Discrete Mathematics 346 (2023) 113208

Contents lists available at ScienceDirect

Discrete Mathematics

www.elsevier.com/locate/disc

The maximum size of a nonhamiltonian graph with given order and connectivity

Xingzhi Zhan, Leilei Zhang *

Department of Mathematics, East China Normal University, Shanghai 200241, China

ARTICLE INFO

Article history:
Received 22 August 2021
Received in revised form 29 August 2022
Accepted 24 September 2022
Available online xxxx

Keywords:
Connectivity
Hamiltonian graph
Traceable graph
Size
Extremal graph

ABSTRACT

Motivated by work of Erdős, Ota determined the maximum size $g(n,k)$ of a k-connected nonhamiltonian graph of order n in 1995. But for some pairs n,k, the maximum size is not attained by a graph of connectivity k. For example, $g(15,3)=77$ is attained by a unique graph of connectivity 7, not 3. In this paper we obtain more precise information by determining the maximum size of a nonhamiltonian graph of order n and connectivity k, and determining the extremal graphs. Consequently we solve the corresponding problem for nontraceable graphs.

1. Introduction

We consider finite simple graphs, and use standard terminology and notations. The *order* of a graph is its number of vertices, and the *size* its number of edges. For graphs we will use equality up to isomorphism, so $G_1 = G_2$ means that G_1 and G_2 are isomorphic. For a given graph G, let $\overline{G}$ denote the complement of G. For two graphs G and H, $G \vee H$ denotes the *join* of G and H, which is obtained from the disjoint union $G + H$ by adding edges joining every vertex of G to every vertex of H. Let K_n denote the complete graph of order n.

One way to understand hamiltonian graphs is to investigate nonhamiltonian graphs. In 1961 Ore [8] determined the maximum size of a nonhamiltonian graph with a given order and also determined the extremal graphs.

Lemma 1. *(Ore [8]) The maximum size of a nonhamiltonian graph of order $n \geq 3$ is $\binom{n-1}{2} + 1$ and this size is attained by a graph G if and only if $G = K_1 \vee (K_{n-2} + K_1)$ or $G = K_2 \vee \overline{K_3}$.*

Bondy [1] gave a new proof of Lemma 1. It is natural to ask the same question by putting constraints on the graphs. In 1962 Erdős [6] determined the maximum size of a nonhamiltonian graph of order n and minimum degree at least k, while in 1995 Ota [9] determined the maximum size $g(n,k)$ of a k-connected nonhamiltonian graph of order n. But for some pairs n,k, the maximum size is not attained by a graph of connectivity k. For example, $g(15,3)=77$ is attained by a unique graph of connectivity 7, not 3.

* Corresponding author.
E-mail addresses: zhan@math.ecnu.edu.cn (X. Zhan), mathdzhang@163.com (L. Zhang).

https://doi.org/10.1016/j.disc.2022.113208
0012-365X/

In this paper we obtain more precise information by determining the maximum size of a nonhamiltonian graph of order n and connectivity k, and determining the extremal graphs, from which Ota's result can be deduced. Consequently we solve the corresponding problem for nontraceable graphs.

2. Main results

Denote by $V(G)$ and $E(G)$ the vertex set and edge set of a graph G, respectively. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S. Let $\deg(v)$ denote the degree of a vertex v, and let $\delta(G)$ denote the minimum degree of a graph G. Let $K_{s,t}$ denote the complete bipartite graph whose partite sets have cardinality s and t, respectively.

We denote by $\kappa(G)$ and $\alpha(G)$ the connectivity and independence number of a graph G, respectively.

We will need the following lemmas.

Lemma 2. *(Chvátal [4]) Let G be a graph with degree sequence $d_1 \le d_2 \le \cdots \le d_n$ where $n \ge 3$. If there is no integer k with $1 \le k < n/2$ such that $d_k \le k$ and $d_{n-k} < n-k$, then G is hamiltonian.*

Lemma 2 can also be found in [3, p. 488].

Lemma 3. *(Chvátal-Erdős [5]) Let G be a graph of order at least three. If $\kappa(G) \ge \alpha(G)$, then G is hamiltonian.*

Lemma 3 can also be found in [3, p. 488] and [10, p. 292].

Given a graph G and a positive integer s with $s \le \alpha(G)$, denote

$$\sigma_s(G) = \min\left\{ \sum_{v \in T} \deg(v) \;\middle|\; T \subseteq V(G) \text{ is an independent set and } |T| = s \right\}.$$

The following result is a special case of Ota's theorem.

