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ON A PROBLEM OF ERDŐS ABOUT GRAPHS WHOSE SIZE IS THE TURÁN NUMBER PLUS ONE

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Abstract

We consider finite simple graphs. Given a graph H and a positive integer n, the Turán number of H for the order n, denoted $\mathrm{ex}(n, H)$, is the maximum size of a graph of order n not containing H as a subgraph. Erdős asked: 'For which graphs H is it true that every graph on n vertices and $\mathrm{ex}(n, H) + 1$ edges contains at least two H's? Perhaps this is always true.' We solve this problem in the negative by proving that for every integer $k \geq 4$ there exists a graph H of order k and at least two orders n such that there exists a graph of order n and size $\mathrm{ex}(n, H) + 1$ which contains exactly one copy of H. Denote by C_4 the 4-cycle. We also prove that for every integer n with $6 \leq n \leq 11$ there exists a graph of order n and size $\mathrm{ex}(n, C_4) + 1$ which contains exactly one copy of C_4, but, for $n = 12$ or $n = 13$, the minimum number of copies of C_4 in a graph of order n and size $\mathrm{ex}(n, C_4) + 1$ is two.

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1. Introduction and statement of the main results

We consider finite simple graphs and use standard terminology and notation. The *order* of a graph is its number of vertices and the *size* its number of edges. Denote by $V(G)$ and $E(G)$ the vertex set and edge set of a graph G, respectively. For graphs we will use equality up to isomorphism, so $G_1 = G_2$ means that G_1 and G_2 are isomorphic. Given graphs H and G, a *copy* of H in G is a subgraph of G that is isomorphic to H. We denote by K_p and C_p the complete graph of order p and cycle of length p, respectively, and $K_{s,t}$ denotes the complete bipartite graph on s and t vertices. In particular, $K_{1,p}$ is the *star* of order $p + 1$. A *triangle-free* graph is one that contains no triangles.

In 1907, Mantel [10] proved that the maximum size of a triangle-free graph of order n is $\lfloor n^2/4 \rfloor$ and the balanced complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ is the unique extremal graph. Later, in 1941, Turán [12] solved the corresponding problem with the triangle replaced by a general complete graph.

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DEFINITION 1.1. Given a graph H and a positive integer n, the *Turán number of H for the order n*, denoted $\mathrm{ex}(n, H)$, is the maximum size of a simple graph of order n not containing H as a subgraph.

Thus, Mantel's theorem says that $\mathrm{ex}(n, K_3) = \lfloor n^2/4 \rfloor$ and Turán determined $\mathrm{ex}(n, K_p)$. Determining the Turán number for various graphs H is one of the main topics in extremal graph theory [1]. Note that a triangle is both K_3 and C_3. It is natural to extend Mantel's theorem to the case of larger cycles. This is difficult for even cycles. For example, the values $\mathrm{ex}(n, C_4)$ have not been determined for a general order n. Precise values are known only for some orders of special forms (see [6, 7]). A conjecture of Erdős and Simonovits on $\mathrm{ex}(n, C_6)$ was refuted in [8].

On the other hand, Rademacher (orally to Erdős who gave a simple proof in 1955 [4]) proved that every graph of order n and size $\mathrm{ex}(n, K_3) + 1$ contains at least $\lfloor n/2 \rfloor$ triangles. A similar result for a general K_p was proved by Moon [11].

PROBLEM 1.2. In 1990, Erdős posed the following problem in [5, pages 472–473]: *For which graphs H is it true that every graph on n vertices and* $\mathrm{ex}(n, H) + 1$ *edges contains at least two H's? Perhaps this is always true.*

We solve this problem in the negative using the class of book graphs.

DEFINITION 1.3. The *book* with p pages, denoted B_p, is the graph that consists of p triangles sharing a common edge.

Our main results are as follows.

THEOREM 1.4. *Let p be an even integer and let n be an odd integer with $n \geq p + 1 \geq 5$. Then there exists a graph of order n and size* $\mathrm{ex}(n, K_{1,p}) + 1$ *which contains exactly one copy of $K_{1,p}$.*

Note that in Theorem 1.4, for any fixed star $K_{1,p}$, there are infinitely many orders n such that the conclusion holds.

THEOREM 1.5. *Let p be an even positive integer. Then there exists a unique graph of order $p + 2$ and size* $\mathrm{ex}(p + 2, B_p) + 1$ *which contains exactly one copy of B_p and there exists a unique graph of order $p + 3$ and size* $\mathrm{ex}(p + 3, B_p) + 1$ *which contains exactly one copy of B_p.*

Combining Theorems 1.4 and 1.5 yields the following corollary.

