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Note

The maximum degree of a minimally hamiltonian-connected graph

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ABSTRACT

We determine the possible maximum degrees of a minimally hamiltonian-connected graph with a given order. This answers a question posed by Modalleliyan and Omoomi in 2016. We also pose two unsolved problems.

We consider finite simple graphs and follow the book [5] for terminology and notations. The *order* of a graph is its number of vertices, and the *size* is its number of edges. We denote by $V(G)$ and $E(G)$ the vertex set and edge set of a graph G respectively. For two graphs G and H, $G \vee H$ denotes the *join* of G and H, which is obtained from the disjoint union $G + H$ by adding edges joining every vertex of G to every vertex of H. K_n and C_n denote the complete graph of order n and the cycle of order n respectively. The *wheel* of order n, denoted by W_n, is the graph $K_1 \vee C_{n-1}$.

A graph is called *hamiltonian-connected* if between any two distinct vertices there is a Hamilton path. Obviously, any hamiltonian-connected graph of order at least 4 is 3-connected, and hence has minimum degree at least 3.

Definition. A hamiltonian-connected graph G is said to be *minimally hamiltonian-connected* if for every edge $e \in E(G)$, the graph $G - e$ is not hamiltonian-connected.

Clearly every hamiltonian-connected graph contains a minimally hamiltonian-connected spanning subgraph. Concerning the maximum degree of a minimally hamiltonian-connected graph with a given order, Modalleliyan and Omoomi [2] proved the following results: (1) The maximum degree of any minimally hamiltonian-connected graph of order n is not equal to $n-2$; (2) the wheel W_n is the only minimally hamiltonian-connected graph of order n with maximum degree $n-1$; (3) for every integer $n \geq 6$ and any integer Δ with $\lceil n/2 \rceil \leq \Delta \leq n-3$, there exists a minimally hamiltonian-connected graph of order n with maximum degree Δ. They [2] posed the question of whether for Δ in the range $3 \leq \Delta < \lceil n/2 \rceil$, there exists a minimally hamiltonian-connected graph of order n with maximum degree Δ. In this note we answer the question affirmatively. Our construction covers the whole range $3 \leq \Delta \leq n-1$, not only $3 \leq \Delta < \lceil n/2 \rceil$.

Theorem 1. *Let $n \geq 4$ be an integer. There exists a minimally hamiltonian-connected graph of order n with maximum degree Δ if and only if $3 \leq \Delta \leq n-1$ and $\Delta \neq n-2$, where $\Delta = 3$ occurs only if n is even.*

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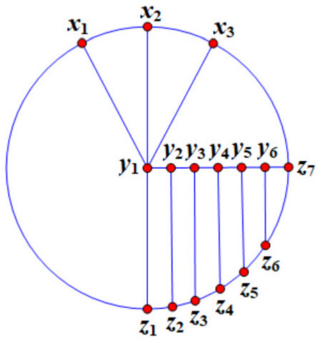

Fig. 1. The graph G(16,5).

Proof. Suppose there exists a minimally hamiltonian-connected graph G of order n with maximum degree Δ. Then G is 3-connected, implying that $\Delta \geq \delta \geq 3$ where δ denotes the minimum degree of G. Modalleliyan and Omoomi [2] proved that $\Delta \neq n-2$. If $\Delta = 3$, then G is cubic and hence its order n is even.

Conversely suppose $3 \leq \Delta \leq n-1$ and $\Delta \neq n-2$, and when $\Delta = 3$, n is even. We will construct a minimally hamiltonian-connected graph of order n with maximum degree Δ. If $\Delta = n-1$, the wheel graph W_n is a minimally hamiltonian-connected graph of order n with maximum degree $n-1$. Next suppose $3 \leq \Delta \leq n-3$. We will distinguish the two cases when $n-\Delta$ is odd and when $n-\Delta$ is even. For symbols such as x_i below, if i exceeds its valid range, then x_i does not appear.