Lemma 4. *(Ota [9, Theorem 1]) Let G be a k-connected graph of order n where $2 \le k < \alpha(G)$. If for every integer p with $k \le p \le \alpha(G) - 1$ we have $\sigma_{p+1}(G) \ge n + p^2 - p$, then G is hamiltonian.*

A bipartite graph with partite sets X and Y is called *balanced* if $|X| = |Y|$. For $n \ge 3$, we denote by $K_{n,n-2} + 4e$ the bipartite graph obtained from $K_{n,n-2}$ by adding two vertices which are adjacent to two common vertices of degree $n-2$.

Lemma 5. *(Liu-Shiu-Xue [7]) Given an integer $n \ge 4$, let $\Omega(n)$ denote the set of all nonhamiltonian balanced bipartite graphs of order $2n$ with minimum degree at least 2, and let $\Omega(3)$ denote the set of all nonhamiltonian balanced bipartite graphs of order 6. Then for any $n \ge 3$, the maximum size of a graph in $\Omega(n)$ is $n^2 - 2n + 4$ and this maximum size is uniquely attained by the graph $K_{n,n-2} + 4e$.*

The case $n \ge 4$ of Lemma 5 is proved in [7, p. 257] and the case $n = 3$ can be verified easily. We will use this lemma with all $n \ge 3$ cases.

Lemma 6. *(Bondy [2]) Let G be a graph of order n with degree sequence $d_1 \le d_2 \le \cdots \le d_n$ and let k be an integer with $0 \le k \le n-2$. If for each integer j with $1 \le j \le n - 1 - d_{n-k}$ we have $d_j \ge j + k$, then G is $(k+1)$-connected.*

Lemma 7. *Let $G = K_s \vee \overline{K_t}$ or $G = K_s \vee (K_2 + \overline{K_t})$ where $t \ge 2$ in both cases, and let $F \subseteq E(G)$. If $\kappa(G - F) = k$, then $|F| \ge s - k$, with equality if and only if all the edges in F are incident to one common vertex in $\overline{K_t}$.*

Proof. We prove the case when $G = K_s \vee \overline{K_t}$. The case when $G = K_s \vee (K_2 + \overline{K_t})$ can be proved similarly.

It is easy to see that $\kappa(G) = s$. Since deleting one edge reduces the connectivity by at most one [10, p. 169], we have $|F| \ge s - k$.

Denote $f = |F|$. Next we use induction on f to prove the equality condition; i.e., $\kappa(G - F) = s - f$. First consider the case $f = 1$. Let $e \in E(G)$. It is easy to check that $\kappa(G - e) = s - 1$ if and only if e has one endpoint in K_s and the other endpoint in $\overline{K_t}$. Now let $F \subseteq E(G)$ with $|F| = f \ge 2$ and suppose that for any $A \subseteq E(G)$ with $|A| = f - 1$, $\kappa(G - A) = s - (f - 1)$ if and only if all the edges in A are incident to one common vertex in $\overline{K_t}$.

If all edges in F are incident to one common vertex in $\overline{K_t}$, it is easy to verify that $\kappa(G - F) = s - f$. Conversely, suppose $\kappa(G - F) = s - f$. Let $F = \{e_1, e_2, \ldots, e_f\}$ and denote $F' = F \setminus \{e_f\}$. Then $\kappa(G - F') = s - f + 1$. By the induction hypothesis, the edges $e_1, \ldots, e_{f-1}$ are incident to one common vertex w in $\overline{K_t}$. The degree sequence of $G - F'$ is

$$s - f + 1, \underbrace{s, \ldots, s}_{t-1}, \underbrace{n-2, \ldots, n-2}_{f-1}, \underbrace{n-1, \ldots, n-1}_{s-f+1}$$

where $n = s + t$ and $s - f + 1 = \deg(w)$. We assert that e_f is incident to w and consequently all the edges in F are incident to one common vertex in $\overline{K_t}$. Let the degree sequence of $G - F$ be $d_1 \le \cdots \le d_n$. By the above degree sequence of $G - F'$ we deduce that $d_{n-s+f} \ge n - 2$. Thus $n - 1 - d_{n-s+f} \le 1$. If e_f is not incident to w, then we would have $d_1 = s - f + 1 = 1 + (s - f)$. By Lemma 6, $G - F$ is $(s - f + 1)$-connected, contradicting the assumption $\kappa(G - F) = s - f$. This proves that e_f is incident to w. □

Notation 1. We denote by $e(G)$ the size of a graph G.

