Factors of alternating sums of products of binomial and $q$-binomial coefficients

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Abstract. In this paper we study the factors of some alternating sums of products of binomial and $q$-binomial coefficients. We prove that for all positive integers $n_1, \ldots, n_m$, $n_{m+1} = n_1$, and $0 \leq j \leq m - 1$,

$$\left[ \frac{n_1 + n_m}{n_1} \right]^{-1} \sum_{k=-n_1}^{n_1} (-1)^k q^{jk^2 + \binom{k}{2}} \prod_{i=1}^{m} \left[ \frac{n_i + n_{i+1}}{n_i + k} \right] \in \mathbb{N}[q],$$

which generalizes a result of Calkin [Acta Arith. 86 (1998), 17–26]. Moreover, we show that for all positive integers $n$, $r$ and $j$,

$$\left[ \frac{2n}{n} \right]^{-1} \left[ \frac{2j}{j} \right] \sum_{k=-j}^{n} (-1)^{n-k} q^{A} \frac{1-q^{2k+1}}{1-q^{n+k+1}} \left[ \frac{2n}{n-k} \right] \left[ \frac{k+j}{k-j} \right]^r \in \mathbb{N}[q],$$

where $A = (r-1)\binom{n}{2} + r\binom{j+1}{2} + \binom{k}{2} - rjk$, which solves a problem raised by Zudilin [Electron. J. Combin. 11 (2004), #R22].

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1 Introduction

In 1998, Calkin [4] proved that for all positive integers $m$ and $n$,

$$\left( \frac{2n}{n} \right)^{-1} \sum_{k=-n}^{n} (-1)^k \left( \frac{2n}{n+k} \right)^m$$

is an integer by arithmetical techniques. For $m = 1, 2$ and $3$, by the binomial theorem, Kummer’s formula and Dixon’s formula, it is easy to see that (1.1) is equal to 0, 1 and $\binom{3n}{n}$, respectively. Recently in the study of finite forms of the Rogers-Ramanujan identities [9] we stumbled across (1.1) for $m = 4$ and $m = 5$, which gives

$$\sum_{k=0}^{n} \binom{2n+k}{n+k} \binom{2n}{n+k}^2 \quad \text{and} \quad \sum_{k=0}^{n} \binom{3n-k}{n-k} \binom{2n+k}{n+k} \binom{2n}{n+k}^2,$$

respectively. Indeed, de Bruijn [3] has shown that for $m \geq 4$ there is no closed form for (1.1) by asymptotic techniques. Our first objective is to give a $q$-analogue of Calkin’s result, which also implies that (1.1) is positive for $m \geq 2$.

In 2004, Zudilin [14] proved that for all positive integers $n$, $j$ and $r$,

$$\left( \frac{2n}{n} \right)^{-1} \left[ \frac{2j}{j} \right] \sum_{k=-j}^{n} (-1)^{n-k} \frac{2k+1}{n+k+1} \binom{2n}{n-k} \binom{k+j}{k-j}^r \in \mathbb{Z},$$

(1.2)

which was originally observed by Strehl [12] in 1994. In fact, Zudilin’s motivation was to solve the following problem, which was raised by Schmidt [11] in 1993 and was apparently not related to Calkin’s result.
Problem 1.1 (Schmidt [11]). For any integer \( r \geq 2 \), define a sequence of numbers \( \{c_k^{(r)}\}_{k \in \mathbb{N}} \), independent of the parameter \( n \), by
\[
\sum_{k=0}^{n} {n \choose k}^r \left( \frac{n+k}{k} \right)^r = \sum_{k=0}^{n} {n \choose k} \left( \frac{n+k}{k} \right)c_k^{(r)}.
\]

Is it true that all the numbers \( c_k^{(r)} \) are integers?

At the end of his paper, Zudilin [14] raised the problem of finding and solving a \( q \)-analogue of Problem 1.1. Our second objective is to provide such a \( q \)-analogue.

For any integer \( n \), define the \( q \)-shifted factorial \( (a)_n \) by \( (a)_0 = 1 \) and
\[
(a)_n = \begin{cases} 
(1-a)(1-aq)\cdots(1-aq^{n-1}), & n = 1, 2, \ldots, \\
((1-aq^{-1})(1-aq^{-2})\cdots(1-aq^n))^{-1}, & n = -1, -2, \ldots.
\end{cases}
\]

We will also use the compact notations for \( m \geq 1 \):
\[
(a_1, \ldots, a_m)_n := (a_1)_n \cdots (a_m)_n, \quad (a_1, \ldots, a_m)_\infty := \lim_{n \to \infty} (a_1, \ldots, a_m)_n.
\]

The \( q \)-binomial coefficients are defined as
\[
\left[ \frac{n}{k} \right]_q := \frac{(q)_n}{(q)_k (q)_{n-k}}.
\]

Since \( \frac{1}{(q)_k} = 0 \) if \( n < 0 \), we have \( \left[ \frac{n}{k} \right]_q = 0 \) if \( k > n \) or \( k < 0 \).

The following is our first generalization of Calkin’s result.

