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Pairs of a tree and a nontree graph with the same status sequence

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ABSTRACT

The status of a vertex x in a graph is the sum of the distances between x and all other vertices. Let G be a connected graph. The status sequence of G is the list of the statuses of all vertices arranged in nondecreasing order. G is called status injective if all the statuses of its vertices are distinct. Let G be a member of a family of graphs $\mathscr{F}$ and let the status sequence of G be s. G is said to be status unique in $\mathscr{F}$ if G is the unique graph in $\mathscr{F}$ whose status sequence is s. In 2011, J.L. Shang and C. Lin posed the following two conjectures. Conjecture 1: A tree and a nontree graph cannot have the same status sequence. Conjecture 2: Any status injective tree is status unique in all connected graphs. We settle these two conjectures negatively. For every integer $n \geq 10$, we construct a tree T_n and a unicyclic graph U_n, both of order n, with the following two properties: (1) T_n and U_n have the same status sequence; (2) for $n \geq 15$, if n is congruent to 3 modulo 4 then T_n is status injective and among any four consecutive even orders, there is at least one order n such that T_n is status injective.

1. Introduction

We consider finite simple graphs. The *order* of a graph is the number of its vertices. A connected graph is said to be *unicyclic* if it has exactly one cycle. We denote by $V(G)$ and $E(G)$ the vertex set and edge set of a graph G respectively. The distance between two vertices x and y in a graph is denoted by $d(x, y)$. The *status* of a vertex x in a graph G, denoted by $s(x)$, is the sum of the distances between x and all other vertices; i.e.,

$$s(x) = \sum_{y \in V(G)} d(x, y).$$

The *status sequence* of G is the list of the statuses of all vertices of G arranged in nondecreasing order. G is called *status injective* if all the statuses of its vertices are distinct [2, p.185]. Harary [4] investigated the digraph version of the concept of status in a sociometric framework, while Entringer, Jackson and Snyder [3] studied basic properties of this concept for graphs.

A natural question is: Which graphs are determined by their status sequences? Slater [7] constructed infinitely many pairs of non-isomorphic trees with the same status sequence. Shang [5] gave a method for constructing general non-isomorphic graphs with the same status sequence. Let G be a member of a family of graphs $\mathscr{F}$ and let the status sequence of G be s. G is said to be *status unique* in $\mathscr{F}$ if G is the unique graph in $\mathscr{F}$ whose status sequence is s. Here we view two

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isomorphic graphs as the same graph. It is known that [6] spiders are status unique in trees and that [1] status injective trees are status unique in trees.

Shang and Lin [6, p.791] posed the following two conjectures in 2011.

Conjecture 1. *A tree and a nontree graph cannot have the same status sequence.*

Conjecture 2. *Any status injective tree is status unique in all connected graphs.*

In this paper we settle these two conjectures negatively. For every integer $n \geq 10$, we construct a tree T_n and a unicyclic graph U_n, both of order n, with the same status sequence. There are infinitely many odd orders n and infinitely many even orders n such that T_n is status injective.

2. Main results

We will need the following lemmas. For a set S, the notation $|S|$ denotes the cardinality of S.

Lemma 1 ([3, p. 284]). *Suppose x and y are adjacent vertices of a connected graph. Let A be the set of vertices closer to x than y, and B the set of vertices closer to y than x. Then $s(y) = s(x) + |A| - |B|$.*

Lemma 2. *Let $x_0x_1x_2 \ldots x_k$ be a path in a tree and denote $d = s(x_1) - s(x_0)$. Then $s(x_{j+1}) - s(x_j) \geq d + 2j$ for each $j = 1, 2, \ldots, k-1$. Consequently if $s(x_0) \leq s(x_1)$ then $s(x_{j+1}) - s(x_j) \geq 2j$ for each $j = 1, 2, \ldots, k-1$ and in particular, $s(x_1) < s(x_2) < s(x_3) < \cdots < s(x_k)$.*

Proof. It suffices to prove the first assertion. We first show the following

Claim. If xyz is a path in a tree and denote $c = s(y) - s(x)$, then $s(z) - s(y) \geq c + 2$.