Notation 2. For positive integers n and k with n odd and $n \ge 2k + 1$, $G_1(n, k)$ denotes the graph obtained from $K_{(n-1)/2} \vee \overline{K_{(n+1)/2}}$ by deleting $(n - 1)/2 - k$ edges that are incident to one common vertex in $\overline{K_{(n+1)/2}}$; for positive integers n and k with n even and $n \ge 2k + 2$, $G_2(n, k)$ denotes the graph obtained from $K_{(n-2)/2} \vee (K_2 + \overline{K_{(n-2)/2}})$ by deleting $(n - 2)/2 - k$ edges that are incident to one common vertex in $\overline{K_{(n-2)/2}}$.

Note that by Dirac's theorem [3, p. 485], for the existence of a nonhamiltonian graph of order n and connectivity k we necessarily have $n \ge 2k + 1$. Now we are ready to state and prove the main result.

Theorem 8. *Let $f(n, k)$ denote the maximum size of a nonhamiltonian graph of order n and connectivity k. Then*

$$f(n,k) = \begin{cases} \binom{n-k}{2} + k^2 & \text{if } n \text{ is odd and } n \ge 6k-5 \text{ or } n \text{ is even and } n \ge 6k-8, \\ \frac{3n^2-8n+5}{8} + k & \text{if } n \text{ is odd and } 2k+1 \le n \le 6k-7, \\ \frac{3n^2-10n+16}{8} + k & \text{if } n \text{ is even and } 2k+2 \le n \le 6k-10. \end{cases}$$

If $n = 6k - 5$, then $f(n, k)$ is attained by a graph G if and only if $G = K_k \vee (K_{n-2k} + \overline{K_k})$ or $G = G_1(n, k)$. If $n = 6k - 8$, then $f(n, k)$ is attained by a graph G if and only if $G = K_k \vee (K_{n-2k} + \overline{K_k})$ or $G = G_2(n, k)$. If n is odd and $n \ge 6k - 3$ or n is even and $n \ge 6k - 6$, then $f(n, k)$ is attained by a graph G if and only if $G = K_k \vee (K_{n-2k} + \overline{K_k})$. If n is odd and $2k + 1 \le n \le 6k - 7$, then $f(n, k)$ is attained by a graph H if and only if $H = G_1(n, k)$. If n is even and $2k + 2 \le n \le 6k - 10$, then $f(n, k)$ is attained by a graph Z if and only if $Z = G_2(n, k)$.

Proof. The case $k = 1$ of Theorem 8 follows from Lemma 1. Note that the extremal graph $K_2 \vee \overline{K_3}$ of order 5 in Lemma 1 has connectivity 2 and hence it should be excluded.

Next suppose $k \ge 2$. It is easy to verify that the extremal graphs stated in Theorem 8 are nonhamiltonian graphs of order n and connectivity k with size $f(n, k)$. They are nonhamiltonian since any hamiltonian graph must be tough [3, pp. 472-473]. Thus it remains to show that $f(n, k)$ is an upper bound on the size and it can only be attained by these extremal graphs.

Let Q be a nonhamiltonian graph of order n and connectivity k with degree sequence $d_1 \le d_2 \le \cdots \le d_n$. By Lemma 3, $k < \alpha(Q)$ and by Lemma 4, there exists an integer p with $k \le p \le \alpha(Q) - 1$ such that $\sigma_{p+1}(Q) \le n + p^2 - p - 1$. Let S be an independent set of Q with cardinality $p + 1$ whose degree sum is $\sigma_{p+1}(Q)$. Then $e(Q[V(Q) \setminus S]) \le \binom{n-p-1}{2}$. We distinguish four cases.

Case 1. n is odd and $n \ge 6k - 5$.

Subcase 1.1. $p \le (n - 3)/2$.

The conditions $p \le (n - 3)/2$ and $n \ge 6k - 5$ imply $3p + 3k + 1 < 2n$. This, together with the condition $p \ge k$, yields $(p - k)(3p + 3k + 1 - 2n) \le 0$. It follows that

$$e(Q) \le n + p^2 - p - 1 + \binom{n-p-1}{2} \le \binom{n-k}{2} + k^2 \tag{1}$$

and equality holds in the second inequality in (1) if and only if $p = k$.

