Proof of a conjecture of Mircea Merca

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Abstract. We prove that, for any prime $p$ and positive integer $r$ with $p^r > 2$, the number of multinomial coefficients such that
\[
\binom{k}{k_1, k_2, \ldots, k_n} = p^r, \quad \text{and} \quad k_1 + 2k_2 + \cdots + nk_n = n,
\]
is given by
\[
\delta_{p^r, k} \left( \left\lfloor \frac{n-1}{p^r-1} \right\rfloor - \delta_{0, n \mod p^r} \right),
\]
where $\delta_{i,j}$ is the Kronecker delta and $\lfloor x \rfloor$ stands for the largest integer not exceeding $x$. This confirms a recent conjecture of Mircea Merca.

Keywords: multinomial coefficients; binomial coefficients; Fine’s formula

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1 Introduction

The multinomial coefficients are defined by
\[
\binom{k}{k_1, k_2, \ldots, k_n} = \frac{k!}{k_1!k_2!\cdots k_n!},
\]
where $k = k_1 + k_2 + \cdots + k_n$. Fine [1, p. 87] gave a connection between multinomial coefficients and binomial coefficients:
\[
\sum_{k_1 + k_2 + \cdots + k_n = n \atop k_1 + 2k_2 + \cdots + nk_n = n} \binom{k}{k_1, k_2, \ldots, k_n} = \binom{n-1}{k-1}.
\]
(1.1)

Let $M_m(n, k)$ be the number of multinomial coefficients such that
\[
\binom{k}{k_1, k_2, \ldots, k_n} = m, \quad \text{and} \quad k_1 + 2k_2 + \cdots + nk_n = n.
\]

For example, we have $M_6(10, 3) = 4$, since
\[
10 = 1 + 2 + 7 = 1 + 3 + 6 = 1 + 4 + 5 = 2 + 3 + 5.
\]
It is easy to see that $M_1(n, k) = \delta_{0, n \mod k}$. Recently, applying Fine’s formula (1.1), Merca [2] obtained new upper bounds involving $M_m(n, k)$ for the number of partitions of $n$ into $k$ parts. He also proved that

$$M_2(n, k) = \delta_{2, k} \left\lfloor \frac{n-1}{2} \right\rfloor, \quad M_p(n, k) = \delta_{p, k} \left( \left\lfloor \frac{n-1}{p-1} \right\rfloor - \delta_{0, n \mod p} \right),$$

where $p$ is an odd prime.

In this paper, we shall prove the following result, which was conjectured by Merca [2, Conjecture 1].

**Theorem 1.** Let $p$ be a prime and let $n, k, r$ be positive integers with $p^r > 2$. Then

$$M_{p^r}(n, k) = \delta_{p^r, k} \left( \left\lfloor \frac{n-1}{p^r-1} \right\rfloor - \delta_{0, n \mod p^r} \right).$$

Merca [2] pointed out that, when $m$ is not a prime power, the formula for $M_m(n, k)$ is more involved. For example, we have

$$M_{10}(n, k) = \delta_{10, k} \left( \left\lfloor \frac{n-1}{9} \right\rfloor - \delta_{0, n \mod 10} \right) + \delta_{5, k} \left( \left\lfloor \frac{n+1}{6} \right\rfloor - \delta_{0, n \mod 5} - \delta_{0, n \mod 6} \right).$$

### 2 Proof of Theorem 1

We need the following result.

**Lemma 2.** Let $n$ and $k$ be two positive integers with $2 \leq k \leq \frac{n}{2}$. Then the binomial coefficient $\binom{n}{k}$ is not a prime power.