Let T be the tree of order n. Let A and B be the two components of $T - xy$ with $x \in V(A)$ and $y \in V(B)$, and let G and H be the two components of $T - yz$ with $y \in V(G)$ and $z \in V(H)$. By Lemma 1, $s(y) - s(x) = |V(A)| - |V(B)| = c$. We also have $|V(A)| + |V(B)| = n$ since every edge in a tree is a cut-edge. Hence $2|V(A)| = c + n$. Since $V(A) \subset V(G)$ and $y \in V(G)$ but $y \notin V(A)$, we have $|V(G)| \geq |V(A)| + 1$. By Lemma 1 and the relation $|V(G)| + |V(H)| = n$ we deduce

$$s(z) - s(y) = |V(G)| - |V(H)| = 2|V(G)| - n \geq 2|V(A)| + 2 - n = c + 2.$$

This proves the claim.

Applying the claim successively to the path $x_{i-1}x_ix_{i+1}$ for $i = 1, 2, \ldots, k-1$ we obtain the first assertion in Lemma 2. □

Lemma 2 is a generalization and strengthening of a result in [3, p.291], which states that if $x_0x_1 \ldots x_k$ is a path in a tree and x_0 has the minimum status of all vertices, then $s(x_1) < s(x_2) < \cdots < s(x_k)$.

Lemma 3. *The quadratic polynomial equation*

$$p^2 + 5p + 4 = q^2 + q - 6$$

in p and q has no nonnegative integer solution.

Proof. Suppose that p and q are nonnegative integers. If $q \leq p + 2$, then $q^2 + q - 6 \leq (p + 2)^2 + (p + 2) - 6 = p^2 + 5p < p^2 + 5p + 4$. If $q \geq p + 3$, then $q^2 + q - 6 \geq (p + 3)^2 + (p + 3) - 6 = p^2 + 7p + 6 > p^2 + 5p + 4$. Hence the equation cannot have any nonnegative integer solution. □

Remark. It is not hard to prove that the only integer solutions of the equation in Lemma 3 are $(p, q) = (-4, -3)$, $(-4, 2)$, $(-1, -3)$, $(-1, 2)$.

Denote by $\mathbb{N}$ the set of positive integers.

Lemma 4. *Let the two functions $f(p) = p^2+5p+4$ and $h(q) = q^2+q-6$ be defined on the set $\mathbb{N}$. If $p \geq 7$ and $|f(p) - h(q)| \leq 15$, then $q = p + 2$ and $f(p) - h(q) = 4$.*

Proof. If $q \geq p + 3$, then

$$h(q) \geq h(p + 3) = f(p) + 2p + 2 \geq f(p) + 16.$$

If $p - 2 \leq q \leq p + 1$, then

$$f(p) \geq f(q - 1) = h(q) + 2q + 6 \geq h(q) + 2p + 2 \geq h(q) + 16.$$

If $q \leq p - 3$, then

$$f(p) \geq f(q + 3) = h(q) + 10q + 34 \geq h(q) + 44.$$

Hence we must have $q = p + 2$ and in this case, $f(p) - h(q) = 4$. □

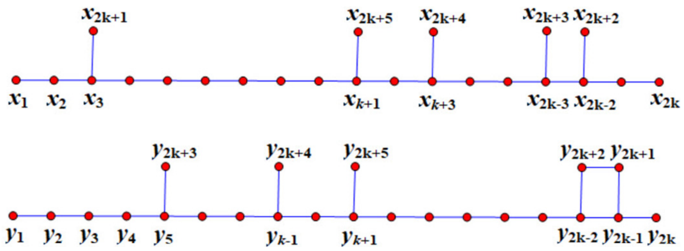

Fig. 1. T_n and U_n with $n = 2k+5$ and $k \geq 7$.

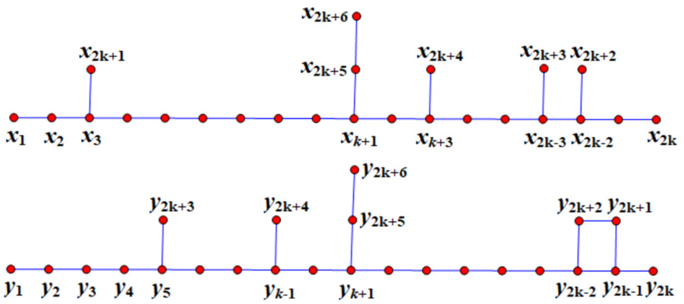

Fig. 2. T_n and U_n with $n = 2k+6$ and $k \geq 7$.