Theorem 1.2. For \( m \geq 3 \) and all positive integers \( n_1, \ldots, n_m \), there holds
\[
\sum_{k=-n_1}^{n_1} (-1)^k q^{(m-1)k^2 + \binom{k}{2}} \prod_{i=1}^{m} \left[ \frac{n_i + n_i + 1}{n_i + k} \right] = \left[ \frac{n_1 + n_m}{n_1} \right] \sum_{\lambda} \prod_{i=1}^{m-2} q^{\frac{1}{\lambda_i}} \left[ \frac{\lambda_1 - 1}{\lambda_i} \right] \left[ \frac{n_i + 1 + n_i + 2}{n_i + 1 - \lambda_i} \right], \tag{1.3}
\]

where \( n_{m+1} = \lambda_0 = n_1 \) and the sum is over all sequences \( \lambda = (\lambda_1, \ldots, \lambda_{m-2}) \) of nonnegative integers such that \( \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{m-2} \).

Calkin [4] has given a partial \( q \)-analogue of (1.1) by considering the alternating sum \( \sum_{k=0}^{n} (-1)^k q^{k \left[ \frac{n}{k} \right]} m \). In this respect, besides (1.3), we shall also prove the following divisibility result.

Theorem 1.3. For all positive integers \( n_1, \ldots, n_m, n_{m+1} = n_1 \), the alternating sum
\[
S(n_1, \ldots, n_m; j, q) := \left[ \frac{n_1 + n_m}{n_1} \right]^{-1} \sum_{k=-n_1}^{n_1} (-1)^k q^{jk^2 + \binom{k}{2}} \prod_{i=1}^{m} \left[ \frac{n_i + n_i + 1}{n_i + k} \right]
\]
is a polynomial in \( q \) with nonnegative integral coefficients for \( 0 \leq j \leq m - 1 \).

We shall give two proofs of Theorem 1.2: The first one is based on a recurrence relation formula for \( S(n_1, \ldots, n_m; j, q) \), which also leads to a proof of Theorem 1.3. The second one follows directly from Andrews’ basic hypergeometric identity between a single sum and a multiple sum [1, Theorem 4].
Theorem 1.4 (Andrews [1]). For every integer \( m \geq 1 \), the following identity holds:

\[
\sum_{k \geq 0} \frac{(aq, aq/\sqrt{a}, -aq/\sqrt{a}, b_1, c_1, ..., b_m, c_m, q^{-N})_k}{(aq/\sqrt{a}, -aq/\sqrt{a}, aq/b_1, aq/c_1, ..., aq/b_m, aq/c_m, aqN)_k} \left( a^m q^{m+N} \right)^k = \frac{(aq, aq/b_m c_m)_N}{(aq/b_m, aq/c_m)_N} \sum_{t_1, ..., t_{m-1} \geq 0} \frac{(aq/b_1 c_1)_{t_1} \cdots (aq/b_{m-1} c_{m-1})_{t_{m-1}}}{(q)_{t_1} \cdots (q)_{t_{m-1}}} (b_2, c_2)_{t_1} \cdots (b_m, c_m)_{t_1} \cdots (aq/b_{m-1} c_{m-1})_{t_1} \cdots (aq/c_{m-1})_{t_{m-1}}
\]

\( (aq/b_m, aq/c_m)_N \)

As will be shown, Theorem 1.6 follows directly from Andrews' identity (1.4).

Therefore, using Theorem 1.4 in its full generality we are able to formulate and prove a \( q \)-analogue of Problem 1.1.

Theorem 1.5. For any integer \( r \geq 1 \), define rational fractions \( c_k^{(r)}(q) \) of the variable \( q \), independent of \( n \), by writing

\[
\sum_{k=0}^{n} q^{r(n-k)+(1-r)(n_k)} \left[ \begin{array}{c} n+k \ni \frac{r}{k} \end{array} \right] = \sum_{k=0}^{n} q^{r(n-k)+(1-r)(n_k)} \left[ \begin{array}{c} n+k \ni \frac{r}{k} \end{array} \right] c_k^{(r)}(q). \quad (1.5)
\]

Then \( c_n^{(r)}(q) \in \mathbb{N}[q] \).

It is interesting to note that Theorem 1.4 is also a key ingredient in Zudilin’s approach to Problem 1.1. Therefore, using Theorem 1.4 in its full generality we are able to formulate and prove a \( q \)-analogue of Problem 1.1.
2 Proof of Theorems 1.2 and 1.3

We will need two known identities in $q$-series. One is the $q$-Pfaff-Saalschütz identity [6, Appendix (II.12)] (see also [7, 13]):

$$
\begin{bmatrix} n_1 + n_2 \\ n_1 + k \end{bmatrix} \begin{bmatrix} n_2 + n_3 \\ n_2 + k \end{bmatrix} \begin{bmatrix} n_3 + n_1 \\ n_3 + k \end{bmatrix} = \sum_{r=0}^{n_1-k} \frac{q^{k^2+2kr}(q)_{n_1+n_2+n_3-k-r}}{(q)_r(q)_{r+2k}(q)_{n_1-2k-r}(q)_{n_2-k-r}(q)_{n_3-k-r}} ,
$$

(2.1)

where $\frac{1}{(q)_n} = 0$ if $n < 0$, and the other is the $q$-Dixon identity:

$$
\sum_{k=-n_1}^{n_1} (-1)^k q^{(3k^2-k)/2} \begin{bmatrix} n_1 + n_2 \\ n_1 + k \end{bmatrix} \begin{bmatrix} n_2 + n_3 \\ n_2 + k \end{bmatrix} \begin{bmatrix} n_3 + n_1 \\ n_3 + k \end{bmatrix} = \frac{(q)_{n_1+n_2+n_3}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}} .
$$

(2.2)

A short proof of (2.2) is given in [8].