Case 1. $n-\Delta$ is odd

We define a graph $G(n,\Delta)$ as follows. Denote $k = \Delta - 2$ and $s = (n-\Delta+1)/2$. We have $k \geq 1$ and $s \geq 2$.

$$\begin{aligned}
V(G(n,\Delta)) &= \{x_1, x_2, \ldots, x_k\} \cup \{y_1, y_2, \ldots, y_s\} \cup \{z_1, z_2, \ldots, z_{s+1}\},\\
E(G(n,\Delta)) &= \{x_i x_{i+1} \mid i = 1, \ldots, k-1\} \cup \{y_i y_{i+1} \mid i = 1, \ldots, s-1\} \cup \{z_i z_{i+1} \mid i = 1, \ldots, s\}\\
&\quad \cup \{y_1 x_i \mid i = 1, \ldots, k\} \cup \{y_i z_i \mid i = 1, \ldots, s\} \cup \{x_1 z_1, x_k z_{s+1}, y_s z_{s+1}\}.
\end{aligned}$$

The graph $G(16,5)$ is depicted in Fig. 1.

Clearly the graph $G(n,\Delta)$ has order n and maximum degree Δ. We first show that the graph $G(n,\Delta)$ is hamiltonian-connected; i.e., for any two distinct vertices u and v, there is a Hamilton (u,v)-path. There are 8 cases for the vertex pairs (u, v). In each case we display a Hamilton (u,v)-path.

We define several symbols to describe the Hamilton paths. For i and j with $i \leq j$, $x_i \overrightarrow{X} x_j$, $y_i \overrightarrow{Y} y_j$ and $z_i \overrightarrow{Z} z_j$ denote the paths $x_i x_{i+1} \ldots x_j$, $y_i y_{i+1} \ldots y_j$ and $z_i z_{i+1} \ldots z_j$, respectively. Throughout $h \overrightarrow{P} g$ will denote a path starting from the vertex h and ending at the vertex g whose vertex set is to be specified. For $1 \leq i \leq s$, let $x_1 \overrightarrow{P} y_i$ and $x_1 \overrightarrow{P} z_i$ denote the paths both with vertex set $\{x_1, y_1, \ldots, y_i, z_1, \ldots, z_i\}$. Similarly, let $y_i \overrightarrow{P} z_{s+1}$ and $z_i \overrightarrow{P} z_{s+1}$ denote the paths both with vertex set $\{y_i, \ldots, y_s, z_i, \ldots, z_s, z_{s+1}\}$. Also, we let $z_{s+1} \overrightarrow{P} z_{s+1} = z_{s+1}$. Finally, given a sequence $p \overrightarrow{S} q$, we denote by $q \overleftarrow{S} p$ its reverse sequence. In the following Hamilton paths, strings like $a_i \overrightarrow{A} a_j$ or $a_j \overleftarrow{A} a_i$ with $i > j$ do not appear.

Case 1.1. (x_i, x_j) with $1 \leq i < j \leq k$.
$x_i \overrightarrow{X} x_{j-1} y_1 x_{i-1} \overleftarrow{X} x_1 z_1 z_2 \overrightarrow{P} z_{s+1} x_k \overleftarrow{X} x_j$.
Case 1.2. (x_i, y_j) with $1 \leq i \leq k$ and $1 \leq j \leq s$.
$x_i \overrightarrow{X} x_k z_{s+1} \overleftarrow{P} z_{j+1} z_j \overleftarrow{Z} z_1 x_1 \overrightarrow{X} x_{i-1} y_1 \overrightarrow{Y} y_j$.
Case 1.3. (x_i, z_j) with $1 \leq i \leq k$ and $1 \leq j \leq s$.
$x_i \overrightarrow{X} x_k z_{s+1} \overleftarrow{P} y_{j+1} y_j \overleftarrow{Y} y_1 x_{i-1} \overleftarrow{X} x_1 z_1 \overrightarrow{Z} z_j$,
where when $j = s$ the string $z_{s+1} \overleftarrow{P} y_{j+1}$ means z_{s+1}.
Case 1.4. (x_i, z_{s+1}) with $1 \leq i \leq k$.
$x_i \overrightarrow{X} x_k y_1 x_{i-1} \overleftarrow{X} x_1 z_1 z_2 \overrightarrow{P} z_{s+1}$.
Case 1.5. (y_i, y_j) with $1 \leq i < j \leq s$.
$y_i \overrightarrow{Y} y_{j-1} z_{j-1} \overleftarrow{Z} z_i z_{i-1} \overleftarrow{P} x_1 x_2 \overrightarrow{X} x_k z_{s+1} \overleftarrow{P} y_j$,
where when $i = 1$ the string $z_{i-1} \overleftarrow{P} x_1$ means x_1.
Case 1.6. (y_i, z_j) with $1 \leq i < j \leq s+1$.
$y_i \overrightarrow{Y} y_{j-1} z_{j-1} \overleftarrow{Z} z_i z_{i-1} \overleftarrow{P} x_1 x_2 \overrightarrow{X} x_k z_{s+1} \overleftarrow{P} z_j$,
where when $i = 1$ the string $z_{i-1} \overleftarrow{P} x_1$ means x_1.

