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Open problems in matrix theory

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Abstract

We survey some open problems in matrix theory by briefly describing their history and current state.

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Sometimes solutions to challenging matrix problems can reveal connections between different parts of mathematics. Two examples of this phenomenon are the proof of the van der Waerden conjecture on permanents (see [47] or [69]) and the recent proof of Horn's conjecture on eigenvalues of sums of Hermitian matrices (see [11] and [32]). Difficult matrix problems can also expose limits to the strength of existing mathematical tools.

We will describe the history and current state of some open problems in matrix theory, which we arrange chronologically in the following sections.

1. Existence of Hadamard matrices

A Hadamard matrix is a square matrix with entries equal to ± 1 whose rows and hence columns are mutually orthogonal. In other words, a Hadamard matrix of order n is a $\{1, -1\}$-matrix A satisfying

$$AA^T = nI$$

where I is the identity matrix. In 1867 Sylvester proposed a recurrent method for construction of Hadamard matrices of order 2^k. In 1893

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Hadamard proved his famous determinantal inequality for a positive semidefinite matrix A :

$$\det A \leq h(A)$$

where $h(A)$ is the product of the diagonal entries of A. It follows from this inequality that if $A = (a_{ij})$ is a real matrix of order n with $|a_{ij}| \leq 1$ then

$$|\det A| \leq n^{n/2};$$

equality occurs if and only if A is a Hadamard matrix. This result gives rise to the term "Hadamard matrix". In 1898 Scarpis proved that if $p \equiv 3 \pmod 4$ or $p \equiv 1 \pmod 4$ is a prime number then there is a Hadamard matrix of order $p+1$ and $p+3$ respectively.

In 1933 Paley stated that the order n $(n \geq 4)$ of any Hadamard matrix is divisible by 4. This is easy to prove. The converse has been a long-standing conjecture.

Conjecture 1 *For every positive integer n, there exists a Hadamard matrix of order $4n$.*

Conjecture 1 has been proved for $4n = 2^k m$ with $m^2 \leq 2^k$. According to [68], the smallest unknown case is now $4n = 668$. See [34, 57, 58, 63, 64].

Hadamard matrices have applications in information theory and combinatorial designs. See [1].

Let $k \leq n$ be positive integers. A square matrix A of order n with entries in $\{0, -1, 1\}$ is called a *weighted matrix with weight k* if

$$AA^T = kI.$$

Geramita and Wallis posed the following more general conjecture in 1976 [33].

Conjecture 2 *If $k \leq n$ are positive integers with $n \equiv 0 \pmod 4$, then there exists a weighted matrix of order n with weight k.*

Note that Conjecture 1 corresponds to the case $k = n$ of Conjecture 2.

2. Characterization of the eigenvalues of nonnegative matrices

In 1937 Kolmogorov asked the question: When is a given complex number an eigenvalue of some (entrywise) nonnegative matrix? The answer is: Every complex number is an eigenvalue of some nonnegative matrix [52, p.166]. Suleimanova [62] extended Kolmogorov's question in 1949 to the following problem which is called the *nonnegative inverse eigenvalue problem.*

Problem 3 *Determine necessary and sufficient conditions for a set of n complex numbers to be the eigenvalues of a nonnegative matrix of order n.*

Problem 3 is open for $n \geq 4$. The case $n = 2$ is easy while the case $n = 3$ is due to Loewy and London [48].

In the same paper [62] Suleimanova also considered the following *real nonnegative inverse eigenvalue problem* and gave a sufficient condition.

Problem 4 *Determine necessary and sufficient conditions for a set of n real numbers to be the eigenvalues of a nonnegative matrix of order n.*

Problem 4 is open for $n \geq 5$. In 1974 Fiedler [29] posed the following *symmetric nonnegative inverse eigenvalue problem.*

Problem 5 *Determine necessary and sufficient conditions for a set of n real numbers to be the eigenvalues of a symmetric nonnegative matrix of order n.*

Problem 5 is open for $n \geq 5$. There are some necessary conditions and many sufficient conditions for these three problems. See the survey paper [27] and the book [52, Chapter VII].