We first establish the following recurrence formula.

Lemma 2.1. Let $m \geq 3$. Then for all positive integers $n_1, \ldots, n_m$ and any integer $j$, the following recurrence holds:

$$
S(n_1, \ldots, n_m; j, q) = \sum_{l=0}^{m} q^2 \begin{bmatrix} n_1 \\ l \end{bmatrix} \begin{bmatrix} n_2 + n_3 \\ n_2 - l \end{bmatrix} S(l, n_3, \ldots, n_m; j-1, q) .
$$

(2.3)

Proof. For any integer $k$ and positive integers $a_1, \ldots, a_l$, let

$$
C(a_1, \ldots, a_l; k) = \prod_{i=1}^{l} \frac{a_i + a_i+1}{a_i + k} ,
$$

where $a_{l+1} = a_1$. Then

$$
S(n_1, \ldots, n_m; j, q) = \frac{(q)_{n_1}(q)_{n_m}}{(q)_{n_1+n_m}} \sum_{k=-n_1}^{n_1} (-1)^k q^{jk^2+(\binom{k}{2})} C(n_1, \ldots, n_m; k) .
$$

(2.4)

We observe that for $m \geq 3$, we have

$$
C(n_1, \ldots, n_m; k) = \frac{(q)_{n_2+n_3}(q)_{n_3+n_1}}{(q)_{n_1+n_2}(q)_{n_m+n_3}} \begin{bmatrix} n_1 + n_2 \\ n_1 + k \end{bmatrix} \begin{bmatrix} n_2 + n_3 \\ n_2 + k \end{bmatrix} C(n_3, \ldots, n_m; k) ,
$$

and, by letting $n_3 \to \infty$ in (2.1),

$$
\begin{bmatrix} n_1 + n_2 \\ n_1 + k \end{bmatrix} \begin{bmatrix} n_2 + n_3 \\ n_2 + k \end{bmatrix} = \sum_{r=0}^{n_1-k} \frac{q^{k^2+2kr}(q)_{n_1+n_2}}{(q)_r(q)_{r+2k}(q)_{n_1-2k-r}(q)_{n_2-k-r}(q)_{n_3-k-r}} .
$$

Plugging these into (2.4) we can write its right-hand side as

$$
R := \sum_{k=-n_1}^{n_1} \sum_{r=0}^{n_1-k} (-1)^k C(n_3, \ldots, n_m; k) \left( \frac{q^{(r+k)^2+(j-1)k^2+(\binom{k}{2})}(q)_{n_2+n_3}(q)_{n_1}(q)_{n_m}}{(q)_r(q)_{r+2k}(q)_{n_1-2k-r}(q)_{n_2-k-r}(q)_{n_3-k-r}} \right) .
$$

Setting $l = r + k$, then $-n_1 \leq l \leq n_1$, but if $l < 0$, at least one of the indices $l + k$ and $l - k$ is negative for any integer $k$, which implies that $\frac{1}{(q)_{l-k}(q)_{l+k}} = 0$ by convention. Therefore, exchanging the order of summation, we have

$$
R = \sum_{l=0}^{n_1} \frac{q^2(q)_{n_2+n_3}(q)_{n_1}(q)_{n_m}}{(q)_{n_1-l}(q)_{n_2-l}(q)_{n_m+n_3}} \sum_{k=-l}^{l} (-1)^k C(n_3, \ldots, n_m; k) \frac{q^{(j-1)k^2+(\binom{k}{2})}}{(q)_{l-k}(q)_{l+k}} .
$$


Now, in the last sum making the substitution

\[ C(n_3, \ldots, n_m; k) = \frac{(q)_{l-k} (q)_{l+k} (q)_{n_m+n_k} C(l, n_3, \ldots, n_m; k)}{(q)_{n_3+l} (q)_{n_m+l}}, \]

we obtain the right-hand side of (2.3).

**First proof of Theorem 1.2.** Letting \( n_3 \to \infty \) in (2.2) yields that

\[ S(n_1, n_2; 1, q) = 1. \]

(5.5)

Theorem 1.2 then follows by iterating \((m - 2) \times \) times formula (2.3).

**Second proof of Theorem 1.2.** Since

\[ \left[ \frac{M}{N+k} \right] = (-1)^k q^{(M-N) k - \binom{k}{2}} \left[ \frac{M}{N} \right] \frac{(q^{-M+N})_k}{(q^{N+1})_k}, \]

by collecting the terms of index \( k \) and \(-k\), the left-hand side of (1.3) can be written as

\[ L := \prod_{i=1}^m \left[ \frac{n_i + n_{i+1}}{n_i} \right] + \sum_{k=1}^{n_1} (1 + q^k) \left( -1 \right)^k q^{(m-1) k^2 + \binom{k}{2}} \prod_{i=1}^m \left[ \frac{n_i + n_{i+1}}{n_i + k} \right] \]

\[ = \prod_{i=1}^m \left[ \frac{n_i + n_{i+1}}{n_i} \right] \left\{ 1 + \sum_{k=1}^{n_1} (1 + q^k) \left( -1 \right)^{(m-1) k} q^{(m-1) \binom{k}{2}} \prod_{i=1}^m q_{n_i} \right\}. \]