COROLLARY 1.6. *For every integer $k \geq 4$ there exists a graph H of order k and at least two orders n such that there exists a graph of order n and size* $\mathrm{ex}(n, H) + 1$ *which contains exactly one copy of H.*

We remark that the conclusion in Theorem 1.4 is false for odd $p \geq 3$ and for the case when both p and n are even. In these two cases, the minimum number of copies of $K_{1,p}$ in a graph of order n and size $\mathrm{ex}(n, K_{1,p}) + 1$ is two. The following result shows that the statement for odd p on books corresponding to Theorem 1.5 is false.

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THEOREM 1.7. *Let $p \geq 3$ be an odd integer. Then the minimum number of copies of B_p in a graph of order $p+2$ and size $\mathrm{ex}(p+2, B_p)+1$ is three and the minimum number of copies of B_p in a graph of order $p+3$ and size $\mathrm{ex}(p+3, B_p)+1$ is $3(p+1)$.*

Recently, He, Ma and Yang [9, Conjecture 10.2] proposed the conjecture that $\mathrm{ex}(q^2+q+2, C_4) = (q(q+1)^2)/2+2$ for large $q = 2^k$. In [9, Proposition 10.3], they proved that if this conjecture is true, then the 4-cycle C_4 would serve as a counterexample to Erdős' Problem 1.2 and that C_4 for the order 22 is such an example [9, page 38]. Our next result shows that C_4 is a counterexample to Problem 1.2 for several low orders.

THEOREM 1.8. *For every integer n with $6 \leq n \leq 11$ there exists a graph of order n and size $\mathrm{ex}(n, C_4)+1$ which contains exactly one copy of C_4, but, for $n = 12$ or $n = 13$, the minimum number of copies of C_4 in a graph of order n and size $\mathrm{ex}(n, C_4)+1$ is two.*

In Section 2 we give proofs of the above results and in Section 3 we make some concluding remarks.

2. Proofs of the main results

We denote by $N(v)$, $N[v]$ and $\deg(v)$ the neighbourhood, closed neighbourhood and degree of a vertex v, respectively. By definition, $N[v] = \{v\} \cup N(v)$. Given a graph G, $\Delta(G)$ and $\overline{G}$ denote the maximum degree of G and the complement of G, respectively. For $S \subseteq V(G)$ we denote by $G[S]$ the subgraph of G induced by S. The *join*, $G \vee H$, of two graphs G and H is obtained from the disjoint union $G+H$ by adding edges joining every vertex of G to every vertex of H. Let P_n denote the path of order n; it has length $n-1$. As usual, qK_2 denotes the graph consisting of q pairwise vertex-disjoint edges.

NOTATION 2.1. For an even positive integer n the notation $K_n - PM$ denotes the graph obtained from the complete graph K_n by deleting all the edges in a perfect matching of K_n. That is, it is the complement of $(n/2)K_2$.

The following lemma is well known [2, pages 12–13], but its hamiltonian part is usually not stated.

LEMMA 2.2. *Let k and n be integers with $1 \leq k \leq n-1$. Then there exists a k-regular graph of order n if and only if kn is even. If kn is even and $k \geq 2$, then there exists a hamiltonian k-regular graph of order n.*

LEMMA 2.3. *Let d and n be integers with $1 \leq d \leq n-1$ and let $f(n,d)$ be the maximum size of a graph of order n with maximum degree $\leq d$. Then*

$$f(n,d) = \begin{cases} (nd-1)/2 & \text{if both } n \text{ and } d \text{ are odd,} \\ nd/2 & \text{otherwise.} \end{cases}$$

PROOF. If at least one of n and d is even, then by Lemma 2.2 there exists a d-regular graph of order n. Hence, the obvious upper bound $nd/2$ can be attained.