3. The permanental dominance conjecture

Let S_n denote the symmetric group on $\{1, 2, \ldots, n\}$ and M_n denote the set of complex matrices of order n. Suppose G is a subgroup of S_n and χ is a character of G. The *generalized matrix function* $d_\chi : M_n \to \mathbf{C}$ is defined by

$$d_\chi(A) = \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)},$$

where $A = (a_{ij})$. Incidental to his work on group representation theory, Schur introduced this notion. For $G = S_n$, if χ is the alternating character then d_χ is the determinant while if χ is the principal character then d_χ is the permanent

$$\text{per}A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}.$$

When χ is the principal character of $G = \{e\}$ where e is the identity permutation in S_n, d_χ is Hadamard's function $h(A)$.

In 1907 Fischer proved that if the matrix

$$A = \begin{pmatrix} A_1 & B \\ B^* & A_2 \end{pmatrix}$$

is positive semidefinite with A_1 and A_2 square, then

$$\det A \le (\det A_1)(\det A_2).$$

Hadamard's inequality follows from this inequality immediately. In 1918 Schur obtained the following generalization of Fischer's inequality:

$$\chi(e)\det A \le d_\chi(A)$$

for positive semidefinite A. Let G be a subgroup of S_n and let χ be an irreducible character of G. The normalized generalized matrix function is defined as

$$\bar{d}_\chi(A) = d_\chi(A)/\chi(e).$$

Since any character of G is a sum of irreducible characters, Schur's inequality is equivalent to

$$\det A \le \bar{d}_\chi(A)$$

for positive semidefinite A. In 1963, M. Marcus proved the permanental analog of Hadamard's inequality

$$\text{per}A \ge h(A)$$

and E.H. Lieb proved the permanental analog of Fischer's inequality

$$\text{per}A \ge (\text{per}A_1)(\text{per}A_2)$$

three years later, where A is positive semidefinite. These results naturally led to the following conjecture which was first published by Lieb [45] in 1966:

Conjecture 6 (The permanental dominance conjecture) *Suppose G is a subgroup of S_n and χ is an irreducible character of G. Then for any positive semidefinite matrix A of order n,*

$$\text{per}A \ge \bar{d}_\chi(A).$$

A lot of work has been done on this conjecture. It has been confirmed for every irreducible character of S_n with $n \le 13$. The reader is referred to [22 section 3] and the references therein for more details and recent progress.

We order the elements of S_n lexicographically to obtain a sequence L_n. For $A = (a_{ij}) \in M_n$ the *Schur power* of A, denoted by $\Pi(A)$, is the matrix of order $n!$ whose rows and columns are indexed by L_n and whose $(\sigma, \tau)-$entry is $\prod_{i=1}^n a_{\sigma(i),\tau(i)}$. Since $\Pi(A)$ is a principal submatrix of $\otimes^n A$, if A is positive semidefinite then so is $\Pi(A)$. It is not difficult to see that both per$A$ and $\det A$ are eigenvalues of $\Pi(A)$. A result of Schur asserts that if A is positive semidefinite then $\det A$ is the smallest eigenvalue of $\Pi(A)$. In 1966, Soules [61] posed the following

Conjecture 7 (The "permanent on top" conjecture) *If the matrix A is positive semidefinite, then* per$A$ *is the largest eigenvalue of* $\Pi(A)$.

Conjecture 7, if true, implies Conjecture 6.

4. The Marcus-de Oliveira conjecture

Let S_n denote the symmetric group on $\{1, 2, \ldots, n\}$ and coΩ denote the convex hull of a set Ω in the complex plane. In 1973 Marcus [50] and in 1982 de Oliveira [56] independently made the following

Conjecture 8 *Let A, B be normal complex matrices of order n with eigenvalues $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ respectively. Then*

$$\det(A + B) \in \mathrm{co} \left\{ \prod_{i=1}^{n} \left(x_i + y_{\sigma(i)}\right) : \sigma \in S_n \right\}.$$

It is known that Conjecture 8 is true in many special cases, e.g., (1) A, B are Hermitian [30]; (2) all the eigenvalues have the same modulus, $|x_1| = \cdots = |x_n| = |y_1| = \cdots = |y_n|$ [8]; (3) $A + B$ is singular [26]. See [6, 7] for more verified cases.

5. Permanents of Hadamard matrices

In 1974 Wang [65] posed the following

Question 9 *Can the permanent of a Hadamard matrix of order n vanish for $n > 2$?*

Wanless [66] showed that the answer is negative for $2 < n < 32$.