Letting \( c_1 = c_2 = \cdots = c_m = c \to \infty \) and \( a \to 1 \) in Andrews’ formula (1.4) we get

\[ 1 + \sum_{k \geq 1} (1 + q^k) \frac{(b_1, \ldots, b_m, q^{-N})_k}{(q/b_1, \ldots, q/b_m, q^{N+1})_k} (-1)^m k q^{m \binom{k}{2}} (q^{m+N})_k \frac{b_1 b_2 \cdots b_m}{(b_1 b_2 \cdots b_m)^k} = \]

\[ \frac{(q)_N}{(q/b_m)_N \prod_{i=1}^{m-1} b_i_{l_i} \cdots b_{m-l_{m-1}}} \sum_{l_{m-1} \leq \cdots \leq l_1 \geq 0} (q)_{l_1} \cdots (q)_{l_{m-1}} (q)_{N-l_1-\cdots-l_{m-1}} \]

\[ \times \prod_{i=1}^{m-1} \frac{(b_{i+1})_{l_1+\cdots+l_i}}{(q/b_i)_{l_1+\cdots+l_i}} \left( \frac{-1}{b_{i+1}} \right)^{l_1+\cdots+l_i} q^{(l_1+\cdots+l_i)+(m-i)l_i}. \]

(2.6)

Now, shifting \( m \) to \( m - 1 \) in (2.6), setting

\[ N = n_m, \quad b_i = q^{-n_i} \quad \text{for} \quad i = 1, \ldots, m - 1, \]

and \( \lambda_i = l_1 + \cdots + l_i \) for \( i = 1, \ldots, m - 2 \), one sees that \( L \) equals

\[ \prod_{i=1}^m \left[ \frac{n_i + n_{i+1}}{n_i} \right] \frac{(q^2)_{n_m}}{(q^{1+n_m})_{n_m}} \sum_{0 \leq \lambda_1 \leq \cdots \leq \lambda_{m-2}} \prod_{i=1}^{m-2} \frac{(q^{-n_{i+1}})_{\lambda_i} (-1)^{\lambda_i} q^{(\lambda_i+1) + n_{i+1} + \lambda_i}}{(q^{1+n_i})_{\lambda_i} \lambda_i^{\lambda_i} - \lambda_{i-1}} \]

\[ \cdot \sum_{0 \leq \lambda_1 \leq \cdots \leq \lambda_{m-2}} \prod_{i=1}^{m-2} \frac{(q^2)_{\lambda_i+1}^{\lambda_i+1}}{(q^{1+n_i})_{\lambda_i} \lambda_i^{\lambda_i}} \left[ \frac{n_i + n_{i+1}}{n_i + \lambda_i} \right] \]

where \( \lambda_0 = 0 \) and \( \lambda_{m-1} = n_m \). The latter identity is clearly equivalent to Theorem 1.2.
In order to prove Theorem 1.3, we shall need the following relation:

\[ S(n_1, \ldots, n_m; 0, q) = S(n_1, \ldots, n_m; m - 1, q^{-1}) q^{n_1 n_2 + n_2 n_3 + \cdots + n_m - 1 - m}. \quad (2.7) \]

As \([n\choose k]_{k^{-1}} = [n\choose k] q^{k(n-k)}\), Eq. (2.7) can be verified by substituting \(q\) by \(q^{-1}\) and then replacing \(k\) by \(-k\) in the definition of \(S(n_1, \ldots, n_m; m - 1, q)\).

**Proof of Theorem 1.3.** We proceed by induction on \(m \geq 1\). By the \(q\)-binomial theorem [6, (II.3)], we have

\[ S(n_1; 0, q) = \sum_{k=-n_1}^{n_1} (-1)^k q^{k}\left[\frac{2n_1}{n+k}\right] = 0. \]

In view of (2.5), it follows from (2.7) that

\[ S(n_1, n_2; 0, q) = S(n_1, n_2; 1, q^{-1}) q^{n_1 n_2} = q^{n_1 n_2}. \]

So the theorem is valid for \(m = 2\).

Now suppose that the expression \(S(n_1, \ldots, n_m-1; j, q)\) is a polynomial in \(q\) with nonnegative integral coefficients for some \(m \geq 3\) and \(0 \leq j \leq m-2\). Then by the recurrence formula (2.3), so is \(S(n_1, \ldots, n_m; j, q)\) for \(1 \leq j \leq m-1\). It remains to show that \(S(n_1, \ldots, n_m; 0, q)\) has the required property. Since the \(q\)-binomial coefficient \([n\choose k]\) is a polynomial in \(q\) of degree \(k(n-k)\) (see [2, p. 33]), it is easy to see from the definition that the degree of the polynomial \(S(n_1, \ldots, n_m; m - 1, q)\) is less than or equal to \(n_1 n_2 + n_2 n_3 + \cdots + n_m - 1 - m\). It follows from (2.7) that \(S(n_1, \ldots, n_m; 0, q)\) is also a polynomial in \(q\) with nonnegative integral coefficients. This completes the inductive step of the proof.

