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Two-degree trees

QIAO Pu[1], ZHAN Xingzhi[2]

(1. *Department of Mathematics*, *East China University of Science and Technology*, *Shanghai* 200237, *China*;
2. *School of Mathematical Sciences*, *East China Normal University*, *Shanghai* 200241, *China*)

Abstract: A graph is called a two-degree graph if its vertices have only two distinct degrees. A two-degree tree of order at least three have two degrees 1 and d for some $d \geqslant 2$; such a tree is called a $(1, d)$-tree. Given a positive integer n, we determine (1) the possible values of d such that there exists a $(1, d)$-tree of order n; (2) the values of d such that there exists a unique $(1, d)$-tree of order n and (3) the maximum diameter of two-degree trees of order n. The results provide a new example showing that sometimes the behavior of graphs is determined by number theoretic properties.

Keywords: two-degree tree; diameter; unique graph

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具有两个度数的树

乔 璞[1], 詹兴致[2]

（1. 华东理工大学 数学系, 上海 200237; 2. 华东师范大学 数学科学学院, 上海 200241）

摘要: 如果一个图只有两个不同的度数, 这个图就称为二度图. 阶数至少为 3 的二度树具有度数 1 和 d, 这里 d 是至少为 2 的整数; 这样的树称为 (1,d)-树. 给定一个正整数 n, 下面的信息被确定了: (1) 存在一个 n 阶 (1,d)-树的可能的 d 的值; (2) 存在唯一的 n 阶 (1,d)-树的可能的 d 的值; (3) n 阶 (1,d)-树的最大可能的直径. 这些结果提供了一个新的例子, 表明有时候图的行为是由数论性质决定的.

关键词: 二度树; 直径; 唯一图

The *order* of a graph is its number of vertices, and the *size* is its number of edges. A *tree* is a connected graph that contains no cycle. Trees are the simplest connected graphs in the sense that they have the least possible size among connected graphs of a given order. A graph is called a *two-degree graph* if its vertices have only two distinct degrees. A *leaf* of a graph is a vertex of degree 1. Every nontrivial tree has at least two leaves [1, p.100]. Thus a two-degree tree has two degrees 1 and d for some $d \geqslant 2$; such a tree is called a $(1, d)$-*tree.* $(1, 3)$-trees are called cubic trees in [2, p.283]. One might think that we may omit “1” and simply call a $(1, d)$-tree a d-tree. This cannot be done, since the term d-tree already has another well accepted meaning (see e.g. [3]).

For a given positive integer n, what are the possible values of d such that there exists a $(1, d)$-tree of order n? Let us consider two examples. There exists a $(1, d)$-tree of order 20 if and only if

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第一作者: 乔 璞, 女, 讲师, 博士, 研究方向为图论. E-mail: pq@ecust.edu.cn

通信作者: 詹兴致, 男, 教授, 研究方向为图论、矩阵论. E-mail: zhan@math.ecnu.edu.cn

$d=2,3,4,7,10,$ or $19,$ while there exists a $(1,d)$-tree of order 39 if and only if $d=2,$ or $38.$ Thus there are six possible values of d for the order $20,$ but there are only two possible values of d for the larger order $39.$ Why?

In this note, for a given positive integer n we determine (1) the possible values of d such that there exists a $(1,d)$-tree of order n; (2) the values of d such that there exists a unique $(1,d)$-tree of order n and (3) the maximum diameter of two-degree trees of order $n.$ These results provide a new example showing that sometimes the behavior of graphs is determined by number theoretic properties.

A *caterpillar* is a tree in which a single path (the *spine*) is incident to or contains every edge; in other words, removal of its leaves yields a path.

Notation. Let d and n be positive integers such that $d-1$ divides $n-2.$ We use $CP(n,d)$ to denote the unique caterpillar of order n which is a $(1,d)$-tree.

The caterpillar $CP(18,5)$ is depicted in Figure 1.

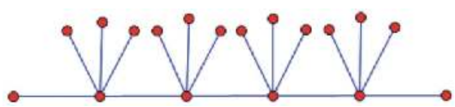

Fig. 1 The caterpillar $CP(18, 5)$

Theorem 1. *Let $n\geqslant 3$ be a positive integer. Then there exists a $(1,d)$-tree of order n if and only if $d-1$ divides $n-2$.*

Proof. Suppose there exists a $(1,d)$-tree of order $n.$ The degree sum formula [1, p.7] proved by Leonhard Euler in 1 736 states that if a graph has vertex set V and size e then

$$\sum_{x\in V}\deg(x)=2e,$$

where $\deg(x)$ denotes the degree of the vertex $x.$ Now suppose in a $(1,d)$-tree of order $n,$ there are k vertices of degree d and hence there are $n-k$ vertices of degree 1. Since a tree of order n has size $n-1,$ by the degree sum formula we have $(n-k)+kd=2(n-1)$; i.e., $(d-1)k=n-2.$ Hence $d-1$ divides $n-2.$

Conversely, suppose $d-1$ divides $n-2.$ Then the caterpillar $CP(n,d)$ is a $(1,d)$-tree of order $n.$

We denote by P_n and S_n the path and star of order n respectively.