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If both n and d are odd, then such a graph has at most $n-1$ vertices with degree d, since the number of odd vertices in any graph is even. It follows that $f(n,d) \le ((n-1)d + (d-1))/2 = (nd-1)/2$. Next we show that this upper bound can be attained. If $d = 1$, the graph $((n-1)/2)K_2 + K_1$ attains the upper bound. Now suppose that $d \ge 2$. By Lemma 2.2, there exists a d-regular graph G of order $n-1$ containing a matching $M = \{x_iy_i \mid i = 1, \dots, (d-1)/2\}$ of size $(d-1)/2$. In G, delete all the edges in M, add a new vertex v and join the edges vx_i, vy_i for $i = 1, \dots, (d-1)/2$. This yields a graph with degree sequence $d, d, \dots, d, d-1$ which has size $(nd-1)/2$. □

Proof of Theorem 1.4. A graph G contains no $K_{1,p}$ if and only if $\Delta(G) \le p-1$. By Lemma 2.3, $\mathrm{ex}(n, K_{1,p}) = (n(p-1)-1)/2$. Note that a graph with degree sequence $d_1 \ge d_2 \ge \cdots \ge d_n$ contains exactly one copy of $K_{1,p}$ if and only if $d_1 = p$ and $d_2 \le p-1$. Hence, a graph of order n and size $\mathrm{ex}(n, K_{1,p}) + 1$ contains exactly one copy of $K_{1,p}$ if and only if its degree sequence is $p, p-1, \dots, p-1$.

Our assumption in Theorem 1.4 implies that $p-1 \ge 3$. By Lemma 2.2, there exists a hamiltonian $(p-1)$-regular graph R of order $n-1$. Obviously R contains a matching $M = \{x_iy_i \mid i = 1, \dots, p/2\}$ of size $p/2$. Deleting all the edges in M, adding a new vertex v and joining the edges vx_i and vy_i for $i = 1, \dots, p/2$, we obtain a graph H with degree sequence $p, p-1, \dots, p-1$. The graph H has order n, size $\mathrm{ex}(n, K_{1,p}) + 1$ and contains exactly one copy of $K_{1,p}$. □

Lemma 2.4. *If p is an even positive integer, then $\mathrm{ex}(p+2, B_p) = p(p+2)/2$ and $\mathrm{ex}(p+3, B_p) = p(p+4)/2$. If p is an odd positive integer, then $\mathrm{ex}(p+2, B_p) = (p+1)^2/2$ and $\mathrm{ex}(p+3, B_p) = (p+1)(p+3)/2$.*

Proof. The proofs for the four Turán numbers have the same pattern, but these results hold for different reasons. Let G be a graph of order n and size e with vertices $v_1, \dots, v_n$ such that $\deg(v_i) = d_i$, $i = 1, \dots, n$, and $d_1 \ge \cdots \ge d_n$. This notation will be used throughout the proof. We assign different values to the order n and size e in different cases.

(1) $\mathrm{ex}(p+2, B_p)$ *for even p*. Suppose that $n = p+2$ and $e \ge (p(p+2)/2) + 1$. Then $\sum_{i=1}^n d_i = 2e \ge p(p+2) + 2$. If $d_2 \le p$, then

$$\sum_{i=1}^n d_i \le (p+1) + (p+1)p = (p+1)^2 < p(p+2) + 2,$$

which is a contradiction. Hence, $d_2 \ge p+1$, implying that $d_1 = d_2 = p+1$. Then G has p triangles sharing the common edge v_1v_2. Thus, G contains B_p. This shows that $\mathrm{ex}(p+2, B_p) \le p(p+2)/2$.

On the other hand, the graph $G_1 = K_{p+2} - PM$ has order $p+2$ and size $p(p+2)/2$ and does not contain B_p. In fact, every edge of G_1 lies in exactly $p-2$ triangles. To see this, let uv be an edge of G_1 and let ux and vy be the two edges in the perfect matching. Then $uvzu$ is a triangle of G_1 if and only if $z \notin \{u, v, x, y\}$ and consequently there are $(p+2) - 4 = p-2$ choices for z. Thus, G_1 yields $\mathrm{ex}(p+2, B_p) \ge p(p+2)/2$.

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Together with the reverse inequality proved above, we obtain the conclusion $\mathrm{ex}(p+2, B_p) = p(p+2)/2$.

(2) $\mathrm{ex}(p+3, B_p)$ *for even* p. Suppose that $n = p+3$ and $e \geq (p(p+4)/2) + 1$. Then $\sum_{i=1}^{n} d_i = 2e \geq p(p+4) + 2$. We distinguish two cases.

Case 1. $d_1 = p+2$. If $d_2 \leq p$, then

$$\sum_{i=1}^{n} d_i \leq (p+2) + (p+2)p = (p+1)(p+2) < p(p+4) + 2,$$

which is a contradiction. Hence, $d_2 \geq p+1$. Let $v_1, w_1, w_2, \ldots, w_p$ be $p+1$ distinct neighbours of v_2. Then G contains the p triangles $v_1 v_2 w_i v_1$, $i = 1, \ldots, p$, sharing the common edge $v_1 v_2$. Thus, G contains B_p.