Now suppose that Q has size $\binom{n-k}{2} + k^2$. Then $p = k$, S has cardinality $k + 1$ and degree sum $n + k^2 - k - 1$, and $V(Q) \setminus S$ is a clique. Since $k + 1 < (n + 1)/2$, we have $d_{(n+1)/2} \ge n - k - 2$. By Lemma 2, there exists i with $i < n/2$ such that $d_i \le i$ and $d_{n-i} \le n - i - 1$. Since n is odd, the condition $i < n/2$ means $i \le (n - 1)/2$. We have

$$e(Q) = \binom{n-k}{2} + k^2 \le [i^2 + (n - 2i)(n - i - 1) + i(n - 1)]/2, \tag{2}$$

where the inequality is equivalent to $(i - k)(2n - 3i - 3k - 1) \le 0$. Since $i \ge d_i \ge \delta(Q) \ge k$, we obtain $i = k$ or $n \le (3i + 3k + 1)/2$.

If $i = k$, equality holds in (2) and hence the degree sequence of Q is

$$\underbrace{k, \ldots, k}_{k}, \underbrace{n-k-1, \ldots, n-k-1}_{n-2k}, \underbrace{n-1, \ldots, n-1}_{k},$$

implying that $Q = K_k \vee (K_{n-2k} + \overline{K_k})$.

Now suppose $i \neq k$. Then we have $n \leq (3i + 3k + 1)/2$. If $i \leq (n-3)/2$, then $n \leq 6k - 7$, contradicting our assumption $n \geq 6k - 5$. Thus $i = (n-1)/2$. We have $n - k - 2 \leq d_{(n+1)/2} \leq (n-1)/2$. Hence $6k - 5 \leq n \leq 2k + 3$, which, together with the condition $k \geq 2$, yields $k = 2$ and $n = 7$. It is easy to check that there are exactly four graphs of order 7 and size 14 with $d_4 = 3$, among which $G_1(7, 2)$ is the only graph that is nonhamiltonian with connectivity 2. Hence $Q = G_1(7, 2)$.

Subcase 1.2. $p \geq (n-1)/2$.

Clearly Q is a spanning subgraph of $R = K_{n-p-1} \vee \overline{K_{p+1}}$. If $p \geq (n+1)/2$, then

$$e(Q) \leq \binom{n-p-1}{2} + (n-p-1)(p+1) < \binom{n-k}{2} + k^2$$

where the second inequality follows from the condition $n \geq 6k - 5$. If $p = (n-1)/2$, we have $\kappa(R) = n - p - 1 > k$. Let $F \subseteq E(R)$ such that $Q = R - F$. Since $\kappa(Q) = k$, by Lemma 7 we have $|F| \geq n - p - 1 - k$. Thus

$$e(Q) \leq \binom{n-p-1}{2} + (n-p-1)(p+1) - (n-p-1-k) \leq \binom{n-k}{2} + k^2 \tag{3}$$

and equality holds in the second inequality of (3) if and only if $n = 6k - 5$.

Suppose $e(Q) = \binom{n-k}{2} + k^2$. Then $n = 6k - 5$ and $|F| = n - p - 1 - k$. By Lemma 7, all the edges in F are incident to one common vertex in $\overline{K_{p+1}}$. Since $p = (n-1)/2$, we have $n - p - 1 = (n-1)/2$, $p + 1 = (n+1)/2$ and $|F| = (n-1)/2 - k$. It follows that $Q = G_1(n, k)$.

Case 2. n is odd and $2k + 1 \leq n \leq 6k - 7$.

Subcase 2.1. $k \leq p < (n-1)/2$.

We have

$$e(Q) \leq n + p^2 - p - 1 + \binom{n-p-1}{2} < \frac{3n^2 - 8n + 5}{8} + k.$$

Subcase 2.2. $p = (n-1)/2$.

In this case Q is a spanning subgraph of $K_p \vee \overline{K_{p+1}}$. Since $\kappa(Q) = k$, by Lemma 7 we obtain

$$e(Q) \leq \binom{p}{2} + p(p+1) - (p-k) = \frac{3n^2 - 8n + 5}{8} + k$$

and equality holds if and only if $Q = G_1(n, k)$.

Subcase 2.3. $p > (n-1)/2$.

In this case Q is a spanning subgraph of $K_{n-p-1} \vee \overline{K_{p+1}}$. Then

$$e(Q) \leq \binom{n-p-1}{2} + (n-p-1)(p+1) < \frac{3n^2 - 8n + 5}{8} + k,$$

where we have used the condition $n \leq 2p - 1$.

Case 3. n is even and $n \geq 6k - 8$.