**Remark.** Though it is not necessary to check the \(m = 3\) case to valid our induction argument, we think it is convenient to include here the formulas for \(m = 3\). First, the \(q\)-Dixon identity (2.2) implies that

\[ S(n_1, n_2, n_3; 1, q) = \left[\frac{n_1 + n_2 + n_3}{n_2}\right]. \]

From (2.3) and (2.5) we derive

\[ S(n_1, n_2, n_3; 2, q) = \sum_{l=0}^{n_1} q^{l^2} \left[\frac{n_1}{l}\right] \left[\frac{n_2 + n_3}{n_2 - l}\right]. \]

Finally, applying (2.7) we get

\[ S(n_1, n_2, n_3; 0, q) = S(n_1, n_2, n_3; 2, q^{-1}) q^{n_1 n_2 + n_2 n_3} \]

\[ = \sum_{l=0}^{n_1} q^{(n_1-l)(n_2-l)+n_3} \left[\frac{n_1}{l}\right] \left[\frac{n_2 + n_3}{n_2 - l}\right]. \]

### 3 Proof of Theorem 1.6

We will distinguish the cases where \(r \geq 2\) is even or odd, and treat separately the values \(r = 2\) and \(r = 3\).
• For \( r = 2 \), apply (1.4) specialized with \( m = 1, \ a = q^{-(2n+1)}, \ N = n - j, \ b_1 = q^{-n} \) and \( c_1 = q^{-(n-j)}. \) The left-hand side of (1.4) is then equal to

\[
\left[ \frac{n + j}{2j} \right]^{-2} q^{-2(n-j) + \binom{n}{2}} t_{n,j}^{(2)}(q).
\]

Equating this with the right-hand side gives

\[
t_{n,j}^{(2)}(q) = \frac{(q)_{2n}(q)_j^2}{(q)_n(q)_{2j}(q)_{2j-n}(q)_{n-j}^2} q^{2(n-j)-\binom{n}{2}},
\]

which shows that \( \left[ \binom{n}{2} \right]^{-1} q^{2(n-j)} t_{n,j}^{(2)}(q) \in \mathbb{N}[q]. \)

• For \( r = 3 \), apply (1.4) specialized with \( m = 1, \ a = q^{-(2n+1)}, \ N = n - j \) and \( b_1 = c_1 = q^{-(n-j)}. \) This yields in that case

\[
t_{n,j}^{(3)}(q) = \frac{(q)_{2n}}{(q)_{3j-n}(q)_{n-j}^2} q^{3(n-j)-2\binom{n}{3}},
\]

which shows that \( \left[ \binom{n}{3} \right]^{-1} q^{2(n-j)} t_{n,j}^{(3)}(q) \in \mathbb{N}[q]. \)

• For \( r = 2s \geq 4 \), apply (1.4) with \( m = s \geq 2, \ a = q^{-(2n+1)}, \ N = n - j, \ b_1 = q^{-n} \) and \( c_1 = c_i = q^{-(n-j)}, \ \forall i \in \{2, \ldots, s\} \) to get

\[
q^{(2s-1)\binom{n}{2}-2s(n-j)} t_{n,j}^{(2s)}(q) = \frac{(q)_{2n}(q)_j}{(q)_n(q)_{2j}(q)_{n-j}} \sum_{l_1 \geq 0} \left[ \frac{j}{l_1} \right] \left[ \frac{n-l_1}{j} \right] \left[ \frac{n-l_1+j}{n-l_1-j} \right] t_{l_1}^{(2s-1)}(q)_{l_1+j+2j(s-1)l_1+(j+1-n)l_1}
\times \sum_{l_2 \geq 0} \left[ \frac{2j}{l_2} \right] \left[ \frac{n-l_1-l_2+j}{n-l_1-l_2-j} \right] t_{l_2}^{(2s-1)}(q)_{l_2+2j(s-2)l_2+(j+1-n)l_2+\ldots}
\times \sum_{l_{s-1} \geq 0} \left[ \frac{2j}{l_{s-1}} \right] \left[ \frac{n-l_1-\cdots-l_{s-1}+j}{n-l_1-\cdots-l_{s-1}+j} \right] t_{l_{s-1}}^{(2s-1)}(q)_{l_{s-1}+2jl_{s-1}+(j+1-n)l_{s-1}}
\times \left[ \frac{n-l_1-\cdots-l_{s-1}+j}{2j} \right] t_{n-j}^{(2s)}(q).
\]

As the condition \( l_1 + \cdots + l_{s-1} \leq n - j \) holds in the last summation, we can see that for \( s \geq 2, \ \left[ \binom{n}{2s} \right]^{-1} q^{(2s-1)\binom{n}{2}} t_{n,j}^{(2s)}(q) \in \mathbb{N}[q]. \)

• For \( r = 2s + 1 \geq 5 \), apply (1.4) with \( m = s \geq 2, \ a = q^{-(2n+1)}, \ N = n - j, \)
and \( b_i = c_i = q^{-(n-j)}, \forall i \in \{1, \ldots, s\} \) to get

\[
q^{2s(n_j^2)-(2s+1)(n_j)} i_{n,j}^{(2s+1)}(q) =
\frac{(q)_{2n}}{(q)_{2j}(q)_{n-j}} \sum_{l_1 \geq 0} \left[ \begin{array}{c} 2j \\ l_1 \end{array} \right] \frac{n-l_1+j}{n-l_1-j} q^{l_1} \times \sum_{l_2 \geq 0} \left[ \begin{array}{c} 2j \\ l_2 \end{array} \right] \frac{n-l_1-l_2+j}{n-l_1-l_2-j} q^{l_2} \times \ldots
\]

As the condition \( l_1 + \cdots + l_{s-1} \leq n-j \) holds in the last summation, we can see that for \( s \geq 2 \), \([2j] \frac{q^{-(n-j)}}{[n]} i_{n,j}^{(2s+1)}(q) \in \mathbb{N}[q]. \)

