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Inverse invariant zero–nonzero patterns ☆

Chao Ma, Xingzhi Zhan *

Department of Mathematics, East China Normal University, Shanghai 200241, China

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ABSTRACT

We determine the irreducible zero–nonzero patterns A such that for any nonsingular matrix B over a field with zero–nonzero pattern A, the inverse B^{-1} has the same zero–nonzero pattern A. We also determine the zero–nonzero patterns P such that for any nonsingular matrix Q over a field with zero–nonzero pattern P, the transpose of the inverse $(Q^{-1})^T$ has the same zero–nonzero pattern P. One application of these results is to deduce the corresponding results on sign patterns.

1. Introduction

A matrix with entries from the set $\{0, *\}$ is called a *zero–nonzero pattern*. We denote by $M_{n,k}(F)$ the set of all $n \times k$ matrices over a field F. Given a field F and an $n \times k$ zero–nonzero pattern $A = (a_{ij})$, we denote by $Z_F(A)$ the set of all $n \times k$ matrices over F with zero–nonzero pattern A, i.e.,

$$Z_F(A) = \{B = (b_{ij}) \in M_{n,k}(F) \mid b_{ij} = 0 \text{ if and only if } a_{ij} = 0\}.$$

Thus $*$ indicates nonzero entries.

A square zero–nonzero pattern A is said to be *intrinsically singular* over a field F if every matrix in $Z_F(A)$ is singular. A zero–nonzero pattern that is not intrinsically singular is called *potentially nonsingular*. Given a field F, we denote by $SI_n(F)$ the set of all zero–nonzero patterns A of order n which are potentially nonsingular and satisfy the condition that for every nonsingular matrix $B \in Z_F(A)$, $B^{-1} \in Z_F(A)$. This notation suggests that the inverse invariant patterns are also called "self-inverse"

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* Corresponding author.
E-mail addresses: machao0923@163.com (C. Ma), zhan@math.ecnu.edu.cn (X. Zhan).

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patterns. We use G^T to mean the transpose of a matrix G. We denote by $IT_n(F)$ the set of all zero–nonzero patterns P of order n which are potentially nonsingular and satisfy the condition that for every nonsingular matrix $Q \in Z_F(P)$, $(Q^{-1})^T \in Z_F(P)$. This notation suggests that the inverse has the transposed pattern. Note that the patterns in $IT_n(F)$ are zero–nonzero pattern analogs of orthogonal matrices.

The purpose of this paper is to determine the irreducible patterns in $SI_n(F)$. To do so, we first characterize the set $IT_n(F)$. The corresponding problems on sign patterns are studied in [2] and [3]. Note that sign patterns are special (more precise) zero–nonzero patterns.

2. Main results

Let A be an $m \times n$ matrix and let $\alpha = (i_1, \dots, i_s)$, $\beta = (j_1, \dots, j_t)$ where $1 \leqslant i_1 < \cdots < i_s \leqslant m$ and $1 \leqslant j_1 < \cdots < j_t \leqslant n$ are integers. We denote by $A[\alpha|\beta]$ the $s \times t$ submatrix of A that lies in the rows $i_1, \dots, i_s$ and columns $j_1, \dots, j_t$, and denote by $A(\alpha|\beta)$ the $(m-s) \times (n-t)$ submatrix of A obtained by deleting the rows $i_1, \dots, i_s$ and columns $j_1, \dots, j_t$. We abbreviate $A[\alpha|\alpha]$ and $A(\beta|\beta)$ to $A[\alpha]$ and $A(\beta)$ respectively. The symbol $A(i, j)$ denotes the entry of A in the i-th row and j-th column.

For positive integers $m \leqslant n$, let $S_{m,n}$ be the set of injections from $\{1, 2, \dots, m\}$ to $\{1, 2, \dots, n\}$. Let $A = (a_{ij})$ be an $m \times n$ matrix with $m \leqslant n$. If $\sigma \in S_{m,n}$, then the sequence $a_{1\sigma(1)}, a_{2\sigma(2)}, \dots, a_{m\sigma(m)}$ is called a *transversal* of A. A *nonzero transversal* is a transversal in which every entry is nonzero. We denote by $|S|$ the cardinality of a set S.

Lemma 1. *Let F be a field with $|F| \geqslant 3$. If a square zero–nonzero pattern A has a nonzero transversal, then A is potentially nonsingular over F.*

Proof. Let A be of order n. We use induction on n. For $n = 1$ the assertion is trivial. Now let $n \geqslant 2$ and assume that the lemma holds for matrices of order $n-1$.

