Transonic Flows in Nozzles

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Introduction

We consider...total pressure...sonic flow. It is well-known from experiments and numerical simulations in gas dynamics that, for given appropriate total pressure (pressure at the entry) and back pressure (pressure at the exit), subsonic-supersonic continuous flow may appear near the throat of the nozzle, while a supersonic-subsonic transonic shock may appear in the divergent part of the nozzle. (See the following picture, which belongs to P. Sohn et al. [8].)

This poster is to introduce many results obtained by the author and his collaborators — Stability: Two-Dimensional Full Euler Flow

Transonic Shocks in Straight Duct

Existence: In straight duct with constant cross-section (see the picture below), one can construct a class of uniform transonic shocks with planar shock fronts by using the Rankine-Hugoniot conditions of either full Euler system or potential flow equation (cf. [10]).

Stability: Two-Dimensional Full Euler Flow

For the stability studied via potential flow equation, see papers of G.-Q. Chen, M. Feldman, Z. Xin, H. Yin etc.

Uniqueness: Three-Dimensional Potential Flow

In [2] we proved that, for given uniform supersonic flow at the entry of a finitely long straight duct with arbitrary $C^0$ cross-section, and a uniform back pressure at the exit of the duct, there is no any transonic shock in the duct, or there exist a family of transonic shocks in the duct which are translations of the special uniform transonic-shocks we constructed before.

In the case of infinitely long straight duct with arbitrary $C^0$ cross-section, for given uniform supersonic flow at the entry, and assuming the flow behind the shock front is always subsonic, we showed that the only transonic shock solution is the special uniform transonic shock modulo a translation.

The proof depends on maximum principles of elliptic equations, an extreme principle for solutions of elliptic equations in unbounded domain, and judicious choices of special solutions as comparison functions. This method of proving uniqueness of free boundary problems of elliptic equations may be used to other problem which has a family of special solutions with fine structure (cf. [5]).

For the stability studied via potential flow equation, see papers of G.-Q. Chen, M. Feldman, Z. Xin, H. Yin etc.

Stability: Two-Dimensional Full Euler Flow

In [4], we showed the spherical symmetric transonic shocks are stable with respect to the perturbations of incoming potential function at the entry, and the back pressure.

New Ingredients: (a) Nonclassical local elliptic problems arise due to the interaction of the "elliptic part" and "hyperbolic part" of the steady subsonic Euler system; (b) Determination of the position of the shock front by using integral like stability conditions of boundary value problems like the Neumann problem of Poisson equation. Z. Xin, H. Yin etc. also studied the stability of this class of spherical transonic shocks in nozzles for two-dimensional Euler flows, but requiring the open angle to be small.

Uniqueness: Three-Dimensional Potential Flow

In [5], upon generalizing the methods in [4], we showed that for given spherically symmetric supersonic data at the entry of a divergent finitely long-straight nozzle with arbitrary smooth cross-section, and spherically symmetric back pressure, the solution must be the spherically symmetric transonic shock constructed before.

Subsonic–Supersonic Flows in Approximate Nozzles

Quasi One-Dimensional Model as Compressible Flows in Riemannian Manifold

From mathematical point of view, the quasi one-dimensional model of nozzle flows in gas dynamics may be regarded as compressible flows in Riemannian manifold like $(\mathcal{M}, G)$

\[ \mathcal{M} = \{(x, y) \in [-1, 1] \times \mathbb{S}^1\} \]

\[ G = dx \otimes dx + n^2(x) dy \otimes dy. \]

with $\mathbb{S}^1$ the unit circle, $n(x)$ a smooth positive strictly convex function on $[-1, 1]$ satisfying $n''(x) < 0$ on $(-1, 0)$ and $n''(x) > 0$ on $(0, 1)$.

Example: The potential flow equation on $(\mathcal{M}, G)$ is given by

\[ \Delta_{G} \psi + \nabla_{G} \mu \cdot \nabla_{G} \psi = 0, \]

\[ -\frac{1}{2} \Delta_{G} \psi - \mu \cdot \nabla_{G} \psi + \frac{\rho^{2}}{G} \psi = 0, \]

with $\psi$ the velocity potential function, $\mu$ the density, $\gamma > 1$, and $\nabla_{G}, \cdot, \cdot, \cdot$ respectively the divergence operator, gradient operator and inner product with respect to the metric $G$.

Existence: Subsonic–Supersonic Flow in Convergent Divergent Approximate Nozzles

This is done by constructing symmetric special solutions (i.e., these solutions depend only on $x$). See [11]. See also L. M. Sibner and R. J. Sibner [7] for earlier works on potential flows in terms.


In [12], by utilizing methods of A. Kuz'min [3], we proved that the symmetric subsonic-supersonic potential flow is stable with respect to the perturbation of potential function $\psi$ at the entry \( x = -1 \).

Uniqueness: Subsonic–Supersonic Potential Flows in Convergent Approximate Nozzles

The left part of $(\mathcal{M}, G)$ (with $-1 \leq x \leq 0$) is called convergent approximative nozzle. In [6] we showed, for given appropriate total pressure such that a symmetric subsonic-sonic flow exists in the nozzle, and the flow at the exit \( x = 0 \) is sonic, then if the flow is accelerating at the exit (i.e., $\partial_{x} \psi > 0$), then any solution of this boundary value problem must be the symmetric subsonic-sonic flow.

New Ingredient: A generalized Hopf boundary point lemma for degenerate elliptic operators, which is applicable to characteristic degenerate boundary point.

Two Unsolved Big Problems

Stability and uniqueness of spherically symmetric transonic shocks in three-dimensional flow with given supersonic upstream and back pressure.

Existence and uniqueness of subsonic-supersonic potential flows in two-dimensional physical convergent-divergent nozzles.

Literature