**Remark.** In the special case \( r = 2 \), our proof gives the following expression for the coefficients \( c_n^{(2)}(q) \):

\[
c_n^{(2)}(q) = \sum_{j=0}^{n} \left[ \begin{array}{c} 2j \\ n \end{array} \right] \left[ \begin{array}{c} n \\ j \end{array} \right] q^{2(n-j)}. \tag{3.1}
\]

These coefficients are \( q \)-analogues of the famous \( c_n^{(2)}(1) \) involved in Apéry’s proof of the irrationality of \( \zeta(3) \):

\[
c_n^{(2)}(1) = \sum_{j=0}^{n} \left[ \begin{array}{c} 2j \\ n \end{array} \right] \left[ \begin{array}{c} n \\ j \end{array} \right] = \sum_{j=0}^{n} \left( \begin{array}{c} n \\ j \end{array} \right)^3. \tag{3.2}
\]

As explained in [12], when \( q = 1 \), one can derive the last expression from (3.1) in an elementary way (by two iteration of the Chu-Vandermonde formula). But our \( q \)-analogue (3.1) does not lead to a natural \( q \)-analogue of (3.2).

### 4 Consequences of Theorems 1.2 and 1.3

Letting \( q = 1 \) in Theorem 1.2 we obtain a direct generalization of Calkin’s result (1.1).

**Theorem 4.1.** For \( m \geq 3 \) and all positive integers \( n_1, \ldots, n_m \), there holds

\[
\sum_{k=-n_1}^{n_1} (-1)^k \prod_{i=1}^{m} \binom{n_i+n_{i+1}}{n_i+k} = \binom{n_1+n_m}{n_1} \sum_{\lambda} \prod_{i=1}^{m-2} \binom{\lambda_{i-1}}{\lambda_i} \binom{n_{i+1}+n_{i+2}}{n_{i+1}-\lambda_i}, \tag{4.1}
\]

where \( n_{m+1} = \lambda_0 = n_1 \) and the sum is over all sequences \( \lambda = (\lambda_1, \ldots, \lambda_{m-2}) \) of nonnegative integers such that \( \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{m-2} \).

**Remark.** For \( m = 1 \) and 2, it is easy to see that the left-hand side of (4.1) is equal to 0 and \( \binom{n_1+n_2}{n_1} \), respectively. Calkin’s result follows from (4.1) by setting \( n_i = n \) for all \( i = 1, \ldots, m \).
Letting \( n_1 = \cdots = n_m = n \) in Theorem 1.3, we obtain a complete \( q \)-analogue of Calkin’s result.

**Corollary 4.2.** For all positive \( m, n \) and \( 0 \leq j \leq m - 1 \),

\[
\label{eq:cor42}
\left[ \begin{array}{c} 2n \\ n \end{array} \right]^{-1} \sum_{k=-n}^{n} (-1)^k q^{jk^2 + \binom{k}{2}} \left[ \begin{array}{c} 2n \\ n+k \end{array} \right]^m
\]

is a polynomial in \( q \) with nonnegative integral coefficients.

Letting \( n_{2i-1} = m \) and \( n_{2i} = n \) for \( 1 \leq i \leq r \) in Theorem 1.3, we obtain

**Corollary 4.3.** For all positive \( m, n, r \) and \( 0 \leq j \leq 2r - 1 \),

\[
\label{eq:cor43}
\left[ \begin{array}{c} m+n \\ m \end{array} \right]^{-1} \sum_{k=-m}^{m} (-1)^k q^{jk^2 + \binom{k}{2}} \left[ \begin{array}{c} m+n \\ m+k \end{array} \right]^r \left[ \begin{array}{c} m+n \\ n+k \end{array} \right]^r
\]

is a polynomial in \( q \) with nonnegative integral coefficients. In particular,

\[
\sum_{k=-m}^{m} (-1)^k \left( \frac{m+n}{m+k} \right)^r \left( \frac{m+n}{n+k} \right)^r
\]

is divisible by \( \binom{m+n}{m} \).

Letting \( n_{3i-2} = l, n_{3i-1} = m \) and \( n_{3i} = n \) for \( 1 \leq i \leq r \) in Theorem 1.3, we obtain

**Corollary 4.4.** For all positive \( l, m, n, r \) and \( 0 \leq j \leq 3r - 1 \),

\[
\label{eq:cor44}
\left[ \begin{array}{c} l+n \\ n \end{array} \right]^{-1} \sum_{k=-l}^{l} (-1)^k q^{jk^2 + \binom{k}{2}} \left[ \begin{array}{c} l+m \\ l+k \end{array} \right]^r \left[ \begin{array}{c} m+n \\ m+k \end{array} \right]^r \left[ \begin{array}{c} n+l \\ n+k \end{array} \right]^r
\]

is a polynomial in \( q \) with nonnegative integral coefficients. In particular,

\[
\sum_{k=-l}^{l} (-1)^k \left( \frac{l+m}{l+k} \right)^r \left( \frac{m+n}{m+k} \right)^r \left( \frac{n+l}{n+k} \right)^r
\]

is divisible by \( \binom{l+m}{l}, \binom{m+n}{m} \) and \( \binom{n+l}{n} \).