Subcase 3.1. $p < (n-2)/2$.

Since n is even, the condition $p < (n-2)/2$ means $p \leq (n-2)/2 - 1 = (n/2) - 2$. The assumptions imply $3p + 3k + 1 - 2n < 0$. We have

$$e(Q) \leq n + p^2 - p - 1 + \binom{n-p-1}{2} \leq \binom{n-k}{2} + k^2 \tag{4}$$

where the second inequality is equivalent to

$$(p-k)(3p + 3k + 1 - 2n) \leq 0.$$

Thus equality holds in the second inequality in (4) if and only if $p = k$.

Suppose $e(Q) = \binom{n-k}{2} + k^2$. Then $p = k$ and Q has a clique of cardinality $n - p - 1$ and an independent set of cardinality $p + 1$ whose degree sum equals $n + p^2 - p - 1$. Also $d_{(n+2)/2} \geq n - k - 2$. By Lemma 2, there exists $i < n/2$ such that $d_i \leq i$ and $d_{n-i} \leq n - i - 1$. We have

$$e(Q) = \binom{n-k}{2} + k^2 \leq [i^2 + (n-2i)(n-i-1) + i(n-1)]/2,$$

where the inequality is equivalent to

$$(i-k)(2n - 3i - 3k - 1) \leq 0. \tag{5}$$

Note that $i \geq k$ since $i \geq d_i \geq \delta(Q) \geq k$. If $i = k$, then the degree sequence of Q is

$$\underbrace{k, \ldots, k}_{k}, \underbrace{n-k-1, \ldots, n-k-1}_{n-2k}, \underbrace{n-1, \ldots, n-1}_{k},$$

implying $Q = K_k \vee (K_{n-2k} + \overline{K_k})$.

Next suppose $i \neq k$. Then the inequality (5) implies $2n - 3i - 3k - 1 \leq 0$. If $i \leq (n-4)/2$, we deduce that $n \leq 6k - 10$, a contradiction. Hence $i = (n-2)/2$. Now the conditions $d_{n-i} \leq n-i-1$ and $d_{(n+2)/2} \geq n-k-2$ yield $n-k-2 \leq d_{(n+2)/2} \leq n/2$. Thus $6k - 8 \leq n \leq 2k + 4$. It follows that $k = 2$ and $4 \leq n \leq 8$ or $k = 3$ and $n = 10$. The possibility $n = 4$ contradicts $k < n/2$ and $n = 6$ contradicts $i \neq k$. Only the two pairs $(k, n) = (2, 8), (3, 10)$ can occur.

If $k = 2$ and $n = 8$, by the conditions $n = 8$, $e = 19$, $d_5 = 4$ and being nonhamiltonian we deduce that $Q = K_3 \vee (K_2 + \overline{K_3})$ which has connectivity 3; if $k = 3$ and $n = 10$, the conditions $n = 10$, $e(Q) = 30$, $k = 3$, $d_6 = 5$ and being nonhamiltonian force $Q = G_2(10, 3)$. These two facts can be verified by simple computer programs. Thus, in the case $k = 2$ and $n = 8$, no more extremal graphs exist, while in the case $k = 3$ and $n = 10$, a second extremal graph exists.

Subcase 3.2. $p = (n-2)/2$.

Clearly $\alpha(Q) \geq p + 1$. We further distinguish two cases.

If $\alpha(Q) \geq p + 2$, then Q is a spanning subgraph of $K_p \vee \overline{K_{p+2}}$. By Lemma 7 we have

$$e(Q) \leq \binom{p}{2} + p(p+2) - (p-k) < \binom{n-k}{2} + k^2. \tag{6}$$

The second inequality in (6) is equivalent to $p^2 + (5-4k)p + 3k^2 - 5k + 2 > 0$ which is guaranteed by $p = (n-2)/2 \geq 3k - 5$.

If $\alpha(Q) = p + 1$, then Q is a spanning subgraph of $K_{p+1} \vee \overline{K_{p+1}}$. Let Q' denote the graph obtained from Q by deleting all the edges in K_{p+1}. Then Q' is a nonhamiltonian balanced bipartite graph. There are two cases.