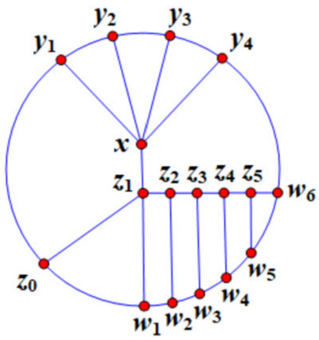

Fig. 2. The graph H(17,5).

Case 1.7. (y_i, z_j) with $1 \le j \le i \le s$.
$y_i \overleftarrow{Y} y_j y_{j-1} \overleftarrow{P} x_1 x_2 \overrightarrow{X} x_k z_{s+1} \overleftarrow{P} z_{i+1} z_i \overleftarrow{Z} z_j$,
where when $j = 1$ the string $y_{j-1} \overleftarrow{P} x_1$ means x_1.
Case 1.8. (z_i, z_j) with $1 \le i < j \le s + 1$.
$z_i \overrightarrow{Z} z_{j-1} y_{j-1} \overleftarrow{Y} y_i y_{i-1} \overleftarrow{P} x_1 x_2 \overrightarrow{X} x_k z_{s+1} \overleftarrow{P} z_j$,
where when $i = 1$ the string $y_{i-1} \overleftarrow{P} x_1$ means x_1.

We have shown that $G(n, \Delta)$ is hamiltonian-connected. Recall that any hamiltonian-connected graph of order at least 4 is 3-connected and hence has minimum degree at least 3. Note that every edge of $G(n, \Delta)$ has one endpoint of degree 3. Thus for any $e \in E(G(n, \Delta))$, $G(n, \Delta) - e$ has a vertex of degree 2, implying that it is not hamiltonian-connected. This completes the proof that $G(n, \Delta)$ is minimally hamiltonian-connected.

Case 2. $n - \Delta$ is even

We define a graph $H(n, \Delta)$ as follows. Denote $k = \Delta - 1$ and $s = (n - \Delta - 2)/2$. Since $n - \Delta$ is even and $\Delta \le n - 3$, we have $k \ge 3$ and $s \ge 1$.

$$V(H(n, \Delta)) = \{x\} \cup \{y_1, y_2, \dots, y_k\} \cup \{z_0, z_1, z_2, \dots, z_s\} \cup \{w_1, w_2, \dots, w_{s+1}\},$$
$$E(H(n, \Delta)) = \{y_i y_{i+1} \mid i = 1, \dots, k-1\} \cup \{z_i z_{i+1} \mid i = 0, 1, \dots, s-1\} \cup \{w_i w_{i+1} \mid i = 1, \dots, s\}$$
$$\cup \{x y_i \mid i = 1, \dots, k\} \cup \{z_i w_i \mid i = 1, \dots, s\} \cup \{x z_1, y_1 z_0, z_0 w_1, y_k w_{s+1}, z_s w_{s+1}\}.$$

The graph $H(17, 5)$ is depicted in Fig. 2.

Clearly the graph $H(n, \Delta)$ has order n and maximum degree Δ. We first show that the graph $H(n, \Delta)$ is hamiltonian-connected; i.e., for any two distinct vertices u and v, there is a Hamilton (u, v)-path. There are 18 cases for the vertex pairs (u, v). In each case we display a Hamilton (u, v)-path.