Case 2. $d_1 \leq p+1$. If $d_{n-1} \leq p$, then

$$\sum_{i=1}^{n} d_i \leq (n-2)(p+1) + 2p = (p+1)^2 + 2p < p(p+4) + 2,$$

which is a contradiction. Hence, $d_1 = d_2 = \cdots = d_{n-1} = p+1$. If $d_n \leq p-1$, then again $\sum_{i=1}^{n} d_i \leq (p+2)(p+1) + p - 1 < p(p+4) + 2$, which is a contradiction. Thus, $d_n \geq p$. But $d_n \neq p+1$, since otherwise G would be a $(p+1)$-regular graph of odd degree $p+1$ and odd order $p+3$, which is impossible by Lemma 2.2. We must have $d_n = p$. Without loss of generality, suppose that $N(v_n) = \{v_1, \ldots, v_p\}$. It follows that $N(v_{n-2}) = \{v_1, \ldots, v_{n-3}, v_{n-1}\}$ and $N(v_{n-1}) = \{v_1, \ldots, v_{n-2}\}$. Clearly, G has the p triangles $v_{n-1} v_{n-2} v_i v_{n-1}$ for $i = 1, 2, \ldots, p$ sharing the common edge $v_{n-1} v_{n-2}$. Hence, G contains B_p. This shows that $\mathrm{ex}(p+3, B_p) \leq p(p+4)/2$.

The graph $G_2 = \overline{K}_3 \vee (K_p - PM)$ has order $p+3$ and size $p(p+4)/2$. Next we show that G_2 does not contain B_p. Let V_1 be the vertex set of the subgraph of G_2 isomorphic to $\overline{K}_3$ and let V_2 be the vertex set of the subgraph of G_2 isomorphic to $K_p - PM$. Let xy be an edge of G_2. If $x, y \in V_2$, then xy lies in exactly $(p-4) + 3 = p - 1$ triangles; if $x \in V_1$ and $y \in V_2$, then xy lies in exactly $p-2$ triangles. Thus, G_2 yields $\mathrm{ex}(p+3, B_p) \geq p(p+4)/2$, which, combined with the reverse inequality proved above, shows that $\mathrm{ex}(p+3, B_p) = p(p+4)/2$.

(3) $\mathrm{ex}(p+2, B_p)$ *for odd* p. Suppose that $n = p+2$ and $e \geq ((p+1)^2/2) + 1$. Then $\sum_{i=1}^{n} d_i = 2e \geq (p+1)^2 + 2$. If $d_2 \leq p$, then

$$\sum_{i=1}^{n} d_i \leq (p+1) + (p+1)p = (p+1)^2 < (p+1)^2 + 2,$$

which is a contradiction. Hence, $d_1 = d_2 = p+1$. Then G contains p triangles sharing the common edge $v_1 v_2$, which form a B_p. This shows that $\mathrm{ex}(p+2, B_p) \leq (p+1)^2/2$.

The graph $G_3 = K_1 \vee (K_{p+1} - PM)$ has order $p+2$ and size $(p+1)^2/2$. Let f be an edge of G_3. If f is incident with the vertex of K_1, then f lies in exactly $p-1$ triangles; if f is an edge of $K_{p+1} - PM$, then f lies in exactly $p-2$ triangles. Thus, G_3 contains

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no B_p and G_3 shows that $\mathrm{ex}(p+2, B_p) \geq (p+1)^2/2$. This inequality, together with the reverse inequality proved above, shows that $\mathrm{ex}(p+2, B_p) = (p+1)^2/2$.

(4) $\mathrm{ex}(p+3, B_p)$ *for odd* p. Suppose $n = p+3$ and $e \geq ((p+1)(p+3)/2) + 1$. Then $\sum_{i=1}^{n} d_i = 2e \geq (p+1)(p+3) + 2$. If $d_2 \leq p+1$, then

$$\sum_{i=1}^{n} d_i \leq (p+2) + (p+2)(p+1) = (p+2)^2 < (p+1)(p+3) + 2,$$

which is a contradiction. Hence, $d_1 = d_2 = p+2$. Then the edge v_1v_2 lies in $p+1$ triangles, implying that G contains a B_{p+1} and hence a B_p. Thus, it follows that $\mathrm{ex}(p+3, B_p) \leq (p+1)(p+3)/2$.