It suffices to exhibit a nonsingular matrix $B \in Z_F(A)$. Suppose the entry $A(1, j)$ lies on a nonzero transversal of A. Then $A(1|j)$ is of order $n-1$ and it also has a nonzero transversal. By the induction hypothesis, $A(1|j)$ is potentially nonsingular over F; i.e., there is a nonsingular matrix $B_1 \in Z_F(A(1|j))$. We define a matrix B by setting $B(1|j) = B_1$ and letting all the nonzero entries in row 1 and in column j of B except $B(1, j)$ be 1. Since F has at least three elements, it has at least two nonzero elements. By considering the Laplace expansion of $\det B$ along the first row, we see that there is a nonzero value of $B(1, j)$ in F such that B is nonsingular. Obviously $B \in Z_F(A)$. Thus A is potentially nonsingular over F. $\square$

By Lemma 1 and the Frobenius–König theorem [5, p. 46] we deduce the following result.

Corollary 2. *Let F be a field with $|F| \geqslant 3$ and let A be a zero–nonzero pattern of order n. Then the following statements are equivalent:*

(i) *A is intrinsically singular over F.*
(ii) *Every transversal of A contains at least one zero entry.*
(iii) *A has an $r \times s$ zero submatrix with $r + s = n + 1$.*

We remark that Lemma 1 and Corollary 2 do not hold for the field F_2 with only two elements. For example, consider the zero–nonzero pattern of order $n \geqslant 2$ with each entry being $*$.

Lemma 3. *Let F be a field and let A be a zero–nonzero pattern of order n, $\alpha, \beta \subseteq \{1, 2, \dots, n\}$ and $|\alpha| = |\beta|$. If A and $A[\alpha|\beta]$ are potentially nonsingular over F, then there is a nonsingular matrix $B \in Z_F(A)$ such that $B[\alpha|\beta]$ is also nonsingular.*

Proof. If F is the field with only two elements, then the lemma holds trivially. Next suppose F has at least three elements. We use induction on the order n. For $n = 1$ the assertion is trivial. Now let

$n \geqslant 2$ and assume that the lemma holds for matrices of order $n-1$. Since A is potentially nonsingular, A has a nonzero transversal. We distinguish two cases.

Case 1. There exist $i \notin \alpha$, $j \notin \beta$ such that the entry $A(i, j)$ lies on a nonzero transversal of A. Then $A(i|j)$ also has a nonzero transversal. By Lemma 1, $A(i|j)$ is potentially nonsingular over F. Note that $A[\alpha|\beta]$ is a submatrix of $A(i|j)$. Let $A[\alpha|\beta] = A(i|j)[\alpha'|\beta']$. By the induction hypothesis, there is a nonsingular matrix $B_1 \in Z_F(A(i|j))$ such that $B_1[\alpha'|\beta'] \in Z_F(A[\alpha|\beta])$ is also nonsingular. We define a matrix B by setting $B(i|j) = B_1$ and letting all the nonzero entries in row i and in column j of B except $B(i, j)$ be 1. Since F has at least two nonzero elements, there is a nonzero value of $B(i, j)$ in F such that B is nonsingular. Obviously $B \in Z_F(A)$ and $B[\alpha|\beta] = B_1[\alpha'|\beta']$ is also nonsingular.

Case 2. There exist no $i \notin \alpha$, $j \notin \beta$ such that $A(i, j)$ lies on a nonzero transversal of A. If $|\alpha| = |\beta| = n$, then $A[\alpha|\beta] = A$. The lemma holds trivially. Next suppose $|\alpha| = |\beta| \leqslant n-1$. Choose $j_1 \notin \beta$. Then there exists $i_1 \in \alpha$ such that $A(i_1, j_1)$ lies on a nonzero transversal of A. Since $A[\alpha|\beta]$ is potentially nonsingular, there exists $j_2 \in \beta$ such that $A(i_1, j_2)$ lies on a nonzero transversal of $A[\alpha|\beta]$.