We define several symbols to describe the Hamilton paths. For i and j with $i \le j$, $y_i \overrightarrow{Y} y_j$, $z_i \overrightarrow{Z} z_j$ and $w_i \overrightarrow{W} w_j$ denote the paths $y_i y_{i+1} \dots y_j$, $z_i z_{i+1} \dots z_j$ and $w_i w_{i+1} \dots w_j$, respectively. Throughout $h \overrightarrow{P} g$ will denote a path starting from the vertex h and ending at the vertex g whose vertex set is to be specified. For $1 \le i \le s$, let $z_0 \overrightarrow{P} z_i$ and $z_0 \overrightarrow{P} w_i$ denote the paths both with vertex set $\{z_0, z_1, \dots, z_i, w_1, \dots, w_i\}$. Similarly, let $z_i \overrightarrow{P} w_{s+1}$ and $w_i \overrightarrow{P} w_{s+1}$ denote the paths both with vertex set $\{z_i, \dots, z_s, w_i, \dots, w_s, w_{s+1}\}$. Also, we let $z_0 \overrightarrow{P} z_0 = z_0$ and $w_{s+1} \overrightarrow{P} w_{s+1} = w_{s+1}$. Finally, given a sequence $p \overrightarrow{S} q$, we denote by $q \overleftarrow{S} p$ its reverse sequence. In the following Hamilton paths, strings like $a_i \overrightarrow{A} a_j$ or $a_j \overleftarrow{A} a_i$ with $i > j$ do not appear.

Case 2.1. (y_1, y_j) with $2 \le j \le k$.
$y_1 \overrightarrow{Y} y_{j-1} x z_1 z_0 w_1 w_2 \overrightarrow{P} w_{s+1} y_k \overleftarrow{Y} y_j$.
Case 2.2. (y_i, y_j) with $2 \le i < j \le k$.
$y_i \overrightarrow{Y} y_{j-1} x y_{i-1} \overleftarrow{Y} y_1 z_0 z_1 \overrightarrow{P} w_{s+1} y_k \overleftarrow{Y} y_j$.
Case 2.3. (y_1, x).
$y_1 z_0 z_1 \overrightarrow{P} w_{s+1} y_k \overleftarrow{Y} y_2 x$.
Case 2.4. (y_i, x) with $2 \le i \le k$.
$y_i \overrightarrow{Y} y_k w_{s+1} \overleftarrow{P} z_1 z_0 y_1 \overrightarrow{Y} y_{i-1} x$.
Case 2.5. (y_i, z_j) with $1 \le i \le k - 1$ and $0 \le j \le s$.
$y_i \overleftarrow{Y} y_1 x y_{i+1} \overrightarrow{Y} y_k w_{s+1} \overleftarrow{P} w_{j+1} w_j \overleftarrow{W} w_1 z_0 \overrightarrow{Z} z_j$.