Now we are ready to state and prove the main result.

Theorem 5. *For every integer $n \geq 10$, there exist a tree T_n and a unicyclic graph U_n, both of order n, with the following two properties:*

(1) T_n and U_n have the same status sequence;

(2) for $n \geq 15$, if $n \equiv 3 \pmod 4$ then T_n is status injective and among any four consecutive even orders, there is at least one order n such that T_n is status injective.

Proof. For the orders $n \geq 19$ we have a uniform construction of T_n and U_n, and we treat this case first. For the orders $10 \leq n \leq 18$, the graphs will be constructed individually and they appear at the end of this proof.

Now suppose $n \geq 19$. We distinguish the odd orders and the even orders. Let $n = 2k+5$ with $k \geq 7$. We define T_n and U_n as follows. $V(T_n) = \{x_i \mid i = 1, 2, \ldots, 2k+5\}$ and $E(T_n) =$

$$\{x_i x_{i+1} \mid i = 1, 2, \ldots, 2k-1\} \cup \{x_3x_{2k+1}, x_{k+1}x_{2k+5}, x_{k+3}x_{2k+4}, x_{2k-3}x_{2k+3}, x_{2k-2}x_{2k+2}\}.$$

$V(U_n) = \{y_i \mid i = 1, 2, \ldots, 2k+5\}$ and $E(U_n) = \{y_i y_{i+1} \mid i = 1, 2, \ldots, 2k-1\} \cup \{y_5y_{2k+3}, y_{k-1}y_{2k+4}, y_{k+1}y_{2k+5}, y_{2k-2}y_{2k+2}, y_{2k-1}y_{2k+1}, y_{2k+1}y_{2k+2}\}$. Note that T_n is a caterpillar of maximum degree 3 and U_n is a unicyclic graph. T_n and U_n are illustrated in Fig. 1.

It can be checked directly that $s(x_i) = s(y_i)$ for $i = 1, 2, 3, k+1, 2k-1, 2k, \ldots, 2k+5$ and $s(x_i) = s(y_{2k+2-i})$ for $4 \leq i \leq 2k-2$. Hence, T_n and U_n have the same status sequence. For the even orders $n = 2k+6$ with $k \geq 7$, T_n is obtained from T_{n-1} defined above by adding the edge $x_{2k+5}x_{2k+6}$, and U_n is obtained from U_{n-1} defined above by adding the edge $y_{2k+5}y_{2k+6}$. T_n and U_n are illustrated in Fig. 2.

We check easily that $s(x_i) = s(y_i)$ for $i = 1, 2, 3, k+1, 2k-1, 2k, \ldots, 2k+6$ and $s(x_i) = s(y_{2k+2-i})$ for $4 \leq i \leq 2k-2$. Thus T_n and U_n also have the same status sequence.

Next we prove that the trees T_n satisfy condition (2) in Theorem 5. In fact, we will determine precisely for which orders n, T_n is status injective.

First consider the case when n is odd and let $n = 2k+5$ with $k \geq 7$. Denote $a = s(x_{k+1}) = k^2 + 3k - 2$. We have

$$s(x_{k-p}) = \begin{cases} a + (p+2)^2 - 1 & \text{if } 0 \leq p \leq k-3, \\ a + k^2 + 1 & \text{if } p = k-2, \\ a + (k+1)^2 + 3 & \text{if } p = k-1; \end{cases}$$

$$s(x_{k+q}) = \begin{cases} a + 1 & \text{if } q = 2, \\ a + q^2 - 5 & \text{if } 3 \leq q \leq k-3, \\ a + (q+2)^2 - 4k + 1 & \text{if } k-2 \leq q \leq k; \end{cases}$$

$$s(x_{2k+r}) = \begin{cases} a+(k+1-r)^2+3 & \text{if } \ 1 \le r \le 3, \\ a+2k+7 & \text{if } \ r=4, \\ a+2k+3 & \text{if } \ r=5. \end{cases}$$