The graph $G_4 = K_{p+3} - PM$ has order $p+3$ and size $(p+1)(p+3)/2$. Every edge of G_4 lies in exactly $p-1$ triangles. Hence, G_4 contains no B_p and G_4 yields $\mathrm{ex}(p+3, B_p) \geq (p+1)(p+3)/2$. We thus conclude that $\mathrm{ex}(p+3, B_p) = (p+1)(p+3)/2$. □

Proof of Theorem 1.5. By Lemma 2.4, $\mathrm{ex}(p+2, B_p) = p(p+2)/2$. Now let $G_5 = K_2 \vee (K_p - PM)$; that is, G_5 is the complement of the graph $(p/2)K_2 + \overline{K}_2$. Note that G_5 is obtained from the graph G_1 in the proof of Lemma 2.4 by adding one edge: $G_5 = G_1 + f$. Then G_5 has order $p+2$ and size $(p(p+2)/2) + 1$ and f is the unique edge of G_5 which lies in p triangles. Hence, G_5 contains exactly one copy of B_p.

Conversely, let Y be a graph of order $p+2$ and size $(p(p+2)/2) + 1$ which contains exactly one copy of B_p. Let uv be the unique edge of Y that lies in exactly p triangles. Then $\deg(u) = \deg(v) = p+1$. Since Y contains only one copy of B_p, every vertex of Y other than u and v has degree at most p. The degree sum of Y is $p(p+2)+2$, implying that the degree sequence of Y must be $p+1, p+1, p, p, \ldots, p$. It follows that Y is the complement of $(p/2)K_2 + \overline{K}_2$. Hence, $Y = G_5$.

Next we consider the order $p+3$. By Lemma 2.4, $\mathrm{ex}(p+3, B_p) = p(p+4)/2$. Let G_6 be the complement of $(p/2)K_2 + P_3$, where P_3 is the path of order three. Note that G_6 is obtained from the graph G_2 in the proof of Lemma 2.4 by adding one edge to the subgraph $\overline{K}_3$: $G_6 = (K_1 + K_2) \vee (K_p - PM)$. Thus, G_6 has order $p+3$ and size $(p(p+4)/2) + 1$. Let h be the edge of G_6 corresponding to K_2, that is, the edge added to G_2. It is easy to check that h is the unique edge of G_6 that lies in p triangles. Hence, G_6 contains exactly one copy of B_p.

Conversely, let Z be a graph of order $p+3$ and size $(p(p+4)/2) + 1$ which contains exactly one copy of B_p. Let $v_1, \ldots, v_{p+3}$ be the vertices of Z such that v_1 and v_2 are adjacent, $N(v_1) \cap N(v_2) = \{v_3, v_4, \ldots, v_{p+2}\}$ and $v_{p+3} \notin N(v_1) \cap N(v_2)$. Thus, v_1v_2 is the unique edge of Z that lies in p triangles. If $\deg(v_i) = p+2$ for some i with $3 \leq i \leq p+2$, then Z would contain at least three copies of B_p, which is a contradiction. Hence, $\deg(v_i) \leq p+1$ for $3 \leq i \leq p+2$. We also have $\deg(v_{p+3}) \leq p$, since otherwise Z would contain at least two copies of B_p.

We assert that $v_{p+3} \notin N(v_1) \cup N(v_2)$. To the contrary, without loss of generality suppose that v_{p+3} is adjacent to v_1. First consider the case $p = 2$. Then $v_{p+3} = v_5$ cannot

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be adjacent to either of v_3 and v_4. Also, v_3 and v_4 are not adjacent, since otherwise Z would contain at least two copies of B_2. But then Z has size $6 < 7 = (2(2+4)/2)+1$, which is a contradiction. Next assume that $p \geq 4$. If $\deg(v_{p+3}) \leq p-2$, then the degree sum of Z is at most $(p+2)+(p+1)(p+1)+(p-2) = p^2+4p+1 < p(p+4)+2$, which is the degree sum of Z, which is a contradiction. Thus, $\deg(v_{p+3}) \geq p-1$, implying that v_{p+3} has at least $p-2$ neighbours among the vertices $v_3, \ldots, v_{p+2.}$ At least one, say v_j, of these $p-2$ neighbours of v_{p+3} has degree $\geq p+1$, since otherwise the degree sum of Z is at most

$$(p+2)+3(p+1)+(p-2)p+p = p^2+3p+5 < p(p+4)+2,$$

which is a contradiction. But now the edge v_1v_j lies in p triangles, yielding another copy of B_p, which is a contradiction. This proves that $v_{p+3} \notin N(v_1) \cup N(v_2)$. It follows that $\deg(v_1) = \deg(v_2) = p+1$.