Letting \( m = 2r + s, n_1 = n_3 = \cdots = n_{2r-1} = n + 1 \) and let all the other \( n_i \) be \( n \) in Theorem 4.1, we get

**Corollary 4.5.** For all positive \( r, s \) and \( n \),

\[
\sum_{k=-n}^{n} (-1)^k \left( \frac{2n+1}{n+k+1} \right)^r \left( \frac{2n+1}{n+k} \right)^r \left( \frac{2n}{n+k} \right)^s
\]

is divisible by both \( \binom{2n}{n} \) and \( \binom{2n+1}{n} \), and is therefore divisible by \( (2n+1)\binom{2n}{n} \).

However, the following result is not a special case of Theorem 4.1.
Corollary 4.6. For all nonnegative \(r\) and \(s\) and positive \(t\) and \(n\),

\[
\sum_{k=-n}^{n} (-1)^k \binom{2n+1}{n+k+1}^r \binom{2n+1}{n+k}^s \binom{2n}{n+k}^t.
\]

is divisible by \(\binom{2n}{n}\).

Proof. We proceed by induction on \(|r-s|\). The \(r = s\) case is clear from Corollary 4.5. Suppose the statement is true for \(|r-s| \leq m-1\). By Theorem 4.1, one sees that

\[
\sum_{k=-n}^{n} (-1)^k \binom{2n+1}{n+k+1}^m \binom{2n+1}{n+k}^s \binom{2n}{n+k}^t
= \frac{2n+2}{2n+1} \sum_{k=-n}^{n} (-1)^k \binom{2n+1}{n+k+1}^{m-1} \binom{2n+1}{n+k}^{s+1} \binom{2n+1}{n+k}^{s+1} \binom{2n}{n+k}^{t-1},
\]

(4.2)

where \(m, t \geq 1\), is divisible by

\[
\frac{2n+2}{2n+1} \binom{2n+1}{n} = \binom{2n}{n}.
\]

By the binomial theorem, we have

\[
\binom{2n+1}{n+k+1} = \binom{2n+1}{n+k+1} + \binom{2n+1}{n+k},
\]

\[
= \sum_{i=0}^{m} \binom{m}{i} \binom{2n+1}{n+k+1}^i \binom{2n+1}{n+k}^{m-i}.
\]

(4.3)

Substituting (4.3) into the left-hand side of (4.2) and using the induction hypothesis and symmetry, we find that

\[
\sum_{k=-n}^{n} (-1)^k \binom{2n+1}{n+k+1}^{m+s} \binom{2n+1}{n+k}^{s} \binom{2n}{n+k}^t
+ \sum_{k=-n}^{n} (-1)^k \binom{2n+1}{n+k+1}^s \binom{2n+1}{n+k}^{m+s} \binom{2n}{n+k}^t
\]

is divisible by \(2\binom{2n}{n}\). However, replacing \(k\) by \(-k\), one sees that

\[
\sum_{k=-n}^{n} (-1)^k \binom{2n+1}{n+k+1}^{m+s} \binom{2n+1}{n+k}^{s} \binom{2n}{n+k}^t
= \sum_{k=-n}^{n} (-1)^k \binom{2n+1}{n+k+1}^s \binom{2n+1}{n+k}^{m+s} \binom{2n}{n+k}^t.
\]

This proves that the statement is true for \(|r-s| = m\).

It is clear that Theorems 1.3 and 4.1 can be restated in the following forms.
Theorem 4.7. For all positive integers $n_1, \ldots, n_m$ and $0 \leq j \leq m - 1$, the alternating sum
\[
(q)_{n_1} \prod_{i=1}^{m} (q)_{n_i + n_{i+1}} \sum_{k=-n_1}^{n_1} (-1)^k q^{j k^2 + (k^2) \sum_{i=1}^{m} \left[ \frac{2n_i}{n_i + k} \right]},
\]
where $n_{m+1} = 0$, is a polynomial in $q$ with nonnegative integral coefficients.

Theorem 4.8. For all positive integers $n_1, \ldots, n_m$, we have
\[
n_1! \prod_{i=1}^{m} (n_i + n_{i+1})! \sum_{k=-n_1}^{n_1} (-1)^k \prod_{i=1}^{m} \left( \frac{2n_i}{n_i + k} \right) \in \mathbb{N},
\]
where $n_{m+1} = 0$.

It is easy to see that, for all positive integers $m$ and $n$, the expression $(\frac{(2m)!}{(2n)!})^{(m+n)!} m! n!$ is an integer by considering the power of a prime dividing a factorial. Letting $n_1 = \cdots n_r = m$ and $n_{r+1} = \cdots = n_{r+s} = n$ in Theorem 4.8, we obtain

Corollary 4.9. For all positive $m$, $n$, $r$ and $s$,
\[
\sum_{k=-m}^{m} (-1)^k \left( \frac{2m}{m+k} \right)^r \left( \frac{2n}{n+k} \right)^s
\]
is divisible by $(\frac{(2m)!}{(2n)!})^{(m+n)!} m! n!$.

In particular, we find that
\[
\sum_{k=-n}^{n} (-1)^k \left( \frac{4n}{2n+k} \right)^r \left( \frac{2n}{n+k} \right)^s
\]
is divisible by $(\frac{4n}{n})$, and
\[
\sum_{k=-n}^{n} (-1)^k \left( \frac{6n}{3n+k} \right)^r \left( \frac{2n}{n+k} \right)^s
\]
is divisible by $(\frac{6n}{n})^{(3n)!} (2n)!$.