6. The Bessis-Moussa-Villani trace conjecture

In 1975, while studying partition functions of quantum mechanical systems, Bessis, Moussa and Villani [10] formulated the conjecture that if A, B are Hermitian matrices of the same order with B positive semidefinite then the function

$$f(t) = \mathrm{Tr} \exp(A - tB)$$

is the Laplace transform of a positive measure on $[0, \infty)$, where t is a real variable and Tr means trace. Recently, Lieb and Seiringer [46] has proved that this conjecture is equivalent to the following

Conjecture 10 (Bessis-Moussa-Villani) *Let A, B be positive semidefinite matrices of order n and let k be a positive integer. Then the polynomial $p(t) = \mathrm{Tr}\,(A + tB)^k$ has all nonnegative coefficients.*

The following cases of this conjecture are proved: (1) $k \leq 5$ and all n; (2) $n = 2$ and any k. See [38] and the references therein for many partial results and recent advances.

7. The S-matrix conjecture

An S-matrix of order n is a 0-1 matrix formed by taking a Hadamard matrix of order $n+1$ in which the entries in the first row and column are 1, changing 1's to 0's and -1's to 1's, and deleting the first row and column. Let $\|\cdot\|_F$ denote the Frobenius norm. In 1976 Sloane and Harwit [60] made the following conjecture. See also [37, p.59].

Conjecture 11 *If A is a nonsingular matrix of order n all of whose entries are in the interval* $[0, 1]$, *then*

$$\|A^{-1}\|_F \geq \frac{2n}{n+1}.$$

Equality holds if and only if A is an S-matrix.

I do not know of any results on Conjecture 11. This problem arose from weighing designs in optics and statistics.

8. Ryser's conjecture on minimum values of permanents

Let Λ_n^k be the set of those $n \times n$ 0-1 matrices with each row and column having exactly k 1's. In 1978 Ryser (in [53]) posed the following

Conjecture 12 *If Λ_v^k contains incidence matrices of (v, k, λ)-designs, then the permanent takes its minimum in Λ_v^k at one of these incidence matrices.*

Wanless [67] showed by computer enumeration that this conjecture is true for $v \leq 12$.

9. Foregger's conjecture on minimum values of permanents

A fully indecomposable square matrix A is called *nearly decomposable* if whenever a nonzero entry of A is replaced with a 0, the resulting matrix is partly decomposable. In 1980 Foregger [31] made the following

Conjecture 13 *If A is a nearly decomposable doubly stochastic matrix of order n, then*

$$\text{per}A \geq 2^{1-n}.$$

Note that this lower bound can be attained at $A = (I + P)/2$ where P is the permutation matrix corresponding to the permutation cycle $(1234 \cdots n)$. Foregger [31] proved the cases $2 \leq n \leq 9$.

10. Dittert's conjecture on permanents

Let K_n be the set of $n \times n$ nonnegative matrices with the sum of their entries equal to n. Define the function ϕ on $n \times n$ matrices by

$$\phi(A) = \prod_{i=1}^{n} r_i + \prod_{j=1}^{n} c_j - \mathrm{per}A$$

where $r_1, \ldots, r_n$ and $c_1, \ldots, c_n$ are the row and column sums of A respectively. In 1983 Dittert (in [54]) posed the following

Conjecture 14

$$\max\{\phi(A) : A \in K_n\} = 2 - \frac{n!}{n^n},$$

and the maximum is attained only for the matrix with each entry equal to $1/n$.

Sinkhorn [59] proved the case $n = 2$ and Hwang [40] proved the case $n = 3$.

11. The Brualdi-Li conjecture on tournament matrices

A tournament matrix is a square 0-1 matrix A satisfying $A + A^T = J - I$ where J is the all ones matrix. Such matrices arise from the results of round robin competitions. An $n \times n$ tournament matrix A is called *regular* if each of the row sums of A is $(n-1)/2$. For even n, an $n \times n$ tournament matrix is called *almost regular* if half of its row sums are $(n-2)/2$ and the other half are $n/2$. It is known [19] that for odd n the regular tournament matrices maximize the Perron root over the class of $n \times n$ tournament matrices. For even n, it is not known which tournament matrices maximize the Perron root. Let U_k be the strictly upper triangular matrix of order k with ones above the main diagonal. In 1983, Brualdi and Li [20] made the following

Conjecture 15 *For even* n, *the matrix*

$$\begin{bmatrix} U_{n/2} & U_{n/2}^T \\ U_{n/2}^T + I & U_{n/2} \end{bmatrix}$$

maximizes the Perron root over the class of tournament matrices of order n.

