Note

The maximum girth and minimum circumference of graphs with prescribed radius and diameter

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Abstract

Ostrand posed the following two questions in 1973. (1) What is the maximum girth of a graph with radius $r$ and diameter $d$? (2) What is the minimum circumference of a graph with radius $r$ and diameter $d$? Question 2 has been answered by Hrnčiar who proves that if $d \leq 2r - 2$ the minimum circumference is $4r - 2d$. In this note we first answer Question 1 by proving that the maximum girth is $2r + 1$. This improves on the obvious upper bound $2d + 1$ and implies that every Moore graph is self-centered. We then prove a property of the blocks of a graph which implies Hrnčiar’s result.

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1. Introduction

We consider finite simple graphs. Ostrand [6, p.75] posed the following two questions in 1973.

Question 1. What is the maximum girth of a graph with radius $r$ and diameter $d$?

Question 2. What is the minimum circumference of a graph with radius $r$ and diameter $d$?

Question 2 has been answered by Hrnčiar [5] who proves that if $d \leq 2r - 2$ the minimum circumference is $4r - 2d$. In this note we first answer Question 1 by proving that the maximum girth is $2r + 1$. This improves on the obvious upper bound $2d + 1$ and implies that every Moore graph is self-centered. We then prove a property of the blocks of a graph which implies Hrnčiar’s result.

Google shows 63 citations of Ostrand’s paper [6] and MathSciNet shows 7 citations. It seems that Question 1 has not been treated.

For terminology and notations we follow the books [1,3,8]. We denote by $V(G)$ the vertex set of a graph $G$ and by $d(u, v)$ the distance between two vertices $u$ and $v$. The eccentricity, denoted by $e(v)$, of a vertex $v$ in a graph $G$ is the distance to a vertex farthest from $v$. Thus $e(v) = \max\{d(v, u) | u \in V(G)\}$. If $e(v) = d(v, x)$, then the vertex $x$ is called an eccentric vertex of $v$. The radius of a graph $G$, denoted $\text{rad}(G)$, is the minimum eccentricity of all the vertices in $V(G)$, whereas the diameter of $G$, denoted $\text{diam}(G)$, is the maximum eccentricity. A vertex $v$ is a central vertex of $G$ if $e(v) = \text{rad}(G)$. When $H$ is a subgraph of a graph $G$ and $u, v \in V(H)$, $d_H(u, v)$ and $e_H(v)$ will mean the distance and eccentricity in $H$ respectively.

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2. Main results

Recall that a block of a graph $G$ is a maximal connected subgraph of $G$ that has no cut-vertex. Thus in a connected graph, a block is either a cut-edge or a maximal 2-connected subgraph of order at least 3. If $H$ is a subgraph of a graph $G$ and $v$ is a vertex of $G$, then the distance between $v$ and $H$, denoted $d(v, H)$, is defined as $d(v, H) = \min\{d(v, x) | x \in V(H)\}$. Throughout the symbol $C_k$ denotes a cycle of length $k$, and $P_t$ denotes a path of order $t$ (and hence of length $t - 1$).

**Lemma 1.** If $B$ is a block of a graph $G$, then $\text{rad}(B) \leq \text{rad}(G)$.

**Proof.** Let $c$ be a central vertex of $G$. If $c \in V(B)$, then $\text{rad}(B) \leq e_B(c) \leq e_G(c) = \text{rad}(G)$. Thus $\text{rad}(B) \leq \text{rad}(G)$. Now suppose $c \notin V(B)$. In the block-cutvertex tree of $G$ ([1, p.121] or [8, p.156]), from any block containing $c$ there is a unique path to $B$. Hence there is a unique vertex $v$ of $B$ such that $d(c, B) = d(c, v)$. Note that $v$ is a cut-vertex of $G$. Let $u$ be an eccentric vertex of $v$ in $B$; i.e., $e_B(v) = d(v, u)$. Then we have
\[
\text{rad}(G) = e_G(c) \geq d(c, u) = d(c, v) + d(v, u) \\
\geq 1 + e_B(v) \geq 1 + \text{rad}(B),
\]
showing that $\text{rad}(G) > \text{rad}(B)$ in this case. \hfill $\Box$

We remark that in *Lemma 1*, if we replace the block $B$ by a generic subgraph, the conclusion may not be true. To see this, consider the wheel graph $W_n$ of order $n \geq 5$ with central vertex $c$. We have $\text{rad}(W_n - c) = \lfloor (n - 1)/2 \rfloor > 1 = \text{rad}(W_n)$.