Case 2.6. (y_k, z_0).
$y_k \overleftarrow{Y} y_1 x z_1 \overrightarrow{Z} z_s w_{s+1} \overleftarrow{W} w_1 z_0$.
Case 2.7. (y_k, z_j) with $1 \le j \le s$.
$y_k \overleftarrow{Y} y_2 x y_1 z_0 \overrightarrow{P} w_{j-1} w_j \overrightarrow{W} w_{s+1} z_s \overleftarrow{Z} z_j$,
where when $j = 1$ the string $z_0 \overrightarrow{P} w_{j-1}$ means z_0.
Case 2.8. (y_1, w_1).
$y_1 z_0 z_1 x y_2 \overrightarrow{Y} y_k w_{s+1} \overleftarrow{P} w_2 w_1$.
Case 2.9. (y_1, w_j) with $2 \le j \le s+1$.
$y_1 z_0 w_1 \overrightarrow{W} w_{j-1} z_{j-1} \overleftarrow{Z} z_1 x y_2 \overrightarrow{Y} y_k w_{s+1} \overleftarrow{P} w_j$.
Case 2.10. (y_i, w_j) with $2 \le i \le k$ and $1 \le j \le s+1$.
$y_i \overrightarrow{Y} y_k x y_{i-1} \overleftarrow{Y} y_1 z_0 \overrightarrow{P} z_{j-1} z_j \overrightarrow{Z} z_s w_{s+1} \overleftarrow{W} w_j$.
Case 2.11. (x, z_j) with $0 \le j \le s$.
$x y_1 \overrightarrow{Y} y_k w_{s+1} \overleftarrow{P} w_{j+1} w_j \overleftarrow{W} w_1 z_0 \overrightarrow{Z} z_j$.
Case 2.12. (x, w_j) with $1 \le j \le s+1$.
$x y_k \overleftarrow{Y} y_1 z_0 \overrightarrow{P} z_{j-1} z_j \overrightarrow{Z} z_s w_{s+1} \overleftarrow{W} w_j$.
Case 2.13. (z_i, z_j) with $0 \le i < j \le s$.
$z_i \overleftarrow{P} z_0 y_1 x y_2 \overrightarrow{Y} y_k w_{s+1} \overleftarrow{P} w_{j+1} w_j \overleftarrow{W} w_{i+1} z_{i+1} \overrightarrow{Z} z_j$.
Case 2.14. (z_0, w_1).
$z_0 z_1 x y_1 \overrightarrow{Y} y_k w_{s+1} \overleftarrow{P} w_2 w_1$.
Case 2.15. (z_0, w_j) with $2 \le j \le s+1$.
$z_0 w_1 \overrightarrow{W} w_{j-1} z_{j-1} \overleftarrow{Z} z_1 x y_1 \overrightarrow{Y} y_k w_{s+1} \overleftarrow{P} w_j$,
Case 2.16. (z_i, w_j) with $1 \le i < j \le s+1$.
$z_i \overrightarrow{Z} z_{j-1} w_{j-1} \overleftarrow{W} w_i w_{i-1} \overleftarrow{P} z_0 y_1 x y_2 \overrightarrow{Y} y_k w_{s+1} \overleftarrow{P} w_j$,
where when $i = 1$ the string $w_{i-1} \overleftarrow{P} z_0$ means z_0.
Case 2.17. (z_i, w_j) with $1 \le j \le i \le s$.
$z_i \overleftarrow{Z} z_j z_{j-1} \overleftarrow{P} z_0 y_1 x y_2 \overrightarrow{Y} y_k w_{s+1} \overleftarrow{P} w_{i+1} w_i \overleftarrow{W} w_j$.
Case 2.18. (w_i, w_j) with $1 \le i < j \le s+1$.
$w_i \overrightarrow{W} w_{j-1} z_{j-1} \overleftarrow{Z} z_i z_{i-1} \overleftarrow{P} z_0 y_1 x y_2 \overrightarrow{Y} y_k w_{s+1} \overleftarrow{P} w_j$.

Thus we have shown that $H(n, \Delta)$ is hamiltonian-connected. Recall that any hamiltonian-connected graph of order at least 4 is 3-connected and hence has minimum degree at least 3. Since the graph $H(n, \Delta) - xz_1$ has connectivity 2, it is not hamiltonian-connected. For every edge $e \in E(H(n, \Delta))$ with $e \ne xz_1$, e has one endpoint of degree 3. Therefore $H(n, \Delta) - e$ has a vertex of degree 2, implying that it is not hamiltonian-connected. This completes the proof that $H(n, \Delta)$ is minimally hamiltonian-connected, and the theorem is proved. □

Remark. The graphs $G(n, \Delta)$ and $H(n, \Delta)$ constructed in the above proof of Theorem 1 have degree sequences $\Delta, 3, 3, \ldots, 3$ and $\Delta, 4, 3, \ldots, 3$ respectively, and hence they have the minimum possible sizes among all graphs of order n with maximum degree Δ and minimum degree at least 3 in the two cases when $n - \Delta$ is odd and when $n - \Delta$ is even respectively. The graph constructed in [2] for Δ in the range $\lceil n/2 \rceil \le \Delta \le n-3$ has degree sequence $\Delta, n-\Delta, 3, \ldots, 3$.

Finally we pose two unsolved problems.

Problem 1. Let $n \ge 4$ be a given integer. What are the possible values of the minimum degree of a minimally hamiltonian-connected graph of order n?

A computer search shows that every minimally hamiltonian-connected graph of order n with $4 \le n \le 10$ has minimum degree 3. The author does not know of an example of a minimally hamiltonian-connected graph with minimum degree at least 4. The following easier problem is of a more basic nature.

Problem 2. Does there exist a minimally hamiltonian-connected graph with minimum degree at least 4?

There are some sufficient conditions for hamiltonian-connected graphs; for recent ones see [1,3] and [4]. But very little is known about necessary conditions. Restrictions on the maximum or minimum degree of a minimally hamiltonian-connected graph may be viewed as necessary conditions for this smaller class of graphs.

Declaration of competing interest

There is no conflict of interest in this work.

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