See [28] and the references therein for some partial results. Kirkland [43] has proved that for all sufficiently large even orders n the maximizers are almost regular.

12. A possible generalization of the Perron-Frobenius theorem

Let $A = (A_{ij})_{n\times n}$ be a block matrix of order nm, where each A_{ij} is a positive semidefinite matrix of order m. Let us call such matrices *block positive semidefinite (BPSD).* Note that when $m = 1$, A is a nonnegative matrix, while when $n = 1$, A is a positive semidefinite matrix. Thus BPSD matrices interpolate two familiar classes of matrices. Both nonnegative matrices and positive semidefinite matrices have the Perron-Frobenius property: The spectral radius is an eigenvalue.

Numerical experiments show that some BPSD matrices have the Perron-Frobenius property while some others do not. In 1988, Roger A. Horn posed the following problem in an unpublished note.

Problem 16 *Let A be a BPSD matrix. Give necessary and/or sufficient conditions on A such that the spectral radius $\rho(A)$ is an eigenvalue of A. More generally, study the properties of the eigenvalues and eigenvectors of BPSD matrices.*

13. The Grone-Merris conjecture on Laplacian spectra

Let G be a graph of order n and let $d(G) = (d_1, \ldots, d_n)$ be the degree sequence of G, where $d_1, \ldots, d_n$ are the degrees of the vertices of G. Let $A(G)$ be the adjacency matrix of G and denote $D(G) = \mathrm{diag}(d_1, \ldots, d_n)$. Then the matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G, and $s(G) = (\lambda_1, \ldots, \lambda_n)$ is called the *Laplacian spectrum,* where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $L(G)$.

For a sequence $x = (x_1, \ldots, x_n)$ with nonnegative integer components, the *conjugate sequence* of x is $x^\star = (x_1^\star, \ldots, x_n^\star)$, where

$$x_j^\star = |\{i : x_i \geq j\}|.$$

Denote by $d^\star(G)$ the conjugate sequence of the degree sequence $d(G)$. Use $x \prec y$ to mean that x is majorized by y. In 1994, Grone and Merris [35] made the following

Conjecture 17 *Let G be a connected graph. Then $s(G) \prec d^\star(G)$.*

If we arrange the components of $s(G)$ and $d^\star(G)$ in decreasing order, then it is known that

$$\lambda_1 \leq d_1^\star, \quad \lambda_1 + \lambda_2 \leq d_1^\star + d_2^\star.$$

14. The CP-rank conjecture

An $n \times n$ real matrix A is called *completely positive* (CP) if, for some m, there exists an $n \times m$ nonnegative matrix B such that $A = BB^T$. The smallest such m is called the *CP-rank* of A. CP matrices have applications in block designs. In 1994, Drew, Johnson and Loewy [25] posed the following

Conjecture 18 *If A is a CP matrix of order $n \geq 4$, then*

$$\text{CP-rank}(A) \leq \lfloor n^2/4 \rfloor.$$

This conjecture is true in the following cases: (1) $n = 4$ [51]; (2) $n = 5$ and A has at least one zero entry [49]; (3) the graph of A does not contain an odd cycle of length greater than 4 [24, 9]. It is known [25, p.309] that for each $n \geq 4$ the conjectured upper bound $\lfloor n^2/4 \rfloor$ can be attained.

15. Bhatia-Kittaneh's question on singular values

Denote by $s_1(X) \geq s_2(X) \geq \cdots$ the ordered singular values of a complex matrix X. In 2000, Bhatia and Kittaneh [17] asked the following

Question 19 *Let A, B be positive semidefinite matrices of order n. Is it true that*

$$s_j^{1/2}(AB) \leq \frac{1}{2} s_j(A+B), \quad j = 1, 2, \ldots, n?$$

The case $n = 2$ is known to be true [17]. Since the square function $f(t) = t^2$ is operator convex on $\mathbf{R}$, this inequality is stronger than the known inequality

$$2s_j(XY^*) \leq s_j(X^*X + Y^*Y), \quad j = 1, 2, \ldots, n$$

for any complex matrices X, Y of order n due to the same authors [18].