We make the convention that the girth of an acyclic graph is undefined. Thus whenever we speak of the girth of a graph, we have already implicitly assumed that the graph contains at least one cycle. We will need the well-known fact (e.g. [6]) that if $r$ and $d$ are the radius and diameter of a graph respectively, then $r \leq d \leq 2r$. The following result answers *Question 1*.

**Theorem 2.** The maximum girth of a graph with radius $r$ and diameter $d$ is $2r + 1$.

**Proof.** Let $G$ be a graph with radius $r$ and diameter $d$ containing at least one cycle. Every cycle lies within one block. Let $B$ be a block of $G$ containing a cycle. Then $B$ is 2-connected. Let $x$ be a central vertex of the subgraph $B$. There is at least one cycle containing $x$, since $B$ is 2-connected [8, p.162]. Among all the cycles containing $x$, we choose one of the shortest length and denote it by $C$. Let $s = \text{rad}(B)$ and denote the length of $C$ by $q$. We assert that $q \leq 2s + 1$. To the contrary suppose $q \geq 2s + 2$. Since $C$ is a shortest cycle containing $x$, $C$ contains a vertex $y$ such that
\[
\text{d}(x, y) = d(x, y) = |q/2| \geq s + 1,
\]
implying that $s = e_B(x) \geq d_B(x, y) \geq s + 1$, a contradiction. By *Lemma 1*, $s \leq r$. We deduce that $G$ has girth at most $q$ and $q \leq 2s + 1 \leq 2r + 1$.

Conversely, given any positive integers $r$ and $d$ with $r \leq d \leq 2r$, let $M(r, d)$ be the monocle graph which is the union of the cycle $C_{2r - 1}$ and the path $P_{d - r + 1}$, the cycle and the path having only one common vertex which is an end vertex of the path. Then $M(r, d)$ has radius $r$, diameter $d$ and girth $2r + 1$. This completes the proof. \hfill $\Box$

Theorem 2 improves on the obvious upper bound $2d + 1$. A *Moore graph* was originally defined in [4] as a graph of diameter $d$, maximum degree $\Delta$ and the largest possible order $1 + \Delta \sum_{i=0}^{d-1}(\Delta - 1)^{i-1}$. An equivalent definition [2,7] of a Moore graph is a graph of diameter $d$ and girth $2d + 1$. A graph $G$ is said to be self-centered if $\text{rad}(G) = \text{diam}(G)$. Thus self-centered graphs are those graphs in which every vertex is a central vertex. The following result is deduced in [3, pp.100–101] by using the concept of the distance degree sequence of a vertex. Now it becomes obvious.

**Corollary 3.** Every Moore graph is self-centered.

**Proof.** Let $G$ be a Moore graph of radius $r$, diameter $d$ and girth $2d + 1$. By *Theorem 2* and the fact $r \leq d$ we have $2d + 1 \leq 2r + 1 \leq 2d + 1$. Hence $r = d$. \hfill $\Box$

Now we prove a property of the blocks of a graph. In the proof we will use an idea in [5].

**Theorem 4.** Every graph of radius $r$ and diameter $d$ has a block whose diameter is at least $2r - d$.

**Proof.** We first prove the following

**Claim.** Let $H$ be a graph of radius $r$ and diameter $d$. If $B$ is a block of $H$ with $\text{diam}(B) < 2r - d$ and $u$ is a vertex of $H$ such that $d(u, B) = \max\{d(x, B) | x \in V(H)\}$, then $d(u, B) \geq r$.