Summarising the above analysis, we have obtained $\deg(v_1) = \deg(v_2) = p+1$, $\deg(v_i) \leq p+1$ for $3 \leq i \leq p+2$ and $\deg(v_{p+3}) \leq p$. These restrictions on the degrees, together with the condition that the degree sum of Z is $p(p+4)+2$, imply that the degree sequence of Z must be $p+1, p+1, \ldots, p+1, p$. It remains to show that G_6 is the only graph with such a degree sequence, implying that $Z = G_6$. The complement of such a graph has degree sequence $2, 1, \ldots, 1$. Clearly, $(p/2)K_2 + P_3$ is the only graph with degree sequence $2, 1, \ldots, 1$. This completes the proof. □

Proof of Corollary 1.6. Theorem 1.5 covers the case when k is even and Theorem 1.4 covers the case when k is odd. □

Proof of Theorem 1.7. Let G be a graph of order n and size $\mathrm{ex}(n, B_p)+1$ with vertices $v_1, \ldots, v_n$ such that $\deg(v_i) = d_i$, $i = 1, \ldots, n$, and $d_1 \geq \cdots \geq d_n$.

First we suppose that $n = p+2$. By Lemma 2.4, $\mathrm{ex}(n, B_p) = (p+1)^2/2$. Then the degree sum of G is $\sum_{i=1}^n d_i = (p+1)^2+2$. If $d_3 \leq p$, then $\sum_{i=1}^n d_i \leq 2(p+1) + p \times p < (p+1)^2+2$, which is a contradiction. Thus, we have $d_1 = d_2 = d_3 = p+1$. Each of the three edges v_1v_2, v_2v_3 and v_3v_1 lies in p triangles. Hence, G contains at least three copies of B_p. The number 3 is attained by the graph $K_3 \vee (K_{p-1} - PM)$, which has order $p+2$ and size $((p+1)^2/2)+1$, and contains exactly three copies of B_p.

Next we suppose that $n = p+3$. By Lemma 2.4, $\mathrm{ex}(n, B_p) = (p+1)(p+3)/2$. We have $\sum_{i=1}^n d_i = (p+1)(p+3)+2$. If $d_2 \leq p+1$, then

$$\sum_{i=1}^{n} d_i \leq (p+2)+(p+2)(p+1) = (p+2)^2 < (p+1)(p+3)+2,$$

which is a contradiction. Hence, $d_1 = d_2 = p+2$. We distinguish two cases.

Case 1. $d_3 = p+2$. Then each of the three edges v_1v_2, v_2v_3 and v_3v_1 lies in $p+1$ triangles, implying that G contains three copies of B_{p+1}. Since one copy of B_{p+1} contains $p+1$ copies of B_p, G contains at least $3(p+1)$ copies of B_p.

Case 2. $d_3 \leq p+1$. The degree sum $(p+1)(p+3)+2$ of G requires that $d_i = p+1$ for each $i = 3, 4, \ldots, n$. Now, in G, the edge v_1v_2 lies in $p+1$ triangles, yielding $p+1$

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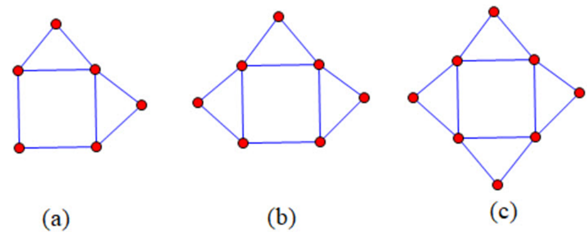

FIGURE 1. Graphs for the orders $n = 6, 7, 8$.

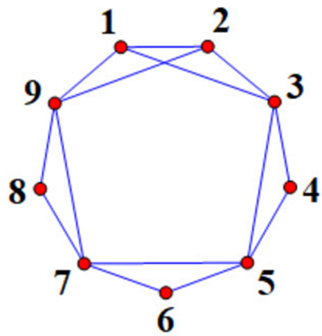

FIGURE 2. The graph for order $n = 9$.

copies of B_p. For $i = 3, \dots, p+3$, each of the two edges v_iv_1 and v_iv_2 lies in p triangles, yielding $2(p+1)$ copies of B_p. Altogether, G contains at least $3(p+1)$ copies of B_p.

