THE RELATION BETWEEN THE NUMBER OF LEAVES OF A TREE AND ITS DIAMETER

Pu Qiao, Xingzhi Zhan, Shanghai

Received November 12, 2020. Published online September 7, 2021.

Abstract. Let $L(n,d)$ denote the minimum possible number of leaves in a tree of order n and diameter d. Lesniak (1975) gave the lower bound $B(n,d)=\lceil 2(n-1)/d\rceil$ for $L(n,d)$. When d is even, $B(n,d)=L(n,d)$. But when d is odd, $B(n,d)$ is smaller than $L(n,d)$ in general. For example, $B(21,3)=14$ while $L(21,3)=19$. In this note, we determine $L(n,d)$ using new ideas. We also consider the converse problem and determine the minimum possible diameter of a tree with given order and number of leaves.

Keywords: leaf; diameter; tree; spider

MSC 2020: 05C05, 05C35, 05C12

A *leaf* in a graph is a vertex of degree 1. For a real number r, $\lfloor r\rfloor$ denotes the largest integer less than or equal to r, and $\lceil r\rceil$ denotes the least integer larger than or equal to r. Let $L(n,d)$ denote the minimum possible number of leaves in a tree of order n and diameter d. In 1975 Lesniak in [1], Theorem 2, page 285 gave the lower bound $B(n,d)=\lceil 2(n-1)/d\rceil$ for $L(n,d)$. When d is even, $B(n,d)=L(n,d)$. But when d is odd, $B(n,d)$ is smaller than $L(n,d)$ in general. For example, $B(21,3)=14$ while $L(21,3)=19$. The proof in [1] uses two lemmas, treating the even case and odd case of the number of leaves separately and showing that in both cases there exists a set of paths with certain special properties.

In this note we first determine $L(n,d)$. We use ideas different from those in [1]. The proof also makes it clear why $L(n,d)$ has such an expression. We then determine the minimum possible diameter of a tree with given order and number of leaves.

The authors were supported by the NSFC grants 11671148 and 11771148 and Science and Technology Commission of Shanghai Municipality (STCSM) grant 18dz2271000.

DOI: 10.21136/CMJ.2021.0492-20

We remark that the corresponding maximum problems are trivial. The maximum possible number of leaves in a tree of order n and diameter d is $n - d + 1$ and the maximum possible diameter of a tree of order n with exactly f leaves is $n - f + 1$.

We make the necessary preparation. For terminology and notation we follow the books [2] and [3]. We denote by $V(G)$ the vertex set of a graph G and by $d(u, v)$ the distance between two vertices u and v. For vertices x and y, an (x, y)-*path* is a path with end vertices x and y. We denote by $\deg(v)$ the degree of a vertex v.

Let P be a path in a tree T and we call P the *stem* of T. For every vertex $x \in V(T)$ there is a unique (x, y)-path Q such that $V(Q) \cap V(P) = \{y\}$. We say that x *originates from* y. Note that by definition, a vertex on the stem originates from itself. A *diametral path* of a tree T is a path of length equal to the diameter of T.

A *spider* is a tree with at most one vertex of degree larger than 2 and this vertex, if it exists, is called the *branch vertex.* If no vertex has degree larger than 2, then any vertex may be specified as the branch vertex. Thus, a spider is a subdivision of a star. A *leg* of a spider is a path from the branch vertex to a leaf.

We will need the following lemma.

Lemma 1 ([2], page 63)**.** *A path $P = v_0 v_1 v_2 \dots v_k$ in a tree is a diametral path if and only if for every vertex x,*

$$d(x, v_i) \leqslant \min\{i, k - i\},$$

where x originates from v_i with P as the stem.

The case $d = 1$ for $L(n, d)$ is trivial, since the only tree of diameter 1 is K_2 which has two leaves. Thus, it suffices to consider the case $d \geqslant 2$.