In the block-cutvertex tree of $H$ ([1, p.121] or [8, p.156]), from the block (necessarily an end block) containing $u$ to $B$ there is a unique path. Thus there is a unique vertex $v \in V(B)$ such that $d(u, B) = d(u, v)$ and $v$ is a cut-vertex of $H$. Denote $d(u, v) = a$. We need to prove $a \geq r$. To the contrary, suppose $a < r$. We will show that $e(v) < r$, which is a contradiction.
since $r$ is the radius of $H$. Let $H_1$ be the component of $H - v$ containing the vertex $u$. For $w \in V(H)$, if $w \in V(H_1)$ then $d(v, w) \leq d(v, u) = a < r$. If $w \in V(B)$ then $d(v, w) \leq \text{diam}(B) < 2r - d \leq r$ and hence $d(v, w) < r$. Finally suppose $w \in V(H) \setminus (V(H_1) \cup V(B))$. Let $z \in V(B)$ such that $d(w, z) = d(w, B)$. Then $z$ is a cut-vertex of $H$ and $d_0(z, v) = d_H(z, v)$. Denote $d(z, v) = b$ and $d(w, z) = c$. Note that $c \leq a$ and $b \leq \text{diam}(B) \leq 2r - d - 1$. If $b + c \geq r$, then $c \geq r - b$ and

$$d(u, w) = d(u, v) + d(v, z) + d(z, w) = a + b + c \geq 2c + b \geq 2(r - b) + b = 2r - b \geq 2r - (2r - d - 1) = d + 1,$$

which is a contradiction since $d$ is the diameter of $H$. Hence $b + c < r$. Then $d(v, w) = b + c < r$. Thus we have $e(v) < r$, a contradiction. This proves the claim.

Now we prove the theorem. It is well-known (e.g., [6]) that $r \leq d \leq 2r$. First note that if $d = 2r$ or $d = 2r - 1$ then the theorem holds trivially. To the contrary suppose the theorem is false and let $G$ be a counterexample of the smallest order with radius $r$ and diameter $d$ such that every block of $G$ has diameter $< 2r - d$. Then $d \leq 2r - 2$. Choose a central vertex $p$ of $G$. We assert that $G$ has a block containing $p$ and of order at least 3. Otherwise each block containing $p$ is an edge. Let $f = pq$ be an edge of $G$ and let $u \in V(G)$ such that $d(u, f) = \text{max}(d(x, f)|x \in V(G))$. Here we regard the edge $f$ as a subgraph of $G$, and the meaning of $d(u, f)$ is defined at the beginning of Section 2. By the Claim we have $d(u, f) \geq r$. Note that $f$ is a cut-edge of $G$. Let $G_1$ be the component of $G - f$ containing $q$. Then $u \notin V(G_1)$, since otherwise

$$r = e(p) \geq d(p, u) = d(p, q) + d(q, u) = 1 + d(u, f) \geq 1 + r,$$

a contradiction. It follows that $d(u, f) = d(u, p)$. Let $h = pt$ be the end edge of a shortest $(u, p)$-path and let $G_2$ be the component of $G - h$ containing the vertex $t$. $h$ is also a cut-edge of $G$. Let $v \in V(G)$ such that $d(v, h) = \text{max}(d(x, h)|x \in V(G))$. By the Claim, $d(v, h) \geq r$. As argued above, $v \notin V(G_2)$ and $d(v, h) = d(v, p)$. But then

$$d \geq d(u, v) = d(u, p) + d(p, v) \geq r + r = 2r,$$

contradicting the condition that $d \leq 2r - 2$.

Let $B$ be a block of $G$ containing $p$ and of order at least 3. Then $\text{diam}(B) < 2r - d$ since $G$ is a counterexample. Let $u \in V(G)$ such that $d(u, B) = \text{max}(d(x, B)|x \in V(G))$. There is a unique vertex $u'$ of $B$ such that $d(u, u') = d(u, B)$ and $u'$ is a cut-vertex of $G$. By the Claim, $d(u, B) \geq r$. On the other hand, $r = e(p) \geq d(p, u) = d(p, u') + d(u', u) \geq d(p, u') + r$, implying that $u' = p$ and $d(p, u) = r$. Let $G_1$ be the component of $G - p$ containing $u$ and denote $G_2 = G - V(G_1)$. Let $v \in V(G_2)$ such that $d(p, v) = \text{max}(d(p, x)|x \in V(G_2))$. Then the conditions $d(u, v) \leq d \leq 2r - 2$ and $d(u, p) = r$ imply that $d(p, v) = d(u, v) - d(u, p) \leq 2r - 2 - r = r - 2$. Let $Q$ be a shortest $(p, v)$-path and let $L$ be the $(p)$-lobe [8, p.211] of $G$ containing $u$; i.e., $L = G[V(G_1) \cup [p]]$. Define a new graph $G' = L \cup Q$. We will show that $G'$ is a counterexample of a smaller order.