The number $3(p+1)$ is attained by the graph $K_2 \vee (K_{p+1} - PM)$, which has order $p+3$ and size $((p+1)(p+3)/2) + 1$ and contains exactly $3(p+1)$ copies of B_p. This completes the proof. □

PROOF OF THEOREM 1.8. We omit some details of the proof of Theorem 1.8, but give several key facts. Clapham *et al.* [3, page 36] determined the values $\mathrm{ex}(n, C_4)$ for all n up to 21. The eight values we need are $\mathrm{ex}(6, C_4) = 7$, $\mathrm{ex}(7, C_4) = 9$, $\mathrm{ex}(8, C_4) = 11$, $\mathrm{ex}(9, C_4) = 13$, $\mathrm{ex}(10, C_4) = 16$, $\mathrm{ex}(11, C_4) = 18$, $\mathrm{ex}(12, C_4) = 21$ and $\mathrm{ex}(13, C_4) = 24$.

The graph in Figure 1(a) has order six and size $8 = \mathrm{ex}(6, C_4) + 1$ and contains exactly one copy of C_4; the graph in Figure 1(b) has order seven and size $10 = \mathrm{ex}(7, C_4) + 1$ and contains exactly one copy of C_4; the graph in Figure 1(c) has order eight and size $12 = \mathrm{ex}(8, C_4) + 1$ and contains exactly one copy of C_4.

The graph in Figure 2 has order nine and size $14 = \mathrm{ex}(9, C_4) + 1$ and contains exactly one copy of C_4, the cycle $1, 3, 2, 9, 1$.

The graph in Figure 3 has order 10 and size $17 = \mathrm{ex}(10, C_4) + 1$ and contains exactly one copy of C_4, the cycle $1, 2, 6, 10, 1$.

The graph in Figure 4 has order 11 and size $19 = \mathrm{ex}(11, C_4) + 1$ and contains exactly one copy of C_4, the cycle $1, 7, 9, 11, 1$.

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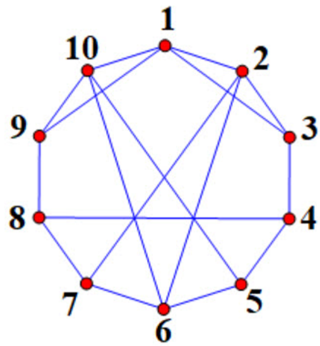

FIGURE 3. The graph for order $n = 10$.

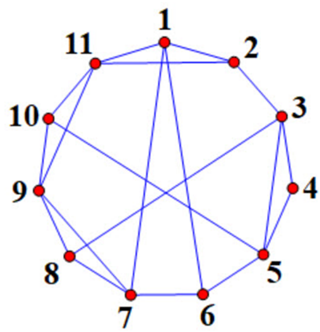

FIGURE 4. The graph for order $n = 11$.

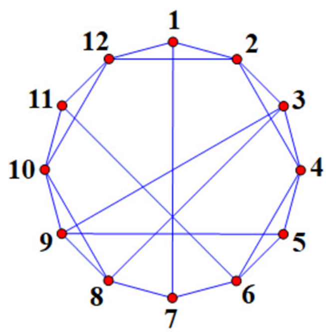

FIGURE 5. A graph of order 12 and size 22 containing two copies of C_4.

Next we treat the case of order $n = 12$. Let G be a graph of order 12 and size $22 = \mathrm{ex}(12, C_4) + 1$. It can be shown, say, using case by case analysis, that G contains at least two copies of C_4. The graph in Figure 5 has order 12 and size 22 and contains exactly two 4-cycles: $3, 4, 5, 9, 3$ and $3, 8, 10, 9, 3$. Thus, the minimum number of copies of C_4 in a graph of order 12 and size 22 is two.

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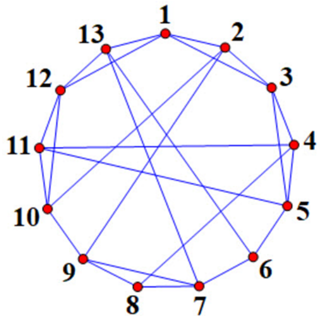

FIGURE 6. A graph of order 13 and size 25 containing two copies of C_4.