In calculating the values $s(x_i)$ for $1 \le i \le 2k$ we have used the fact that if $P = z_1 z_2 \dots z_m$ is a path, then

$$s(z_i) = i(i-m-1) + m(m+1)/2$$

in P, while in calculating the values $s(x_j)$ for $j = 2k+1, \dots, 2k+5$ we have used Lemma 1. From the above expressions it follows that x_{k+1} is the unique vertex with the minimum status, $x_1, x_2, x_3, x_{2k-1}, x_{2k}, x_{2k+1}, x_{2k+2}, x_{2k+3}$ are the vertices with the eight largest statuses, since

$$s(x_1) > s(x_{2k}) > s(x_{2k+1}) > s(x_2) > s(x_{2k+2}) > s(x_{2k-1}) > s(x_3) > s(x_{2k+3}) > s(x_i) \tag{1}$$

for any $i \ne 1, 2, 3, 2k-1, 2k, 2k+1, 2k+2, 2k+3$ and

$$s(x_{2k+1}) > s(x_{2k+2}) > s(x_{2k+3}) > s(x_{2k+4}) > s(x_{2k+5}). \tag{2}$$

Partition the vertex set of T_n into three sets:

$$L = \{x_i | \ 1 \le i \le k\}, \quad R = \{x_i | \ k+1 \le i \le 2k\} \ \text{ and } \ W = \{x_i | \ 2k+1 \le i \le 2k+5\}.$$

The inequalities in (2) show that any two distinct vertices in W have different statuses. Applying Lemma 2 to the two paths $x_{k+1}x_k x_{k-1} \dots x_2 x_1$ and $x_{k+1}x_{k+2} \dots x_{2k-1}x_{2k}$ we see that any two distinct vertices in L or in R have different statuses. Next we show that for any $x \in L$ and $y \in R$, $s(x) \ne s(y)$. By the inequalities in (1) it suffices to prove that $s(x_i) \ne s(x_j)$ for $4 \le i \le k$ and $k+2 \le j \le 2k-2$, which is equivalent to $s(x_{k-p}) \ne s(x_{k+q})$ for $0 \le p \le k-4$ and $2 \le q \le k-2$. We have the expressions $s(x_{k-p}) = a + (p+2)^2 - 1$ for $0 \le p \le k-4$, $s(x_{k+2}) = a+1$, $s(x_{2k-2}) = a + k^2 - 4k + 1$ and $s(x_{k+q}) = a + q^2 - 5$ for $3 \le q \le k-3$. First, $s(x_{k-p}) \ge s(x_k) = a+3 > a+1 = s(x_{k+2})$. The equality $s(x_{k-p}) = s(x_{k+q})$ for $3 \le q \le k-3$ is equivalent to $4 = (q+p+2)(q-p-2)$, which is impossible, since $q+p+2 \ge 5$ and $q-p-2$ is an integer. Also, $s(x_{k-p}) = s(x_{2k-2})$ is equivalent to $2 = (k+p)(k-p-4)$, which is impossible, since $k+p \ge 7$ and $k-p-4$ is an integer. Hence $s(x) \ne s(y)$ for $x \in L$ and $y \in R$.

By the above analysis, it is clear that the only possibilities for two distinct vertices to have the same status are $s(x_{2k+5}) = s(x_i)$ and $s(x_{2k+4}) = s(x_i)$ for $4 \le i \le k$ or $k+2 \le i \le 2k-2$. By the expressions for their status values, it is easy to verify that $s(x_{2k+5}) = s(x_i)$ for some i with $4 \le i \le k$ if and only if $k = 2c^2 - 2$ for some integer c; $s(x_{k+2}) < s(x_{2k+5}) < s(x_{2k-2})$ and $s(x_{2k+5}) = s(x_i)$ for some i with $k+3 \le i \le 2k-3$ if and only if $k = 2c^2-4$ for some integer c; $s(x_{2k+4}) = s(x_i)$ for some i with $4 \le i \le k$ if and only if $k = 2c^2 - 4$ for some integer c; $s(x_{k+2}) < s(x_{2k+4}) < s(x_{2k-2})$ and $s(x_{2k+4}) = s(x_i)$ for some i with $k+3 \le i \le 2k-3$ if and only if $k = 2c^2 - 6$ for some integer c.