We first show that the graph $G'$ has the same radius and diameter as $G$. It is clear that $p$ remains a central vertex of $G'$ with $e_G(p) = d_G(p, u) = r$. Let $x, y \in V(G')$ be a diametral pair of $G'$; i.e., $d(x, y) = d$. It is impossible that both $x$ and $y$ lie in $G_2$, since otherwise

$$d(u, x) = d(u, p) + d(p, x) = r + d(p, x) \geq r + d(x, y) - d(p, y) \geq r + d - d(p, v) \geq r + d - (r - 2) = d + 2,$$

a contradiction. Hence either both $x$ and $y$ lie in $L$ or one of them, say $x$, lies in $L$ and the other $y$ lies in $G_2$. In the former case, $d = d_G(x, y) = d_G(x, y)$ and in the latter case,

$$d = d_G(x, y) = d_G(x, p) + d_G(p, y) \leq d_G(x, p) + d_G(p, v) = d_G(x, v) = d_G(x, v).$$

Thus $\text{diam}(G') \geq \text{diam}(G)$. On the other hand, since for any two vertices $s, t \in V(G')$, $d_G(s, t) = d_G(s, t) \leq d$, we have $\text{diam}(G') \leq \text{diam}(G)$. Hence $\text{diam}(G') = \text{diam}(G)$.

$G'$ has two kinds of blocks: the blocks in $L$ which are also blocks of $G$ and the edges of the path $Q$. Hence every block of $G'$ has diameter $\leq 2r - d$. Since the block $B$ has order $\geq 3$, it is 2-connected and it contains at least one cycle. Let $C$ be a cycle in $B$. Since any shortest path between two vertices cannot contain all the vertices of a cycle, at least one vertex of $C$ does not lie in the path $Q$. It follows that $G'$ has a strictly smaller order than $G$, contradicting the choice of $G$. This completes the proof. \[\Box\]

Next we use Theorem 4 to deduce Hrnčiar’s result [5] which answers Question 2. Note that if $d = 2r$ or $d = 2r - 1$, there is a tree with radius $r$ and diameter $d$. A tree has no cycle. Thus, for Question 2 it suffices to consider the case $d \leq 2r - 2$.

**Theorem 5** (Hrnčiar). The minimum circumference of a graph of radius $r$ and diameter $d$ with $d \leq 2r - 2$ is $4r - 2d$. 

Proof. Let $G$ be a graph of radius $r$ and diameter $d$ with $d \leq 2r - 2$. By Theorem 4, $G$ has a block $B$ with $\text{diam}(B) \geq 2r - d$. Let $u, v \in V(B)$ such that $d(u, v) = \text{diam}(B)$. Since $2r - d \geq 2$, $B$ is 2-connected. Hence there exists a cycle $C$ containing $u$ and $v$. The condition $d(u, v) \geq 2r - d$ implies that the length of $C$ is at least $2(2r - d) = 4r - 2d$, proving that the circumference of $G$ is at least $4r - 2d$. The sun-graph $S_{4r-2d,d-r}$ obtained by attaching a path $P_{d-r+1}$ to each vertex of the cycle $C_{4r-2d}$ has radius $r$, diameter $d$ and circumference $4r - 2d$. □

The above proof of Theorem 5 shows that large diameter of a block implies the existence of a long cycle. The converse need not be true. For example, the complete graph of order $\geq 3$ has a long cycle (Hamilton cycle) but it has a very small diameter ($= 1$). Finally, we remark that the corresponding problems of Question 1 for minimum girth and Question 2 for maximum circumference are trivial. It is easy to see that the minimum girth of a graph with radius $r$ and diameter $d$ is 3, and the circumference can be arbitrarily large.

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References