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Note

Every 3-connected claw-free graph with domination number at most 3 is hamiltonian-connected

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ABSTRACT

We prove that every 3-connected claw-free graph with domination number at most 3 is hamiltonian-connected. The result is sharp and it is inspired by a conjecture posed by Zheng, Broersma, Wang and Zhang in 2020.

1. Introduction

In this paper, by a *graph* we always mean a finite simple undirected graph; if we admit multiple edges, we always speak about a *multigraph*. We follow the book [8] for terminology and notations.

The complete bipartite graph on s and t vertices is denoted by $K_{s,t}$. The graph $K_{1,3}$ is called the *claw*. A graph is called *claw-free* if it contains no induced subgraph isomorphic to the claw. A graph is called *hamiltonian-connected* if between any two distinct vertices there is a hamiltonian path. A subset X of vertices in a graph G is called a *dominating set* if every vertex of G is either contained in X or adjacent to some vertex of X. The *domination number* of G is the size of a smallest dominating set of G.

In 1994, Ageev [1] proved the following sufficient condition for a claw-free graph to be hamiltonian involving the domination number.

Theorem A. *[1]. Every 2-connected claw-free graph with domination number at most 2 is hamiltonian.*

The main result of this note is inspired by the following conjecture posed by Zheng, Broersma, Wang and Zhang [9]. Note that a hamiltonian-connected graph is necessarily 3-connected.

Conjecture B. *[9]. Every 3-connected claw-free graph with domination number at most 2 is hamiltonian-connected.*

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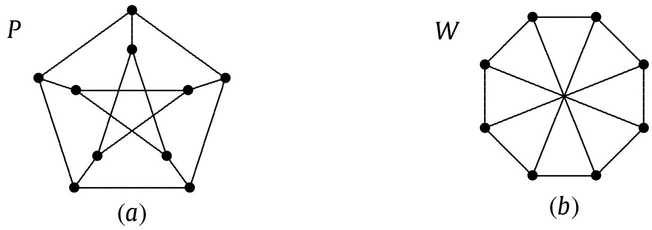

Fig. 1. The Petersen graph P and the Wagner graph W.

We prove the following stronger theorem which is sharp.

Theorem 1. *Every 3-connected claw-free graph with domination number at most 3 is hamiltonian-connected.*

The proof of Theorem 1 is postponed to Section 2. We will first need to recall some necessary known concepts and results.

We say that an edge is *pendant* if it contains a vertex of degree 1, and that a vertex is *simplicial* if its neighbors induce a complete graph. The *line graph* of a multigraph H is the graph $G = L(H)$ with $V(G) = E(H)$, in which two vertices are adjacent if and only if the corresponding edges of H have at least one vertex in common.

The following was proved in [5] using a modification of an approach from [10].

Theorem C. *[5]. Let G be a connected line graph of a multigraph. Then there is, up to an isomorphism, a uniquely determined multigraph $H = L^{-1}(G)$ such that a vertex $e \in V(G)$ is simplicial in G if and only if the corresponding edge $e \in E(H)$ is a pendant edge in H.*

An edge-cut $R \subset E(H)$ of a multigraph H is *essential* if $H - R$ has at least two nontrivial components, and H is *essentially k-edge-connected* if every essential edge-cut of H is of size at least k. It is a well-known fact that a line graph G is k-connected if and only if $L^{-1}(G)$ is essentially k-edge-connected.

A set of vertices $M \subset V(G)$ *dominates* an edge e if e has at least one vertex in M. A closed trail T is a *dominating closed trail* (abbreviated DCT) if T dominates all edges of G and an (e, f)-trail (i.e., a trail with terminal edges e, f) is an *internally dominating (e, f)-trail* (abbreviated (e, f)-IDT) if the set of its interior vertices dominates all edges of G.

Harary and Nash-Williams [2] proved a correspondence between a DCT in H and a hamiltonian cycle in $L(H)$ (the result was established in [2] for graphs, but it is easy to observe that the proof is true also for line graphs of multigraphs). A similar result showing that $G = L(H)$ is hamiltonian-connected if and only if H has an (e_1, e_2)-IDT for any pair of edges $e_1, e_2 \in E(H)$, was given in [3].

Theorem D. *[2,3]. Let H be a multigraph with $|E(H)| \geq 3$ and let $G = L(H)$.*

(i) *[2] The graph G is hamiltonian if and only if H has a DCT.*
(ii) *[3] For every $e_i \in E(H)$ and $a_i = L(e_i)$, $i = 1, 2$, G has a hamiltonian (a_1, a_2)-path if and only if H has an (e_1, e_2)-IDT.*

Let G be a 3-connected line graph and let $H = L^{-1}(G)$. The *core* of H is the multigraph $\mathrm{co}(H)$ obtained from H by removing all pendant edges and suppressing all vertices of degree 2.

Shao [7] proved the following properties of the core of a multigraph.

Lemma E. *[7]. Let H be an essentially 3-edge-connected multigraph. Then*

(i) $\mathrm{co}(H)$ *is uniquely determined,*
(ii) $\mathrm{co}(H)$ *is 3-edge-connected,*
(iii) *if $\mathrm{co}(H)$ has a spanning closed trail, then H has a dominating closed trail.*

We denote by P the Petersen graph and by W the Wagner graph (see Fig. 1).

Let G be a multigraph, $R \subset G$ a spanning subgraph of G, and let $\mathcal{R}$ be the set of components of R. Then G/R is the multigraph with $V(G/R) = \mathcal{R}$, in which, for each edge in $E(G)$ between two components of R, there is an edge in $E(G/R)$ joining the corresponding vertices of G/R. The (multi-)graph G/R is said to be a *contraction* of G. (Roughly, in G/R, components of R are contracted to single vertices while keeping the adjacencies between them.)