From Theorem 4.8 it is easy to see that
\[
n_1! \prod_{i=1}^{m} (n_i + n_{i+1})! \sum_{k=-n_1}^{n_1} (-1)^k \prod_{i=1}^{m} \left( \frac{2n_i}{n_i + k} \right)^{r_i},
\]
where $n_{m+1} = 0$, is a nonnegative integer for all $r_1, \ldots, r_m \geq 1$. For $m = 3$, letting $(n_1, n_2, n_3)$ be $(n, 3n, 2n)$, $(2n, n, 3n)$, or $(2n, n, 4n)$, we obtain the following two corollaries.

Corollary 4.10. For all positive $r$, $s$, $t$ and $n$,
\[
\sum_{k=-n}^{n} (-1)^k \left( \frac{6n}{3n+k} \right)^r \left( \frac{4n}{2n+k} \right)^s \left( \frac{2n}{n+k} \right)^t
\]
is divisible by $(\frac{6n}{n})$ and $(\frac{6n}{3n})$.

Corollary 4.11. For all positive $r$, $s$, $t$ and $n$,
\[
\sum_{k=-n}^{n} (-1)^k \left( \frac{8n}{4n+k} \right)^r \left( \frac{4n}{2n+k} \right)^s \left( \frac{2n}{n+k} \right)^t
\]
is divisible by $(\frac{8n}{3n})$. 

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5 Some open problems

Based on computer experiments, we would like to present four interesting conjectures. The first two are refinements of Corollaries 4.10 and 4.11 respectively.

**Conjecture 5.1.** For all positive \( r, s, t \) and \( n \),
\[
\sum_{k=-n}^{n} (-1)^k \left( \binom{6n}{3n+k} r \binom{4n}{2n+k} s \binom{2n}{n+k} t \right)
\]
is divisible by both \( 2^{(6n)} \) and \( 6^{(6n)} \).

**Conjecture 5.2.** For all positive \( r, s, t \) and \( n \) with \( (r, s, t) \neq (1, 1, 1) \),
\[
\sum_{k=-n}^{n} (-1)^k \left( \binom{8n}{4n+k} r \binom{4n}{2n+k} s \binom{2n}{n+k} t \right)
\]
is divisible by \( 2^{(8n)} \).

Conjectures 5.1 and 5.2 are true for \( r + s + t \leq 10 \) and \( n \leq 100 \).

Let \( \gcd(a_1, a_2, \ldots) \) denote the greatest common divisor of integers \( a_1, a_2, \ldots \).

**Conjecture 5.3.** For all positive \( m \) and \( n \), we have
\[
\gcd \left( \sum_{k=-n}^{n} (-1)^k \left( \binom{2n}{n+k} r \right), r = m, m+1, \ldots \right) = \binom{2n}{n}. \tag{5.1}
\]

Let \( d \geq 2 \) be a fixed integer. Every nonnegative integer \( n \) can be uniquely written as
\[
n = \sum_{i \geq 0} a_i d^i,
\]
where \( 0 \leq a_i \leq d-1 \) for all \( i \) and only finitely many number of \( b_i \) are nonzeros, denoted by \( n = [\cdots a_1 a_0]_d \), in which the first 0's are omitted. Let \( n = [a_1 \cdots a_r]_3 = [b_1 \cdots b_s]_7 = [c_1 \cdots c_t]_{13} \). We now define three statistics \( \alpha(n) \), \( \beta(n) \) and \( \gamma(n) \) as follows.

- Let \( \alpha(n) \) be the number of disconnected 2's in the sequence \( a_1 \cdots a_r \). Here two nonzero digits \( a_i \) and \( a_j \) are said to be disconnected if there is at least one 0 between \( a_i \) and \( a_j \).
- Let \( \beta(n) \) be the number of 1's in \( b_1 \cdots b_s \) which are not immediately followed by a 4, 5, or 6.
- Let \( \gamma(n) \) be the number of 1's in \( c_1 \cdots c_t \) which are immediately followed by one of 7, \ldots, 12, or immediately followed by a number of 6's and then followed by one of 7, \ldots, 12.

For instance, \( [20212]_3 = 185 \) and so \( \alpha(185) = 2 \); \( [10142]_7 = 2480 \), and so \( \beta(2480) = 1 \); \( [1667]_{13} = 3296 \) and so \( \gamma(3296) = 1 \). The first \( n \) such that \( \alpha(n) = 4 \) is \( [202020]_3 = 1640 \); the first \( n \) such that \( \beta(n) = 4 \) is \( [1111]_7 = 400 \); while the first \( n \) such that \( \gamma(n) = 4 \) is \( [17171717]_{13} = 97110800 \).

We end this paper with the following conjecture.
Conjecture 5.4. For every positive integer $n$, we have
\[
\gcd \left( \sum_{k=-n}^{n} (-1)^k \left( \frac{2n}{n+k} \right)^{3r}, r = 1, 2, \ldots \right) = \left( \frac{2n}{n} \right)^{3\alpha(n)},
\]
\[
\gcd \left( \sum_{k=-n}^{n} (-1)^k \left( \frac{2n}{n+k} \right)^{3r+1}, r = 1, 2, \ldots \right) = \left( \frac{2n}{n} \right)^{7\beta(n)} 13^{\gamma(n)},
\]
\[
\gcd \left( \sum_{k=-n}^{n} (-1)^k \left( \frac{2n}{n+k} \right)^{3r+2}, r = 1, 2, \ldots \right) = \left( \frac{2n}{n} \right).
\]

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References


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