16. Convergence of the iterated Aluthge transforms

Every square complex matrix A has the polar decomposition $A = UP$ where U is unitary and P is positive semidefinite. The *Aluthge transform* of A is

$$\Delta(A) = P^{1/2} U P^{1/2}.$$

Though the unitary factor in the polar decomposition is not unique when A is singular, the Aluthge transform is well defined, that is, it does not depend on the choice made for the unitary factor. If A is normal, then $\Delta(A) = A$. For $0 < \lambda < 1$, the λ-*Aluthge transform*

$$\Delta_\lambda(A) = P^\lambda U P^{1-\lambda}$$

is also well defined. Note that $\Delta = \Delta_{1/2}$. Let $B(H)$ be the algebra of bounded linear operators on a Hilbert space H. The Aluthge transform can also be defined for operators in $B(H)$. In 2000, Jung, Ko and Pearcy [41] conjectured that for any $T \in B(H)$, the sequence $\{\Delta^m(T)\}_{m=1}^\infty$ is norm convergent to an operator. Here $\Delta^1(T) = \Delta(T)$ and $\Delta^m(T) = \Delta(\Delta^{m-1}(T))$, $m = 2, 3, \ldots$. However Cho, Jung and Lee [23] showed that this conjecture is false for infinite dimensional Hilbert spaces. So there remains the possibility that it holds in finite dimensions:

Conjecture 20 *Let A be a square complex matrix. Then the sequence $\{\Delta^m(A)\}_{m=1}^\infty$ converges.*

Note that $\Delta(A)$ and A have the same eigenvalues. It is known ([41, Prop. 1.10], [42, Prop. 3.1], [2, Thm 1]) that if the Aluthge sequence of a matrix converges, then the limit matrix is normal. These two properties make the Aluthge transform more interesting.

Ando and Yamazaki [4] verified Conjecture 20 when A is of order 2. See [42] for some special cases. Huang and Tam [39] proved that if the nonzero eigenvalues of A have distinct moduli, then the λ-Aluthge sequence $\{\Delta_\lambda^m(A)\}_{m=1}^\infty$ converges. Huang and Tam [39] also posed the following

Conjecture 21 *For any square complex matrix A and $0 < \lambda < 1$,*

$$\|A^*A - AA^*\|_F \geq \|\Delta_\lambda(A)^*\Delta_\lambda(A) - \Delta_\lambda(A)\Delta_\lambda(A)^*\|_F.$$

17. Expressing real matrices as linear combinations of orthogonal matrices

In 2002, Li and Poon [44] proved that every square real matrix is a linear combination of 4 orthogonal matrices, i.e., given a square real matrix A, there exist real orthogonal matrices Q_i and real numbers r_i, $i = 1, 2, 3, 4$ (depending on A, of course) such that

$$A = r_1Q_1 + r_2Q_2 + r_3Q_3 + r_4Q_4.$$

They asked the following

Question 22 *Is the number 4 of the terms in the above expression least possible?*

18. Sign patterns

Research on sign patterns of matrices is active now and there are many open problems in that field. See [21] and [36].

Let $f(A)$ be the number of positive entries of a nonnegative matrix A. In a talk at the 12th ILAS conference (Regina, Canada, June 26-29, 2005) I posed the following

Problem 23 *Characterize those sign patterns of square nonnegative matrices A such that the sequence $\{f(A^k)\}_{k=1}^{\infty}$ is nondecreasing.*

Sidak observed in 1964 that there exists a primitive nonnegative matrix A of order 9 satisfying

$$18 = f(A) > f(A^2) = 16.$$

This is the motivation for Problem 23.

We may consider the same problem with "nondecreasing" replaced by "nonincreasing". Perhaps the first step is to study the case when A is irreducible.

