Unsolved Matrix Problems

Xingzhi Zhan
zhan@math.ecnu.edu.cn
Department of Mathematics
East China Normal University
Let $\rho(\cdot)$ denote the spectral radius, i.e., the maximum modulus of eigenvalues of a matrix. Let $\circ$ denote the Hadamard product, i.e., entry-wise product of matrices.

**Conjecture 1.** If $A, B$ are nonnegative matrices of the same order, then

$$\rho(A \circ B) \leq \rho(AB).$$

Let $Z(n)$ denote the set of 0-1 matrices of order $n$. Let $f(A)$ denote the number of 1’s in a matrix $A$. For positive integers $n$, $k$ let

$$\gamma(n, k) = \max\{f(A) | A \in Z(n), A^k \in Z(n)\},$$

$$\beta(n, k) = \max\{f(A^k) | A \in Z(n)\}.$$

In 2007 I posed the following two problems at a seminar.

**Problem 2.** Determine $\gamma(n, k)$ and determine the 0-1 matrices that attain this maximum number.

The case $k = 2$ is solved by Wu [7], and the case $k \geq n - 1$ is solved by Huang and Zhan [5].

Let $A = (a_{ij}) \in Z(n)$. The digraph of $A$ is the digraph with vertices $1, \ldots, n$ and in which $(i, j)$ is an arc if and only if $a_{ij} \neq 0$. This gives a bijection between $Z(n)$ and the set of digraphs of order $n$. 
The graph-theoretic interpretation of Problem 2: 
Let $D$ be a digraph of order $n$. If for each pair of vertices $i, j$ of $D$ there is at most one walk of length $k$ from $i$ to $j$, then what is the maximum number of arcs in $D$? Determine the digraphs that attain this maximum number.

**Problem 3.** Determine $\beta(n, k)$ and determine the 0-1 matrices that attain this maximum number. We may also consider the analogous problem for symmetric 0-1 matrices.

The graph-theoretic interpretation of Problem 3: 
Let $D$ be a digraph of order $n$. What is the maximum number of pairs of vertices $i, j$ of $D$ for which there is exactly one walk of length $k$ from $i$ to $j$? Determine the digraphs that attain this maximum number.
It is known [2, 6] that $\beta(n, k) = n^2$ if and only if there is a positive integer $m$ such that $m^k = n$.

We can see that

$$\beta(n, 2) = \begin{cases} n^2 & \text{if } n \text{ is a square}, \\ n^2 - 1 & \text{otherwise}. \end{cases}$$

For a complex matrix $A$, denote the entry-wise complex conjugate, the transpose and the conjugate transpose of $A$ by $\bar{A}$, $A^T$ and $A^*$ respectively.
In 2007 I posed the following conjecture at a seminar.

**Conjecture 4.** *Let $A, B$ be complex matrices of the same order. Then*

$$\| (A \circ B)(A \circ B)^* \| \leq \| (A \circ \bar{A})(B \circ \bar{B})^T \|$$

*for any unitarily invariant norm $\| \cdot \|$.*

Du [3] has proved the special cases when $\| \cdot \|$ is the spectral norm, the trace norm, or the Frobenius norm. See [1, 4] for properties of unitarily invariant norms.

Let $S_n[a, b]$ denote the set of $n \times n$ real symmetric matrices whose entries are in the interval $[a, b]$. For an $n \times n$ real symmetric matrix $A$, we denote the eigenvalues of $A$ in decreasing order by $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$. The spread of such an $A$ is defined to be $s(A) = \lambda_1(A) - \lambda_n(A)$.

I posed the following two problems in [8] in 2006.

**Problem 5.** For a given integer $j$ with $2 \leq j \leq n - 1$, determine
\[
\max\{\lambda_j(A) : A \in S_n[a, b]\},
\]
\[
\min\{\lambda_j(A) : A \in S_n[a, b]\}
\]
and determine the matrices that attain the maximum and the matrices that attain the minimum.

The cases $j = 1, n$ are solved [8].
Problem 6. For generic $a < b$ determine

$$\max\{s(A) : A \in S_n[a, b]\}$$

and determine the matrices that attain the maximum.

The case $a = -b$ is solved [8].
Let $\phi(A)$ be the number of nonzero entries of a matrix $A$. At the 12th ILAS conference, Regina, Canada, 2005 I posed the following problem.

**Problem 7.** Characterize the sign patterns of entry-wise nonnegative matrices $A$ such that the sequence $\{\phi(A^k)\}_{k=1}^{\infty}$ is nondecreasing. We may consider the same problem with “nondecreasing” replaced by “nonincreasing”.

Remark: Sidak observed in 1964 that there exists a primitive matrix $A$ of order 9 satisfying

$$18 = \phi(A) > \phi(A^2) = 16.$$
References

[5] Z. Huang and X. Zhan, Digraphs that have at most one walk of a given length with the same endpoints, to appear.
Thank you!