(a) Suppose $n \geq 8$ and $\delta(Q') \geq 2$ or $n = 6$. By Lemma 5, $e(Q') \leq (p+1)^2 - 2(p+1) + 4 = p^2 + 3$. Hence

$$e(Q) \leq \binom{p+1}{2} + p^2 + 3 \leq \binom{n-k}{2} + k^2. \tag{7}$$

The second inequality in (7) is equivalent to

$$p^2 + (5-4k)p + 3k^2 - 3k - 4 \geq 0 \tag{8}$$

which is implied by the condition $p \geq 3k - 5$. Equality holds in (8) if and only if $k = 2$ and $p = 2$, i.e., Q' is the extremal graph of order 6 defined in Lemma 5. Hence Q has size $\binom{n-k}{2} + k^2$ if and only if $Q = K_2 \vee (K_2 + \overline{K_2})$.

(b) Now suppose $n \geq 8$ and $\delta(Q') \leq 1$. Let $x \in V(Q')$ with $\deg_{Q'}(x) = \delta(Q')$. Starting with the structure $K_{p+1} \vee \overline{K_{p+1}}$, we deduce that x lies in K_{p+1}, since $\delta(Q) \geq 2$. In this case Q is a spanning subgraph of $K_p \vee (K_2 + \overline{K_p})$. By Lemma 7 and using the fact that $p \geq 3k - 5$ we have

$$e(Q) \leq \binom{p+2}{2} + p^2 - (p-k) \leq \binom{n-k}{2} + k^2. \tag{9}$$

Equality in the second inequality in (9) holds if and only if $p = 3k - 5$ or $p = k$. Suppose $p = 3k - 5$. Then $n = 6k - 8$. By (9) and Lemma 7, Q has size $\binom{n-k}{2} + k^2$ if and only if $Q = G_2(n, k)$ with $n = 6k - 8$. If $p = k$, then the conditions $p = (n-2)/2$ and $n \geq 6k - 8$ imply $n \leq 7$, contradicting our assumption $n \geq 8$. Hence the case $p = k$ cannot occur.

Subcase 3.3. $p > (n-2)/2$.

Note that Q is a spanning subgraph of $K_{n-p-1} \vee \overline{K_{p+1}}$. If $p = n/2$, we have $n - p - 1 \geq k$. By Lemma 7

$$e(Q) \leq \binom{n-p-1}{2} + (n-p-1)(p+1) - (n-p-1-k) < \binom{n-k}{2} + k^2. \tag{10}$$

The second inequality in (10) is equivalent to $p^2 + (3-4k)p + 3k^2 - k - 2 > 0$, which is implied by $p = n/2 \geq 3k - 4$.

If $p \geq (n+2)/2$, we have

$$e(Q) \leq \binom{n-p-1}{2} + (n-p-1)(p+1) < \binom{n-k}{2} + k^2. \tag{11}$$

The second inequality in (11) is equivalent to $p^2 + p + 3k^2 + k - 2nk > 0$. To prove this inequality it suffices to show $p^2 + (1-4k)p + 3k^2 + 5k > 0$, which is implied by $p \geq (n+2)/2 \geq 3k - 3$.

Case 4. n is even and $2k + 2 \leq n \leq 6k - 10$.

Denote $m = (n-2)/2$. Then $3 \leq k \leq m$. We distinguish three subcases.

Subcase 4.1. $p < m$.

We have

$$e(Q) \le \binom{n-p-1}{2} + n + p^2 - p - 1 < \frac{3n^2 - 10n + 16}{8} + k. \tag{12}$$

The second inequality in (12) is equivalent to $3p^2 - (4m+3)p + m^2 + 5m - 2k < 0$, which is implied by the conditions $k \le p < m \le 3k - 6$.

Subcase 4.2. $p = m$.

If $\alpha(Q) \ge p+2$, then Q is a spanning subgraph of $K_p \vee \overline{K_{p+2}}$. Recall that $p \ge k$. By Lemma 7 we have

$$e(Q) \le \binom{p}{2} + p(p+2) - (p-k) = (3p^2 + p + 2k)/2 < (3p^2 + p + 2k)/2 + 1$$
$$= \frac{3n^2 - 10n + 16}{8} + k.$$

Now suppose $\alpha(Q) = p+1$. Then Q is a spanning subgraph of $K_{p+1} \vee \overline{K_{p+1}}$. Define the graph Q' as in Subcase 3.2 above.

If $\delta(Q') \ge 2$, by Lemma 5 we have $e(Q') \le p^2 + 3$. Hence

$$e(Q) \le \binom{p+1}{2} + p^2 + 3 < \frac{3n^2 - 10n + 16}{8} + k.$$

If $\delta(Q') \le 1$, then Q is a spanning subgraph of $K_p \vee (K_2 + \overline{K_p})$. By Lemma 7 we obtain

$$e(Q) \le \binom{p+2}{2} + p^2 - (p-k) = \frac{3n^2 - 10n + 16}{8} + k$$

and equality holds if and only if $Q = G_2(n,k)$.