Thus, T_n with $n = 2k+5$ is not status injective if and only if $k = 2c^2 - 2$, $2c^2 - 4$ or $2c^2 - 6$ for some integer c. Since all these values of k are even, it follows that for every odd k, T_n is status injective; i.e., if $n \equiv 3 \pmod 4$ then T_n is status injective.

Next we treat the case when the order n is even. Let $n = 2k+6$ with $k \ge 7$. With $d = s(x_{k+1}) = k^2 + 3k$ we have

$$s(x_{k-p}) = \begin{cases} d+p^2+5p+4 & \text{if } \ 0 \le p \le k-3, \\ d+k^2+k & \text{if } \ p=k-2, \\ d+k^2+3k+4 & \text{if } \ p=k-1; \end{cases}$$

$$s(x_{k+q}) = \begin{cases} d+2 & \text{if } \ q=2, \\ d+q^2+q-6 & \text{if } \ 3 \le q \le k-3, \\ d+q^2+5q-4k+4 & \text{if } \ k-2 \le q \le k; \end{cases}$$

$$s(x_{2k+r}) = \begin{cases} d+k^2+k+2 & \text{if } \ r=1, \\ d+k^2-k+2 & \text{if } \ r=2, \\ d+k^2-3k+4 & \text{if } \ r=3, \\ d+2k+10 & \text{if } \ r=4, \\ d+2k+2 & \text{if } \ r=5, \\ d+4k+6 & \text{if } \ r=6. \end{cases}$$

From the above expressions we deduce that x_{k+1} is the unique vertex with the minimum status d. The case $k=7$ corresponds to $n=20$ and we check directly that T_{20} is status injective. Next suppose $k \ge 8$. Then $x_1, x_2, x_3, x_{2k-1}, x_{2k}, x_{2k+1}, x_{2k+2}, x_{2k+3}$ are the vertices with the eight largest statuses, since

$$s(x_1) > s(x_{2k}) > s(x_{2k+1}) > s(x_2) > s(x_{2k+2}) > s(x_{2k-1}) > s(x_3) > s(x_{2k+3}) > s(x_i) \tag{3}$$

for any $i \ne 1, 2, 3, 2k-1, 2k, 2k+1, 2k+2, 2k+3$. Also

$$s(x_{2k+1}) > s(x_{2k+2}) > s(x_{2k+3}) > s(x_{2k+6}) > s(x_{2k+4}) > s(x_{2k+5}). \tag{4}$$

In considering two vertices with equal status, we can exclude the eight vertices with the eight largest statuses by (3) and the unique vertex x_{k+1} with the minimum status. Denote

$$L' = \{x_i |\, 4 \le i \le k\}, \quad R' = \{x_i |\, k+2 \le i \le 2k-2\} \ \text{ and } \ W' = \{x_i |\, 2k+1 \le i \le 2k+6\}.$$

Let x and y be two distinct vertices with $s(x) = s(y)$. By the inequalities in (4), it is impossible that $x, y \in W'$. By Lemma 2 we cannot have $x, y \in L'$ or $x, y \in R'$. Suppose $x \in L'$ and $y \in R'$. We have $s(x) > s(x_{k+2}), s(x_4) > s(x_{2k-2})$ and $s(x_i) < s(x_{2k-2})$ for $5 \le i \le k$. Thus, $y \ne x_{k+2},\ x_{2k-2}$. We have $x = x_i$ for some i with $4 \le i \le k$ and $y = x_j$ for some j with $k+3 \le j \le 2k-3$. Hence $s(x) = d + p^2 + 5p + 4$ with $0 \le p \le k-4$ and $s(y) = d + q^2 + q - 6$ with $3 \le q \le k-3$. Then $s(x) = s(y)$ yields $p^2 + 5p + 4 = q^2 + q - 6$, which is impossible by Lemma 3.