Finally, we consider the case of order $n = 13$. A graph of order 13 and size $25 = \mathrm{ex}(13, C_4) + 1$ has average degree $50/13 < 4$. Hence, it has a vertex of degree ≤ 3. Deleting that vertex, we obtain a graph of order 12 and size at least 22 which contains at least two 4-cycles by the result proved for the order $n = 12$. On the other hand, the graph in Figure 6 has order 13 and size 25 and contains exactly two 4-cycles: $1, 2, 10, 12, 1$ and $3, 5, 11, 4, 3$. Thus, the minimum number of copies of C_4 in a graph of order 13 and size 25 is two. This completes the proof. □

3. Concluding remarks

We believe that Erdős' intuition which is expressed in Problem 1.2 is almost true; that is, for most graphs H and most positive integers n, every graph of order n and size $\mathrm{ex}(n, H) + 1$ contains at least two copies of H. We pose the following problem.

PROBLEM 3.1. Determine all the pairs (H, n), where H is a graph and n is a positive integer, such that there exists a graph of order n and size $\mathrm{ex}(n, H) + 1$ which contains exactly one copy of H.

It is natural to ask whether Theorem 1.5 on the books B_p can be extended to orders larger than $p + 3$. The answer is no in general. A computer search gives the following information: (1) $\mathrm{ex}(6, B_2) = 9$ and the minimum number of copies of B_2 in a graph of order six and size 10 is two; (2) $\mathrm{ex}(7, B_2) = 12$ and the minimum number of copies of B_2 in a graph of order seven and size 13 is three; (3) $\mathrm{ex}(8, B_4) = 21$ and the minimum number of copies of B_4 in a graph of order eight and size 22 is six; (4) $\mathrm{ex}(9, B_4) = 27$ and the minimum number of copies of B_4 in a graph of order nine and size 28 is 21.

Finally, we pose the following problem.

PROBLEM 3.2. Given positive integers p and n with $n \geq p + 2$, determine the Turán number $\mathrm{ex}(n, B_p)$.

Lemma 2.4 solves the cases $n = p + 2$, $p + 3$ of Problem 3.2. Yan and Zhan [13] determined the Turán numbers $\mathrm{ex}(p + 4, B_p)$, $\mathrm{ex}(p + 5, B_p)$ and $\mathrm{ex}(p + 6, B_p)$

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and characterised the corresponding extremal graphs for the numbers $\mathrm{ex}(n, B_p)$ with $n = p+2,\ p+3,\ p+4,\ p+5$.

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References

[1] B. Bollobás, *Extremal Graph Theory* (Academic Press, London–New York, 1978).
[2] G. Chartrand, L. Lesniak and P. Zhang, *Graphs and Digraphs*, 6th edn (CRC Press, Boca Raton, FL, 2016).
[3] C. R. J. Clapham, A. Flockhart and J. Sheehan, 'Graphs without four-cycles', *J. Graph Theory* **13**(1) (1989), 29–47.
[4] P. Erdős, 'Some theorems on graphs', *Riveon Lematematika* **9** (1955), 13–17.
[5] P. Erdős, 'Some of my favourite unsolved problems', in: *A Tribute to Paul Erdős* (eds. A. Baker, B. Bollabás and A. Hajnal) (Cambridge University Press, Cambridge, 1990).
[6] F. A. Firke, P. M. Kosek, E. D. Nash and J. Williford, 'Extremal graphs without 4-cycles', *J. Combin. Theory Ser. B* **103**(3) (2013), 327–336.
[7] Z. Füredi, 'Graphs without quadrilaterals', *J. Combin. Theory Ser. B* **34**(2) (1983), 187–190.
[8] Z. Füredi, A. Naor and J. Verstraëte, 'On the Turán number for the hexagon', *Adv. Math.* **203**(6) (2006), 476–496.
[9] J. He, J. Ma and T. Yang, 'Stability and supersaturation of 4-cycles', Preprint, 2021, arXiv:1912.00986v3.
[10] W. Mantel, 'Problem 28', *Wisk. Opgaven* **10** (1907), 60–61.
[11] J. W. Moon, 'On the number of complete subgraphs of a graph', *Canad. Math. Bull.* **8** (1965), 831–834.
[12] P. Turán, 'Eine Extremalaufgabe aus der Graphentheorie', *Mat. Fiz. Lapok* **48** (1941), 436–452.
[13] J. Yan and X. Zhan, 'The Turán number of book graphs', Preprint, 2021, arXiv:2010.09973v1.

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