Theorem 2. *Let $L(n, d)$ denote the minimum possible number of leaves in a tree of order n and diameter d with $d \geqslant 2$. Then*

$$L(n,d) = \begin{cases} \left\lceil \dfrac{2(n-1)}{d} \right\rceil & \text{if } d \text{ is even,} \\[2ex] \left\lceil \dfrac{2(n-2)}{d-1} \right\rceil & \text{if } d \text{ is odd.} \end{cases}$$

Proof. The idea is to show that for any tree T there is a corresponding spider with the same order, diameter and number of leaves as T. Hence, to determine $L(n, d)$ it suffices to consider spiders.

If $d = n - 1$, then the tree must be a path which has two leaves. In this case the formula for $L(n, d)$ is true. Note also that a path is a spider. Next we assume $d \leqslant n - 2$.

Let T be a tree of order n and diameter d. Choose a diametral path $P = v_0v_1v_2\dots v_d$ as the stem. Suppose that x is a leaf of T outside P originating from y. There is a unique (x,y)-path Q. Since P is a diametral path, $y \neq v_0, v_d$. Hence, $\deg(y) \geqslant 3$. We define the *first big vertex* of x, denoted by $b(x)$, to be the first vertex of degree at least 3 from x to y on Q.

Denote $c = \lfloor \frac{1}{2}d \rfloor$. Then $c = \frac{1}{2}d$ if d is even and $c = \frac{1}{2}(d-1)$ if d is odd. Let $z = v_c$. If T has a leaf u outside P with $b(u) \neq z$, let w be the neighbor of $b(u)$ on the $(b(u),u)$-path. Since T is a tree, w and z are not adjacent. We delete the edge $wb(u)$ and add the edge wz to obtain a new tree T_1. Since $\min\{i, d-i\} \leqslant \min\{c, d-c\}$ for any $0 \leqslant i \leqslant d$, by Lemma 1 we deduce that P remains a diametral path of T_1. Clearly T_1 and T have the same set of leaves. Hence, T_1 and T have the same order, diameter and number of leaves. We still designate P as the stem of T_1. If T_1 has a leaf outside P whose first big vertex is not z, perform the above operation on T_1 to obtain a tree T_2. Repeating this operation in the resulting trees successively finitely many times, we obtain a tree in which every leaf outside P originates from z and with z as its first big vertex. Such a tree is a spider. An example of the above transformations is depicted in Figure 1.

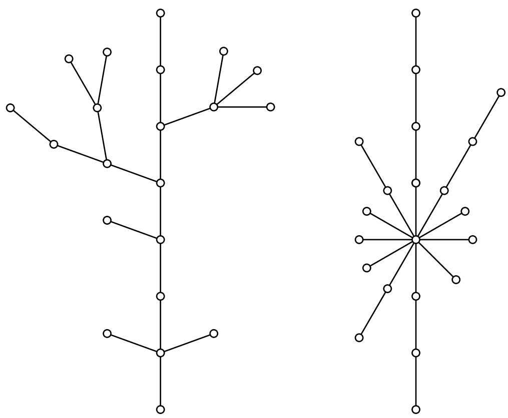

Figure 1. Transforming a general tree to a spider

The above analysis shows that $L(n,d)$ can be attained at a spider S with a diametral path $P = v_0v_1v_2\dots v_d$, where $z = v_c$ is the branch vertex. Clearly, the number of leaves in S is equal to the number of legs of S. To make the number of legs as small as possible, we need to make each leg as long as possible. Since the diameter of S is d, except the leg $v_cv_{c+1}\dots v_d$ when d is odd, every other leg has length at most c. Thus, the minimum possible number of legs of such a spider is $\lceil (n-1)/c \rceil$ when d is even and $\lceil (n-2)/c \rceil$ when d is odd. This completes the proof. $\square$

Next we consider the converse problem: Determine the minimum possible diameter of a tree of order n with exactly f leaves. It suffices to treat the case when $n \geqslant f+1$, since K_2 is the only tree with $n \leqslant f$.