Now, by (3) and the above analysis it is clear that $s(x) = s(y)$ can occur only if $x \in \{x_{2k+4}, x_{2k+5}, x_{2k+6}\}$ and $y \in L' \cup R'$ or the roles of x and y are interchanged. The case $k = 8$ corresponds to $n = 22$, and we check directly that T_{22} is not status injective. Next we suppose $k \ge 9$. Then $s(x_{2k-2}) > s(x_{2k+6}) > s(x_{2k+4}) > s(x_{2k+5})$, and hence x_{2k-2} can be excluded from R'. Similarly, since $s(x_{k+2}) < s(x_k) < s(x_{2k+5}) < s(x_{2k+4}) < s(x_{2k+6})$, x_k can be excluded from L' and x_{k+2} can be excluded from R'. Note that the statuses of the vertices in $L' \setminus \{x_k\}$ have the uniform expression $d + p^2 + 5p + 4$ with $1 \le p \le k-4$ and the statuses of the vertices in $R' \setminus \{x_{k+2},\ x_{2k-2}\}$ have the uniform expression $d + q^2 + q - 6$ with $3 \le q \le k-3$.

Denote the empty set by ϕ, and denote $\Omega_k = \{2k+2,\ 2k+10,\ 4k+6\}$, $\Gamma_k = A_k \cup B_k$ where $A_k = \{p^2+5p+4 |\, 1 \le p \le k-4,\ p \in \mathbb{N}\}$ and $B_k = \{q^2+q-6 |\, 3 \le q \le k-3,\ q \in \mathbb{N}\}$. It follows that when $k \ge 9$, T_n has two distinct vertices with the same status if and only if $\Omega_k \cap \Gamma_k \ne \phi$. Denote $\Gamma = A \cup B$ where $A = \{p^2+5p+4 |\, p \in \mathbb{N}\}$ and $B = \{q^2+q-6 |\, q \in \mathbb{N}\}$. Since $\Omega_k \cap \Gamma_k = \Omega_k \cap \Gamma$, we obtain the following criterion for $k \ge 9$:

T_n is status injective if and only if $\Omega_k \cap \Gamma = \phi$.

The graphs T_n with $15 \le n \le 18$ constructed below are all status injective. Using the above criterion we can check that T_n is status injective for

$$k = 10, 14, 18, 21, 23, 25, 27, 29, 33, 35, 38, 40, 42.$$

Thus the assertion in Theorem 5 on T_n for even n with $k \le 42$ is true.

Next we suppose $k \ge 43$. We will prove that among the four numbers $k, k+1, k+2, k+3$ there is at least one for which T_n is status injective. To do so, consider

$$\begin{aligned} \Omega_k &= \{2k+2,\ 2k+10,\ 4k+6\} \\ \Omega_{k+1} &= \{2k+4,\ 2k+12,\ 4k+10\} \\ \Omega_{k+2} &= \{2k+6,\ 2k+14,\ 4k+14\} \\ \Omega_{k+3} &= \{2k+8,\ 2k+16,\ 4k+18\}. \end{aligned}$$

The numbers in these four sets can be partitioned into two classes:

$$X = \{2k+i |\, i = 2, 4, 6, 8, 10, 12, 14, 16\} \ \text{ and } \ Y = \{4k+j |\, j = 6, 10, 14, 18\}.$$

We claim that

$$|X \cap A| \le 1, \quad |X \cap B| \le 1, \quad |Y \cap A| \le 1, \quad |Y \cap B| \le 1. \tag{5}$$