19. Monotonicity of a geometric mean of positive definite matrices

The following geometric mean of three or more positive definite matrices has recently been defined in [55] and [15] independently using a geometric approach. See also [12, 16]. Here we give only the basic idea. Denote by $\mathbf{P}_n$ the set of positive definite matrices of order n. We consider $\mathbf{P}_n$ as a differentiable manifold. The distance $\delta(A, B)$ between A, $B \in \mathbf{P}_n$ is the infimum of lengths of curves in $\mathbf{P}_n$ that connect A to B. It can be proved that $\delta(A, B) = \|\log(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})\|_F$. Given $A_i \in \mathbf{P}_n$, $i = 1, 2, \ldots, k$, there is a unique matrix in $\mathbf{P}_n$, denoted $G(A_1, \ldots, A_k)$, that minimizes the function

$$f(X) = \sum_{i=1}^{k} \delta^2(A_i, X).$$

Then $G(A_1, \ldots, A_k)$ is called the *geometric mean* of $A_1, \ldots, A_k$. This geometric mean is symmetric, invariant under congruence, and continuous. Let " $\leq$ " be the Loewner partial order. In 2006, Bhatia and Holbrook [15, p.616] made the following

Conjecture 24 *The mean G is monotone with respect to its arguments, i.e., if A_i, $B_i \in \mathbf{P}_n$ satisfy $A_i \leq B_i$, $i = 1, \ldots, k$, then*

$$G(A_1, \ldots, A_k) \leq G(B_1, \ldots, B_k).$$

Another geometric mean of three or more positive definite matrices has been defined in [3].

20. Eigenvalues of real symmetric matrices

Let $S_n[a,b]$ denote the set of $n \times n$ real symmetric matrices whose entries are in the interval $[a, b]$. For an $n \times n$ real symmetric matrix A, we always denote the eigenvalues of A in decreasing order by $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$. The *spread* of an $n \times n$ real symmetric matrix A is $s(A) = \lambda_1(A) - \lambda_n(A)$.

The following two problems were posed in [70].

Problem 25 *For a given j with $2 \leq j \leq n-1$, determine*

$$\max\{\lambda_j(A) : A \in S_n[a,b]\},$$

$$\min\{\lambda_j(A) : A \in S_n[a,b]\}$$

and determine the matrices that attain the maximum and the matrices that attain the minimum.

The cases $j = 1$, n are solved in [70].

Problem 26 *Determine*

$$\max\{s(A) : A \in S_n[a,b]\}$$

and determine the matrices that attain the maximum.

The special case when $a = -b$ is solved in [70].

21. Sharp constants in spectral variation

Let α_j and β_j, $j = 1, \ldots, n$, be the eigenvalues of $n \times n$ complex matrices A and B, respectively, and denote

$$\mathrm{Eig}A = \{\alpha_1, \ldots, \alpha_n\}, \quad \mathrm{Eig}B = \{\beta_1, \ldots, \beta_n\}.$$

The *optimal matching distance* between the spectra of A and B is

$$d(\mathrm{Eig}A, \mathrm{Eig}B) = \min_{\sigma} \max_{1 \leq j \leq n} |\alpha_j - \beta_{\sigma(j)}|$$

where σ varies over all permutations of the indices $\{1, 2, \ldots, n\}$.

Let $\|\cdot\|$ be the spectral norm. It is known [13] that there exists a number c with $1 < c < 3$ such that

$$d(\mathrm{Eig}A, \mathrm{Eig}B) \leq c\|A - B\|$$

for any normal matrices A, B of any order. See [14] for the very interesting history of this result.

Bhatia [13, pp.154–155] posed the following natural

Problem 27 *Determine the best possible constant c such that*

$$d(\mathrm{Eig}A,\, \mathrm{Eig}B) \le c\|A - B\|$$

for any normal matrices A, B of any order.

There are several other such constants whose exact values are not known [13].

22. Singular values of Heinz means

Let A, B be positive semidefinite matrices of order n. For $0 \le t \le 1$, the *Heinz mean* of A and B is

$$H_t(A, B) = (A^t B^{1-t} + A^{1-t} B^t)/2.$$

Note that $H_t = H_{1-t}$, and $H_0 = H_1$ is the arithmetic mean.

Denote the ordered singular values of an $n \times n$ complex matrix X by $s_1(X) \ge s_2(X) \ge \cdots \ge s_n(X)$. It was conjectured in [71] that

$$s_j(A^t B^{1-t} + A^{1-t} B^t) \le s_j(A + B), \quad j = 1, \ldots, n$$

and this has recently been proved by Audenaert [5]. Now I have the following further

Problem 28 *Let A, B be positive semidefinite matrices of order n. For $1 \le j \le n$, investigate the properties of the function*

$$f(t) = s_j(A^t B^{1-t} + A^{1-t} B^t), \quad t \in [0,\, 1].$$

The above inequality says that $f(t)$ attains its maximum value at $t = 0,\, 1$.

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