Subcase 4.3. $p > m$.

Note that Q is a spanning subgraph of $K_{n-p-1} \vee \overline{K_{p+1}}$. We further distinguish two cases.

(a) $p = m+1$. Now the conditions $2k+2 \le n$ and $p = m+1 = n/2$ imply $n - p - 1 = (n-2)/2 \ge k$. By Lemma 7, we have

$$e(Q) \le \binom{n-p-1}{2} + (n-p-1)(p+1) - (n-p-1-k) < \frac{3n^2 - 10n + 16}{8} + k$$

where the second inequality is equivalent to $4p^2 - n^2 + 2n - 4p + 8 > 0$ which holds, since $n = 2p$.

(b) $p \ge m+2$. In this case the following rough estimate suffices:

$$e(Q) \le \binom{n-p-1}{2} + (n-p-1)(p+1) < \frac{3n^2 - 10n + 16}{8} + k. \tag{13}$$

The second inequality in (13) is equivalent to $4p^2 + 4p - n^2 - 6n + 8k + 16 > 0$, which holds, since $n \le 2p - 2$. This completes the proof. □

The following corollary follows from Theorem 8 immediately.

Corollary 9. *Let $f(n,k)$ be defined as in Theorem 8. If G is a graph of order n and connectivity k with size greater than $f(n,k)$, then G is hamiltonian.*

Next we use Theorem 8 to deduce Ota's result.

Corollary 10. *(Ota [9, p. 209]) The maximum size of a k-connected nonhamiltonian graph of order n is*

$$\max\left\{\binom{n-k}{2} + k^2,\ \binom{\lfloor (n+2)/2 \rfloor}{2} + \left\lfloor \frac{n-1}{2} \right\rfloor^2\right\}. \tag{14}$$

Proof. Denote the number in (14) by M. Let $f(n,c)$ be defined as in Theorem 8 with c in place of k there. Note that by Dirac's theorem ([3] or [10]), the connectivity c of a nonhamiltonian graph of order n satisfies $c < n/2$.

If n is odd, by Theorem 8

$$f(n,c) = \begin{cases} \binom{n-c}{2} + c^2 & \text{if } n \ge 6c - 5, \\ \frac{3n^2 - 8n + 5}{8} + c & \text{if } 2c + 1 \le n \le 6c - 7. \end{cases}$$

Thus the maximum size of a k-connected nonhamiltonian graph of order n is

$$\begin{aligned}
&\max\{f(n,c)|k \le c \le (n-1)/2\}\\
=&\max\left\{\max\left\{\binom{n-c}{2}+c^2\middle| k\le c\le \frac{n+5}{6}\right\},\ \max\left\{\frac{3n^2-8n+5}{8}+c\middle|\frac{n+7}{6}\le c\le\frac{n-1}{2}\right\}\right\}\\
=&\max\left\{\binom{n-k}{2}+k^2,\ \frac{3n^2-8n+5}{8}+\frac{n-1}{2}\right\}\\
=&M.
\end{aligned}$$

If n is even, by Theorem 8

$$f(n,c)=\begin{cases}\binom{n-c}{2}+c^2 & \text{if } n\ge 6c-8,\\ \frac{3n^2-10n+16}{8}+c & \text{if } 2c+2\le n\le 6c-10.\end{cases}$$

The maximum size of a k-connected nonhamiltonian graph of order n is

$$\begin{aligned}
&\max\{f(n,c)|k \le c \le (n-2)/2\}\\
=&\max\left\{\max\left\{\binom{n-c}{2}+c^2\middle| k\le c\le \frac{n+8}{6}\right\},\ \max\left\{\frac{3n^2-10n+16}{8}+c\middle|\frac{n+10}{6}\le c\le\frac{n-2}{2}\right\}\right\}\\
=&\max\left\{\binom{n-k}{2}+k^2,\ \frac{3n^2-10n+16}{8}+\frac{n-2}{2}\right\}\\
=&M. \quad \square
\end{aligned}$$

A graph is *traceable* if it contains a Hamilton path; otherwise it is *nontraceable.* Next we turn to nontraceable graphs. The following trick is well-known (e.g. [4, p. 166] or [5, p. 112]).