Corollary 2. *The path P_n and the star S_n are the only two-degree trees of order $n\geqslant 4$ if and only if $n-2$ is a prime number.*

Proof. If $n-2$ is a prime number, 1 and $n-2$ are the only divisors of $n-2.$ By Theorem 1, there exists a $(1,d)$-tree of order n if and only if $d-1=1$ or $d-1=n-2$; i.e., $d=2$ or $d=n-1.$ The path P_n is the unique $(1,2)$-tree of order n and the star S_n is the unique $(1,n-1)$-tree of order $n.$

If $n-2$ is a composite number, let q be a divisor of $n-2$ with $2\leqslant q\leqslant n-3.$ By Theorem 1, there exists a $(1,q+1)$-tree of order $n,$ which is neither the path P_n or the star S_n since $3\leqslant q+1\leqslant n-2$.

Theorem 3. *There exists a unique $(1,d)$-tree of order $n\geqslant 3$ if and only if $d-1$ divides $n-2$ and furthermore $d=2$ or $3d\geqslant n+1$.*

Proof. Suppose there exists a $(1,d)$-tree T of order n. By Theorem 1 and its proof, $d-1$ divides $n-2$ and $(n-2)/(d-1)$ is the number of vertices of degree d. First note that the only $(1,2)$-tree is the path P_n. Next assume $d \geqslant 3$.

Suppose $3d \geqslant n+1$. Then $(n-2)/(d-1) \leqslant 3$. There are three possibilities. If $(n-2)/(d-1) = 1$, i.e., $d = n-1$, then the star S_n is the unique $(1,n-1)$-tree of order n. If $(n-2)/(d-1) = 2$, i.e., $d = n/2$, then the caterpillar $CP(n,n/2)$ (a double-broom) is the unique $(1,n/2)$-tree of order n. If $(n-2)/(d-1) = 3$, let x,y,z be the three vertices of degree d which equals $(n+1)/3$. If any two of the vertices x,y and z do not have a common neighbor, then the tree T would have order at least $3d = n+1$, a contradiction. Without loss of generality, suppose x and y have a common neighbor, which must be z. Now x and y are nonadjacent since T contains no cycles. It follows that $T = CP(n,d)$ and the uniqueness is proved.

Now suppose $3d \leqslant n$. We will show that there exist at least two non-isomorphic $(1,d)$-trees of order n. First, the caterpillar $CP(n,d)$ is such a tree. The condition that $d-1$ divides $n-2$ implies $d-1$ divides $(n-d+1)-2$. Hence by Theorem 1, the caterpillar $CP(n-d+1,d)$, which we denote by G, exists. Let $v_1, v_2, \cdots, v_k$ be the vertices of degree d on the spine of G in order where $k = ((n-d+1)-2)/(d-1)$. Since k is an integer, the assumption that $3d \leqslant n$ implies $k \geqslant 3$. Let $H(n,d)$ be the graph obtained from G by attaching $d-1$ edges to one of the leaf neighbors of v_2. The graphs $CP(17,4)$ and $H(17,4)$ are depicted in Figure 2.

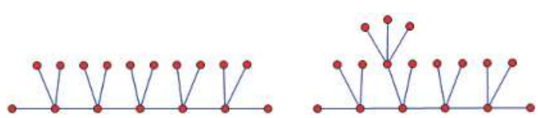

Fig. 2 CP(17, 4) and H(17, 4)

Both $CP(n,d)$ and $H(n,d)$ are $(1,d)$-trees of order n, but they are non-isomorphic, since the former has diameter $(n-2)/(d-1)+1$ while the latter has diameter $(n-2)/(d-1)$. The proof is complete.

Finally we consider the maximum possible diameter of non-path two-degree trees of a given order.

Theorem 4. *Let $n \geqslant 4$ be a positive integer and suppose p is the smallest prime divisor of $n-2$. Then the maximum possible diameter of a non-path two-degree tree of order n is $(n-2)/p+1$.*

Proof. First note that a $(1,d)$-tree of order at least three is a path if and only if $d=2$. Let T be a $(1,d)$-tree of order n with $d \geqslant 3$. Suppose T has diameter k and let $P: x_0, x_1, \cdots, x_k$ be a diametral path of T; i.e, the distance between x_0 and x_k is k. Since each internal vertex x_i of P with $1 \leqslant i \leqslant k-1$ has degree d and any two of them do not have a common neighbor, we have $n \geqslant (k+1)+(k-1)(d-2)$. Hence $k \leqslant (n-2)/(d-1)+1$. On the other hand, the caterpillar $CP(n,d)$ has the diameter $(n-2)/(d-1)+1$. This proves that the maximum diameter of a $(1,d)$-tree of order n is $(n-2)/(d-1)+1$. It follows that the maximum possible diameter of a non-path two-degree tree of order n is

$$\max\left\{\frac{n-2}{d-1}+1 \,\middle|\, d \geqslant 3,\ d-1 \text{ divides } n-2\right\} = \frac{n-2}{p}+1,$$

where p is the smallest prime divisor of $n-2$.

The following corollary is an immediate consequence of Theorem 4.

Corollary 5. *If* $n \geqslant 4$ *is an even positive integer, then the maximum possible diameter of a non-path two-degree tree of order* n *is* $n/2$.

[References]

[1] BONDY J A, MURTY U S R. Graph Theory[M]. GTM 244, New York: Springer, 2008.

[2] BLASS A, HARARY F, MILLER Z. Which trees are link graphs? [J]. J Combin. Theory (Ser B), 1980, 29: 277-292.

[3] BROERSMA H, XIONG L, YOSHIMOTO K. Toughness and hamiltonicity in *K*-trees [J]. Discrete Math, 2007, 307: 832-838.

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