The contraction operation maps $V(G)$ onto $V(G/R)$ (where vertices of a component of R are mapped on a vertex of G/R). If $G/R \simeq F$, then this defines a function $\alpha : G \to F$ which is called a *contraction of G on F*.

The following theorem was proved in [4] (see also [6]).

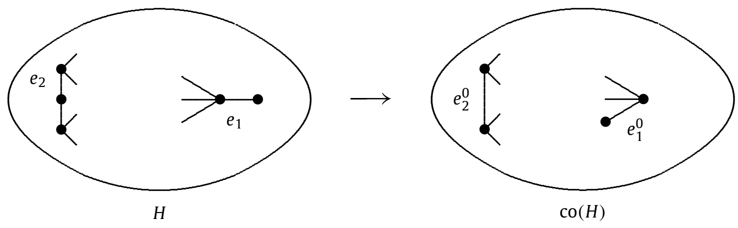

Fig. 2. An example of the construction.

Theorem F. *[4]. Let H be a 3-edge-connected multigraph, $A \subset V(H)$, $|A| = 8$, and let $e \in E(H)$. Then either*

(i) *H contains a closed trail T such that $A \subset V(T)$ and $e \in E(T)$, or*
(ii) *there is a contraction $\alpha : H \to P$ such that $\alpha(e) = xy \in E(P)$ and $\alpha(A) = V(P) \setminus \{x, y\}$.*

In fact, we will need only the following easy corollary.

Corollary 2. *Let H be a 3-edge-connected multigraph, $A \subset V(H)$, $|A| \leq 7$, and let $e \in E(H)$. Then H contains a closed trail T such that $A \subset V(T)$ and $e \in E(T)$.*

The concept of an M-closure $\mathrm{cl}^M(G)$ of a claw-free graph G was defined in [5]. We do not need to know the exact construction of this closure. We will use only the following theorem proved in [5].

Theorem G. *[5]. Let G be a claw-free graph and let the graph $\mathrm{cl}^M(G)$ be its M-closure. Then $\mathrm{cl}^M(G)$ has the following properties:*

(i) *$V(G) = V(\mathrm{cl}^M(G))$ and $E(G) \subset E(\mathrm{cl}^M(G))$,*
(ii) *$\mathrm{cl}^M(G)$ is uniquely determined,*
(iii) *G is hamiltonian-connected if and only if $\mathrm{cl}^M(G)$ is hamiltonian-connected,*
(iv) *$\mathrm{cl}^M(G) = L(H)$, where H is a multigraph.*

2. Proof of Theorem 1

Now we are ready to give a proof of the main result.

Proof of Theorem 1. Suppose Theorem 1 is false and let G be a counterexample to Theorem 1. By Theorem G, $\mathrm{cl}^M(G)$ is a non-hamiltonian-connected line graph of a multigraph. Let $H = L^{-1}(\mathrm{cl}^M(G))$. By Theorem D, there are edges e_1, e_2 such that H has no (e_1, e_2)-IDT. Since G is 3-connected, H is essentially 3-edge-connected.

To reach the contradiction, we first convert the problem into the core of H, and then we find a trail in co(H) such that the corresponding trail in H is an (e_1, e_2)-IDT. We define the edges e_1^0, e_2^0 as follows:

(i) if $e_i, i \in \{1, 2\}$, have in H both vertices of degree at least 3, then $e_i \in E(\mathrm{co}(H))$, and we set $e_i^0 = e_i$, $i = 1, 2$ in co(H);
(ii) if e_i is a pendant edge, then we denote by e_i^0 an arbitrary edge in co(H) containing the vertex of higher degree of e_i;
(iii) finally, if one of the vertices of e_i, say, v_2, has degree 2, we take as e_i^0 the new edge in co(H) resulting by suppressing the vertex v_2.

See the example in Fig. 2, in which e_1 is of type (ii), and e_2 is of type (iii).

Now, in the multigraph co(H), if $e_1^0 = e_2^0$, we set $e_n = e_1^0 = e_2^0$; otherwise we subdivide the edges e_1^0, e_2^0 and join the two new vertices with an edge e_n. In both cases, we denote by H_n the resulting multigraph. In the first case, $H_n = \mathrm{co}(H)$ and it is 3-edge-connected by Lemma E, and it is easy to see that in the second case, H_n is also 3-edge-connected (recall that, in co(H), all pendant edges are removed and all vertices of degree 2 are suppressed).

Let $\{w_1, w_2, w_3\}$ be a dominating set of G. By the definition of a line graph, the three corresponding edges in H, denoted f_1, f_2, f_3, dominate all edges of H. For $f_1, f_2, f_3 \in E(H)$, we find edges f_j^0, $j = 1, 2, 3$ in co(H) by the same rules as e_i^0. Since the three edges f_j^0, $j = 1, 2, 3$ have at most 6 different vertices, denoted $z_1, \ldots, z_6$, by Corollary 2, there is a closed trail in H_n containing the vertices $z_1, \ldots, z_6$ and the edge e_n. It is straightforward to check that for every case of constructing e_i^0 and e_n, we can find an (e_1, e_2)-IDT in H (recall that the corresponding set of vertices $\{z_1, \ldots, z_6\}$ dominates all edges of H). $\square$

The result is sharp. A counterexample for domination number 4 is the line graph of a graph obtained from the Wagner graph by adding at least one pendant edge to each of its vertices.

Declaration of competing interest

We claim that there is no conflict of interest in our paper.

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