Define two polynomials $f(p) = p^2 + 5p + 4$ and $h(q) = q^2 + q - 6$. Then $A = \{f(p) |\, p \in \mathbb{N}\}$ and $B = \{h(q) |\, q \in \mathbb{N}\}$. In the sequel the symbol $\Rightarrow$ means "implies". We first prove $|X \cap A| \le 1$. To the contrary, suppose there exist i, j, p_1, p_2 with $2 \le i < j \le 16$ and $p_1 < p_2$ such that $f(p_1) = 2k + i$ and $f(p_2) = 2k + j$. $k \ge 43$ and $i \ge 2 \Rightarrow f(p_1) = 2k + i \ge 88 \Rightarrow p_1 \ge 7$. We have $f(p_2) - f(p_1) = j - i \le 14$. But on the other hand, $f(p_2) - f(p_1) \ge f(p_1 + 1) - f(p_1) = 2p_1 + 6 \ge 20$, a contradiction. The inequality $|X \cap B| \le 1$ is similarly proved by using the fact that $h(q) \in X \Rightarrow h(q) \ge 88 \Rightarrow q \ge 10$. The inequalities $|Y \cap A| \le 1$ and $|Y \cap B| \le 1$ can also be similarly proved by using the facts that $f(p) \in Y \Rightarrow f(p) \ge 178 \Rightarrow p \ge 11$ and $h(q) \in Y \Rightarrow h(q) \ge 178 \Rightarrow q \ge 14$.

Note that the assumption $k \ge 43$ implies that $\min X \ge 88$ and $\min Y \ge 178$. Hence if $f(p) \in X \cup Y$ we have $p \ge 7$ and Lemma 4 can be applied.

Suppose $\Omega_i \cap \Gamma \ne \phi$ for $i = k, k+1, k+2$. We will show that $\Omega_{k+3} \cap \Gamma = \phi$. Since $\Omega_k \cap \Gamma \ne \phi$, at least one of the two cases $\{2k+2,\ 2k+10\} \cap \Gamma \ne \phi$ and $4k+6 \in \Gamma$ must occur. Recall that $\Gamma = A \cup B$.

Case 1. $\{2k+2,\ 2k+10\} \cap \Gamma \ne \phi$. We first consider the case when $\{2k+2,\ 2k+10\} \cap A \ne \phi$. Denote $\Psi = \{2k+4,\ 2k+12,\ 2k+8,\ 2k+16\}$. By (5), $\Psi \cap A = \phi$. By Lemma 4, $\Psi \cap B = \phi$. It follows that $\Psi \cap \Gamma = \phi$. Since $\Omega_{k+1} \cap \Gamma \ne \phi$ and $\{2k+4,\ 2k+12\} \cap \Gamma = \phi$, we deduce that $4k+10 \in \Gamma$. By (5), $4k+10$ and $4k+18$ cannot be both in A or both in B. Since $4 \ne 8 = (4k+18) - (4k+10) \le 15$, by Lemma 4 it is also impossible that one of $4k+10$ and $4k+18$ is in A and the other in B. But $4k+10 \in \Gamma = A \cup B$. Hence $4k+18 \notin \Gamma$ and we obtain $\Omega_{k+3} \cap \Gamma = \phi$. The case when $\{2k+2,\ 2k+10\} \cap B \ne \phi$ is similar. Again we use (5), Lemma 4 and $\Omega_{k+1} \cap \Gamma \ne \phi$ to deduce $\Omega_{k+3} \cap \Gamma = \phi$.

Case 2. $4k+6 \in \Gamma$. Using (5) and Lemma 4 we deduce that $\{4k+14,\ 4k+18\} \cap \Gamma = \phi$. Then the condition $\Omega_{k+2} \cap \Gamma \ne \phi$ implies $\{2k+6,\ 2k+14\} \cap \Gamma \ne \phi$. Applying (5) and Lemma 4 once more we have $\{2k+8,\ 2k+16\} \cap \Gamma = \phi$. Hence $\Omega_{k+3} \cap \Gamma = \phi$.

This completes the proof of the case $n \ge 19$ of Theorem 5. The graph pairs T_n and U_n with $10 \le n \le 18$ are depicted in Figs. 3–11. They satisfy the condition $s(T_n) = s(U_n)$ and for $15 \le n \le 18$, T_n is status injective. In these graphs, the number beside a vertex is the status of that vertex.

This completes the proof of Theorem 5. □

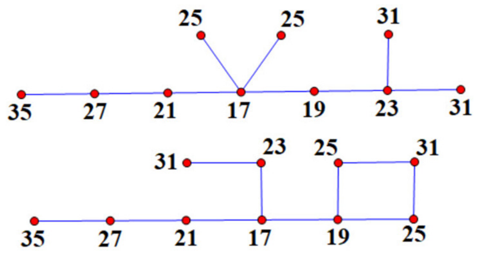

Fig. 3. T_{10} and U_{10}.