**Proof.** For any prime $p$, the $p$-adic order of $n!$ can be given by

$$\text{ord}_p n! = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor.$$

If $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ were a prime power, say $p^r$, then

$$r = \sum_{i=1}^{\infty} \left( \left\lfloor \frac{n}{p^i} \right\rfloor - \left\lfloor \frac{k}{p^i} \right\rfloor - \left\lfloor \frac{n-k}{p^i} \right\rfloor \right).$$

(2.1)

Note that $\lfloor x+y \rfloor - \lfloor x \rfloor - \lfloor y \rfloor \leq 1$. From (2.1) we deduce that $r$ is less than or equal to the largest integer $i$ such that $p^i \leq n$. Namely, $p^r \leq n$. On the other hand, for $2 \leq k \leq \frac{n}{2}$, we have $\binom{n}{k} > n$, a contradiction. Therefore, the initial assumption must be false. □

**Proof of Theorem 1.** Let

$$\binom{k_1 + k_2 + \cdots + k_n}{k_1, k_2, \ldots, k_n} = p^r.$$

(2.2)
We assert that there are exactly two \( i \)’s such that \( k_i \geq 1 \). In fact, if \( k_1, k_2, k_3 \geq 1 \), then either \( \binom{k_1+k_2+k_3}{k_1,k_2,k_3} = \binom{3}{1,1,1} = 6 \), or by Lemma 2, \( \binom{k_1+k_2+k_3}{k_a} \) \((k_a = \max\{k_1, k_2, k_3\})\) is not a prime power. But this is impossible, since both \( \binom{k_1+k_2+k_3}{k_1,k_2,k_3} \) and \( \binom{k_1+k_2+k_3}{k_1,k_2,k_3} \) divide \( \binom{k_1+k_2+\ldots+k_n}{k_1,k_2,\ldots,k_n} \). This proves the assertion. Furthermore, by Lemma 2 again, one of the two non-zero \( k_i \)’s must be 1, and by (2.2), the other non-zero term is equal to \( p^r - 1 \). In other words, the identity (2.2) holds if and only if \((k_1, k_2, \ldots, k_n)\) is a rearrangement of \((p^r - 1, 1, 0, \ldots, 0)\). Consider the equation

\[
(p^r - 1)x + y = n. 
\]  

(2.3)

If \( k = p^r \), then we conclude that \( M_{p^r}(n, k) \) is equal to the number of positive integer solutions \((x, y)\) to (2.3) with \( x \neq y \), i.e.,

\[
M_{p^r}(n, k) = \left\lfloor \frac{n - 1}{p^r - 1} \right\rfloor - \begin{cases} 1, & \text{if } n \equiv 0 \pmod{p^r}, \\ 0, & \text{otherwise}. \end{cases}
\]

If \( k \neq p^r \), then it is obvious that \( M_{p^r}(n, k) = 0 \). This completes the proof. \( \square \)

3 Concluding remarks

Note that, Lemma 2 plays an important part in our proof of Theorem 1. It seems that we may say something more about the factors of \( \binom{n}{k} \) for \( 2 \leq k \leq \frac{n}{2} \). Since \( \binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1} \), we have \( \gcd(\binom{n}{k}, n) > 1 \), where \( \gcd(a, b) \) denotes the greatest common divisor of two integers \( a \) and \( b \). If

\[
\gcd\left(\binom{n}{k}, n - 1\right) > 1,
\]  

(3.1)

then, noticing that \( \gcd(n, n - 1) = 1 \), we immediately deduce that \( \binom{n}{k} \) has at least two different prime factors (namely, Lemma 2 holds). However, in general, the inequality (3.1) does not hold. For example, \( \gcd\left(\binom{14}{3}, 6\right) = 1 \). Similarly, the fact \( \gcd\left(\binom{14}{4}, 12\right) = 1 \) means that we cannot expect

\[
\gcd\left(\binom{n}{k}, n - 2\right) > 1.
\]  

(3.2)

We close our paper with the following conjecture, which asserts that at least one of (3.1) and (3.2) is true.

**Conjecture 3.** Let \( n \) and \( k \) be two positive integers with \( 2 \leq k \leq \frac{n}{2} \). Then

\[
\gcd\left(\binom{n}{k}, \binom{n - 1}{2}\right) > 1.
\]

We have verified the above conjecture for \( n \) up to 5000 via Maple.

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References