Theorem 3. *Let $D(n,f)$ be the minimum possible diameter of a tree of order n with exactly f leaves. Then*

$$D(n,f) = \begin{cases} 2 & \text{if } n = f+1, \\ 2k+1 & \text{if } n = kf+2, \\ 2k+2 & \text{if } kf+3 \leqslant n \leqslant (k+1)f+1. \end{cases}$$

Proof. In the proof of Theorem 2, we showed that for any tree T there is a corresponding spider with the same order, diameter and number of leaves as T. Thus, it suffices to consider spiders. Note that the number of leaves of a spider is equal to its number of legs, which is also true for the case when the spider is a path (corresponding to $f = 2$) if we take a central vertex of the path as its branch vertex. Let S be a spider of order n with exactly f legs whose lengths are $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_f$ arranged in nonincreasing order. Then the diameter of S is $x_1 + x_2$. Hence, our problem is equivalent to minimizing $x_1 + x_2$ under the constraint

$$x_1 + x_2 + x_3 + \ldots + x_f = n - 1, \tag{1}$$

where $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_f$ are positive integers.

If $n = f+1$, then (1) becomes $x_1 + x_2 + x_3 + \ldots + x_f = f$, which has the only solution $x_1 = x_2 = x_3 = \ldots = x_f = 1$. Hence, $x_1 + x_2 = 2$. Let $n = kf + 2$. If $x_1 + x_2 \leqslant 2k$, then $x_2 \leqslant k$ and consequently $x_i \leqslant k$ for each $i = 3, \ldots, f$. It follows that

$$x_1 + x_2 + x_3 + \ldots + x_f \leqslant (x_1 + x_2) + (f-2)k \leqslant 2k + (f-2)k = fk = n - 2,$$

contradicting (1). This shows that $D(n,f) \geqslant 2k+1$. On the other hand, the values $x_1 = k+1$, $x_2 = \ldots = x_f = k$ satisfy (1) and $x_1 + x_2 = 2k+1$. Hence, $D(n,f) = 2k+1$.

Now consider the third case $kf+3 \leqslant n \leqslant (k+1)f+1$. We have $kf+2 \leqslant n-1 \leqslant kf+f$. Thus, there exists an integer r with $2 \leqslant r \leqslant f$ such that $n - 1 = kf + r$. We first show $D(n,f) \geqslant 2k+2$. If $x_1 + x_2 \leqslant 2k+1$, then $x_2 \leqslant k$ and consequently each $x_i \leqslant k$ for $i = 3, \ldots, f$. It follows that

$$\begin{aligned} x_1 + x_2 + x_3 + \ldots + x_f &\leqslant (x_1 + x_2) + (f-2)k \leqslant 2k + 1 + (f-2)k = fk + 1 \\ &< fk + r = n - 1, \end{aligned}$$

contradicting (1). Hence, $D(n,f) \geqslant 2k+2$. On the other hand, the values $x_1 = x_2 = \ldots = x_r = k+1$ and $x_{r+1} = \ldots = x_f = k$ satisfy (1) and $x_1 + x_2 = 2k+2$, which shows $D(n,f) = 2k+2$. This completes the proof. □

References

[1] *L. Lesniak*: On longest paths in connected graphs. Fundam. Math. *86* (1975), 283–286. zbl MR doi

[2] *O. Ore*: Theory of Graphs. Colloquium Publications 38. American Mathematical Society, Providence, 1962. zbl MR doi

[3] *D. B. West*: Introduction to Graph Theory. Prentice Hall, Upper Saddle River, 1996. zbl MR

Authors' addresses: Pu Qiao, Department of Mathematics, East China University of Science and Technology, 130 Meilong Rd, Xuhui District, Shanghai 200237, P. R. China, e-mail: `pq@ecust.edu.cn`; Xingzhi Zhan (corresponding author), Department of Mathematics, East China Normal University, 500 Dongchuan Rd., Shanghai 200241, P. R. China, e-mail: `zhan@math.ecnu.edu.cn`.