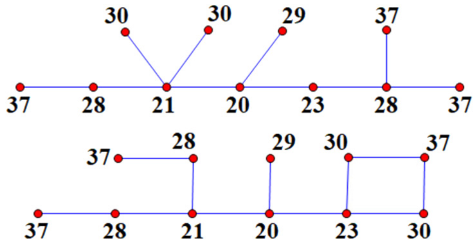

Fig. 4. T_{11} and U_{11}.

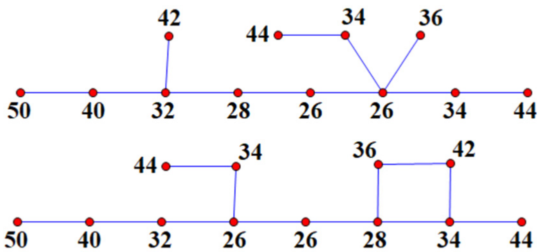

Fig. 5. T_{12} and U_{12}.

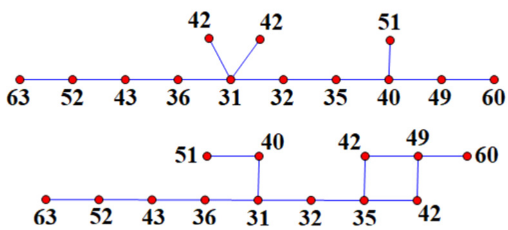

Fig. 6. T_{13} and U_{13}.

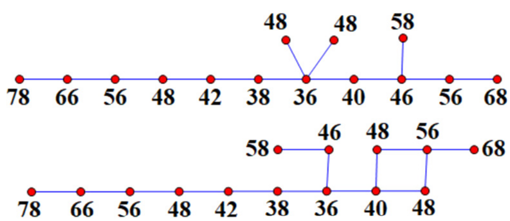

Fig. 7. T_{14} and U_{14}.

Remark. A computer search shows that 10 is the smallest order for the existence of a tree and a nontree graph with the same status sequence.

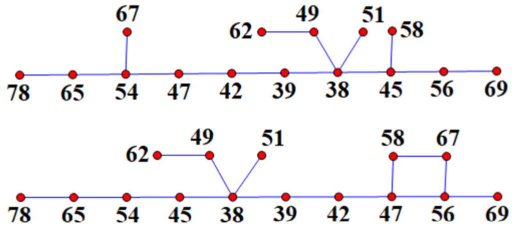

Fig. 8. T_{15} and U_{15}.

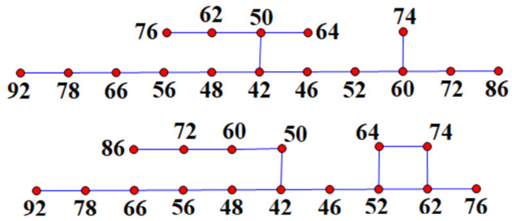

Fig. 9. T_{16} and U_{16}.

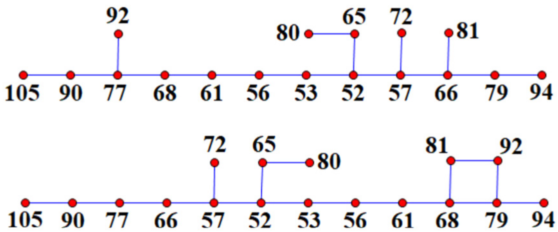

Fig. 10. T_{17} and U_{17}.

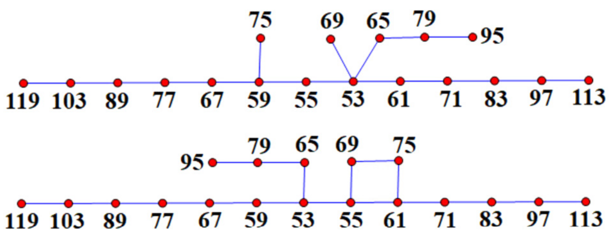

Fig. 11. T_{18} and U_{18}.

Declaration of competing interest

The authors declare that there is no conflict of interest in this paper.

Acknowledgments

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