Lemma 11. *Let G be a graph and denote $H = G \vee K_1$. Then G is traceable if and only if H is hamiltonian, and $\kappa(G) = k$ if and only if $\kappa(H) = k+1$.*

Notation 3. For positive integers n and k with n odd and $n \ge 2k+3$, $H_1(n,k)$ denotes the graph obtained from $K_{(n-3)/2} \vee (K_2 + \overline{K_{(n-1)/2}})$ by deleting $(n-3)/2-k$ edges that are incident to one common vertex in $\overline{K_{(n-1)/2}}$; for positive integers n and k with n even and $n \ge 2k+2$, $H_2(n,k)$ denotes the graph obtained from $K_{(n-2)/2} \vee \overline{K_{(n+2)/2}}$ by deleting $(n-2)/2-k$ edges that are incident to one common vertex in $\overline{K_{(n+2)/2}}$.

By Dirac's theorem [3, p. 485] and Lemma 11, for the existence of a nontraceable graph of order n and connectivity k we must have $n \ge 2k+2$. The next corollary follows from Theorem 8 and Lemma 11 immediately. Note that all extremal graphs in Theorem 8 have a dominating vertex.

Corollary 12. *Let $\varphi(n,k)$ denote the maximum size of a nontraceable graph of order n and connectivity k. Then*

$$\varphi(n,k)=\begin{cases}\binom{n-k-1}{2}+k(k+1) & \text{if } n \text{ is odd and } n\ge 6k-3 \text{ or } n \text{ is even and } n\ge 6k,\\ \frac{3n^2-12n+17}{8}+k & \text{if } n \text{ is odd and } 2k+3\le n\le 6k-5,\\ \frac{3n^2-10n+8}{8}+k & \text{if } n \text{ is even and } 2k+2\le n\le 6k-2.\end{cases}$$

If $n = 6k-3$, then $\varphi(n,k)$ is attained by a graph G if and only if $G = K_k \vee (K_{n-2k-1} + \overline{K_{k+1}})$ or $G = H_1(n,k)$. If $n = 6k$, then $\varphi(n,k)$ is attained by a graph G if and only if $G = K_k \vee (K_{n-2k-1} + \overline{K_{k+1}})$ or $G = H_2(n,k)$. If n is odd and $n \ge 6k-1$ or n is even and $n \ge 6k+2$, then $\varphi(n,k)$ is attained by a graph G if and only if $G = K_k \vee (K_{n-2k-1} + \overline{K_{k+1}})$. If n is odd and $2k+3 \le n \le 6k-5$, then $\varphi(n,k)$ is attained by a graph G if and only if $G = H_1(n,k)$. If n is even and $2k+2 \le n \le 6k-2$, then $\varphi(n,k)$ is attained by a graph G if and only if $G = H_2(n,k)$.

Declaration of competing interest

We claim that there is no conflict of interest in our paper.

Data availability

No data was used for the research described in the article.

Acknowledgement

The authors are grateful to the two anonymous referees for their kind and professional suggestions which have improved the presentation of the paper. This research was supported by the NSFC grant 12271170 and Science and Technology Commission of Shanghai Municipality (STCSM) grant 22DZ2229014.

References

[1] J.A. Bondy, Variations on the Hamiltonian theme, Can. Math. Bull. 15 (1) (1972) 57–62.
[2] J.A. Bondy, Properties of graphs with constraints on degrees, Studia Sci. Math. Hung. 4 (1969) 473–475.
[3] J.A. Bondy, U.S.R. Murty, Graph Theory, GTM, vol. 244, Springer, 2008.
[4] V. Chvátal, On Hamilton's ideals, J. Comb. Theory, Ser. B 12 (1972) 163–168.
[5] V. Chvátal, P. Erdős, A note on Hamiltonian circuits, Discrete Math. 2 (1972) 111–113.
[6] P. Erdős, Remarks on a paper of Pósa, Magy. Tud. Akad. Mat. Kut. Intéz. Közl. 7 (1962) 227–229.
[7] R. Liu, W.C. Shiu, J. Xue, Sufficient spectral conditions on Hamiltonian and traceable graphs, Linear Algebra Appl. 467 (2015) 254–266.
[8] O. Ore, Arc coverings of graphs, Ann. Mat. Pura Appl. 55 (1961) 315–321.
[9] K. Ota, Cycles through prescribed vertices with large degree sum, Discrete Math. 145 (1995) 201–210.
[10] D.B. West, Introduction to Graph Theory, Prentice Hall, Inc., 1996.