((X, \beta)\) be two minimal dynamical systems such that \(A_\alpha\) and \(A_\beta\) have tracial rank zero. Then \(\alpha\) and \(\beta\) are \(C^*\)-strongly approximately flip conjugate if and only if the following hold: There is an sequence of isomorphism \(\chi_n : C(X) \to C(X)\) and \(\theta \in KL(A_\alpha, A_\beta)\) such that \(\Gamma(\theta)\) gives an isomorphism from

\((K_0(A_\alpha), K_0(A_\alpha)_+, [1], K_1(A_\alpha))\) to \((K_0(A_\beta), K_0(A_\beta)_+, [1], K_1(A_\beta))\),

\[ [j_\alpha] \times \theta = [j_\beta \circ \chi_n] \text{ in } KL(C(X), A_\beta) \text{ for all } n \quad \text{and} \]

\[ \lim_{n \to \infty} \| \rho_{A_\beta} \circ j_\beta \circ \chi_n(f) - \theta_\rho \circ \rho_{A_\alpha} \circ j_\alpha(f) \| = 0 \]

for all \(f \in C(X)\).
Cor C. Let $X$ be a compact metric space with torsion free $K$-theory. Let $(X, \alpha)$ and $(X, \beta)$ be two minimal dynamical systems such that $TR(A_\alpha) = TR(A_\beta) = 0$. Suppose that there is a unit preserving order isomorphism

- (i) $\gamma : (K_0(A_\alpha), K_0(A_\alpha)_+, [1_{A_\alpha}], K_1(A_\alpha))$
  \hspace{1cm} $\rightarrow (K_0(A_\beta), K_0(A_\beta)_+, [1_{A_\beta}], K_1(A_\beta))$,

- (ii) $[j_\alpha] \times \theta = [j_\beta \circ \chi]$ in $KL(C(X), A_\beta)$ and

- (iii) $\gamma_\rho \circ j_\alpha = \rho_{A_\beta} \circ j_\beta \circ \chi$

for some isomorphism $\chi : C(X) \rightarrow C(X)$. Then $(X, \alpha)$ and $(X, \beta)$ are $C^*$-strongly approximately flip conjugate.
\[ \star \text{. The Cantor set.} \]

In the case when \( X \) is the Cantor set, \( K_0(C(X)) = C(X, \mathbb{Z}) \). It follows that, if there is \( \theta : K_i(A_{\alpha}) \to K_i(A_{\beta}) \) that is an order and unit preserving isomorphism, then there exists \( \chi : C(X) \to C(X) \) such that

\[
\theta \circ (j_{\alpha})_{*0} = (j_{\beta} \circ \chi)_{*0}.
\]

Moreover, it implies that

\[
\theta_{\rho} \circ \rho_{A_{\alpha}} \circ j_{\alpha} = \rho_{A_{\beta}} \circ j_{\beta} \circ \chi.
\]

In other words, in the case that \( X \) is the Cantor set condition (ii) and (iii) is automatic.

\[ \star \text{. Approximate conjugacy.} \]

Two dynamical systems \((X, \alpha)\) and \((X, \beta)\) are said to be weakly approximately conjugate if there are \( \sigma_n, \gamma_n : X \to X \) such that

\[
\lim_{n \to \infty} \| f(\sigma_n^{-1} \circ \beta \circ \sigma_n) - f(\alpha) \| = 0 \quad \text{and} \quad \lim_{n \to \infty} \| f(\gamma_n^{-1} \circ \alpha \circ \gamma_n) - f(\beta) \| = 0
\]

for all \( f \in C(X) \).
This is too weak since there is no consistency among $\sigma_n$ and $\gamma_n$.

Suppose

$$\lim_{n \to \infty} \| f(\sigma_n \circ \alpha \circ \sigma_n^{-1}) - f(\beta) \| = 0$$

for all $f \in C(X)$. Then there exists a sequence of completely positive linear maps $\psi_n : B \to A$ such that

$$\lim_{n \to \infty} \| \psi_n(ab) - \psi(a)\psi(b) \| = 0$$

for all $a, b \in B$ and

$$\lim_{n \to \infty} \| \psi_n(f) - f \circ \sigma_n \| = 0$$

for all $f \in C(X)$ and

$$\lim_{n \to \infty} \psi_n(u_\beta) = u_\alpha,$$

where $u_\alpha$ and $u_\beta$ denote the implementing unitaries in $C(X) \ltimes_\alpha \mathbb{Z}$ and $C(X) \ltimes_\beta \mathbb{Z}$.

Let $(X, \alpha)$ and $(X, \beta)$ be dynamical systems on compact metrizable spaces $X$ and $Y$. Suppose that a sequence of homeomorphisms $\sigma_n : X \to X$ satisfies $\sigma_n \alpha \sigma_n^{-1} \to \beta$.

Let $\{\psi_n\}$ be the asymptotic morphism arising from $\sigma_n$. 

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★. We say that the sequence \( \{\sigma_n\} \) induces an order and unit preserving homomorphism \( H_* : K_*(C(X) \rtimes_\beta \mathbb{Z}) \to K_*(C(X) \rtimes_\alpha \mathbb{Z}) \) between \( K \)-groups, if for every projection \( p \in M_\infty(C(X) \rtimes_\beta \mathbb{Z}) \) and every unitary \( u \in M_\infty(C(X) \rtimes_\beta \mathbb{Z}) \), there exists \( N \in \mathbb{N} \) such that

\[
[\psi_n(p)] = H_*([p]) \in K_0(C(X) \rtimes_\alpha \mathbb{Z}) \quad \text{and}
\]

\[
[\psi_n(u)] = H_*([u]) \in K_1(C(X) \rtimes_\beta \mathbb{Z})
\]

for every \( n \geq N \).

★. (For torsion free case). We say that \((X, \alpha)\) and \((X, \beta)\) are approximately \( K \)-conjugate, if there exist homeomorphisms \( \sigma_n : X \to X \), \( \gamma_n : X \to X \) and a (unit preserving) order isomorphism \( H_* : K_*(C(X) \rtimes_\beta \mathbb{Z}) \to K_*(C(X) \rtimes_\alpha \mathbb{Z}) \) between \( K \)-groups such that

\[
\lim_{n \to \infty} \|f(\sigma_n \alpha \sigma_n^{-1}) - f(\beta)\| = 0 \quad \text{and}
\]

\[
\lim_{n \to \infty} \|f(\gamma_n \beta \gamma_n^{-1}) - f(\alpha)\| = 0
\]

for all \( f \in C(X) \) and the associated asymptotic morphisms \( \{\psi_n\} : B \to A \) and \( \{\phi_n\} : A \to B \) induce the isomorphisms \( H_* \) and \( H_*^{-1} \).
Theorem E (with H. Matui)

Let $X$ be the Cantor set and $\alpha$ and $\beta$ be minimal homeomorphisms. Then the following are equivalent:

(i) $\alpha$ and $\beta$ are $C^*$-strongly approximately flip conjugate,

(ii) $\alpha$ and $\beta$ are approximately $K$-conjugate,

(iii) $A_\alpha$ and $A_\beta$ are isomorphic,

(iv) $(K_0(A_\alpha), K_0(A_\alpha)_+, [1_{A_\alpha}]) \cong (K_0(A_\beta), K_0(A_\beta)_+, [1_{A_\beta}])$.

By a theorem of Giordano, Putnam and Skau, the above also equivalent to

(v) $(X, \alpha)$ and $(X, \beta)$ are strong orbit equivalent.