Now we consider the matrix $A(i_1|j_1)$. The choice of $A(i_1, j_1)$ implies that $A(i_1|j_1)$ has a nonzero transversal, and the choice of $A(i_1, j_2)$ implies that $A[\alpha \setminus \{i_1\}|\beta \setminus \{j_2\}]$ also has a nonzero transversal. By Lemma 1, $A(i_1|j_1)$ and $A[\alpha \setminus \{i_1\}|\beta \setminus \{j_2\}]$ are potentially nonsingular over F. Note that $A[\alpha \setminus \{i_1\}|\beta \setminus \{j_2\}]$ is a submatrix of $A(i_1|j_1)$. Let $A[\alpha \setminus \{i_1\}|\beta \setminus \{j_2\}] = A(i_1|j_1)[\alpha'|\beta']$. By the induction hypothesis, there is a nonsingular matrix $B_1 \in Z_F(A(i_1|j_1))$ such that $B_1[\alpha'|\beta'] \in Z_F(A[\alpha \setminus \{i_1\}|\beta \setminus \{j_2\}])$ is also nonsingular. We construct a matrix B by setting $B(i_1|j_1) = B_1$ so that $B[\alpha \setminus \{i_1\}|\beta \setminus \{j_2\}] = B_1[\alpha'|\beta']$ and letting all the nonzero entries in row i_1 and in column j_1 except $B(i_1, j_2)$ and $B(i_1, j_1)$ be 1. Since F has at least two nonzero elements, there are nonzero values for $B(i_1, j_2)$ and $B(i_1, j_1)$ in F such that both $B[\alpha|\beta]$ and B are nonsingular. Obviously $B \in Z_F(A)$. □

Two matrices A and B are said to be *permutation equivalent* if there are permutation matrices P and Q such that $PAQ = B$. Note that $IT_n(F)$ is closed under permutation equivalence. The following theorem characterizes the set $IT_n(F)$.

Theorem 4. *Let F be a field with $|F| \geqslant 3$ and let A be a zero–nonzero pattern of order n. Then $A \in IT_n(F)$ if and only if A is permutation equivalent to a block diagonal matrix with each diagonal block being $[*]$ or $\begin{bmatrix} * & * \\ * & * \end{bmatrix}$.*

Proof. The sufficiency can be proved by an easy calculation. Next we prove the necessity.

When $n = 1$, $A \in IT_1(F)$ implies $A = [*]$. When $n = 2$, it is easy to verify that $A \in IT_2(F)$ implies that A must be one of the following patterns:

$$\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}, \qquad \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}, \qquad \begin{bmatrix} * & * \\ * & * \end{bmatrix}.$$

Note that $\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$ and $\begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$ are permutation equivalent. Thus the assertion holds for $n = 1$ and $n = 2$.

When $n \geqslant 3$, we first show that if $A \in IT_n(F)$, then A has at least one zero entry. To the contrary, assume that each entry of A is nonzero. Consider the symmetric matrix

$$B = \begin{bmatrix} a & \cdots & a & 1 \\ \vdots & \iddots & \iddots & 1 \\ a & \iddots & \iddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \in Z_F(A),$$

where $a \neq 0, 1$. A simple calculation shows that $\det B \neq 0$ and $\det B(1|1) = 0$. Then $(B^{-1})^T(1, 1) = B^{-1}(1, 1) = \det B(1|1)/\det B = 0$. Since $A(1, 1) = *$, $(B^{-1})^T \notin Z_F(A)$, which contradicts the condition that $A \in IT_n(F)$.

Suppose $A(i, j) = 0$. We will show that $A(i|j)$ is intrinsically singular over F. To the contrary, assume that $A(i|j)$ is potentially nonsingular over F. By Lemma 3, there is a nonsingular matrix $C \in Z_F(A)$ such that $C(i|j)$ is also nonsingular. Then

$$\left(C^{-1}\right)^T(i,j) = C^{-1}(j,i) = (-1)^{i+j}\frac{\det C(i|j)}{\det C} \neq 0.$$

Since $A(i,j)=0$, $(C^{-1})^T \notin Z_F(A)$, which contradicts $A \in IT_n(F)$. Thus $A(i|j)$ is intrinsically singular over F. By Corollary 2, $A(i|j)$ and hence A has an $r \times s$ zero submatrix with $r+s=n$. This implies that there are permutation matrices P and Q such that $A = P\left[\begin{smallmatrix} A_1 & 0 \\ A_2 & A_3 \end{smallmatrix}\right]Q$, where A_1 and A_3 are square matrices of orders r and $n-r$ respectively. Since A is potentially nonsingular over F, A_1 and A_3 are potentially nonsingular over F.

For any $D_i \in Z_F(A_i)$, $i=1,2,3$, with D_1 and D_3 nonsingular, let $D = P\left[\begin{smallmatrix} D_1 & 0 \\ D_2 & D_3 \end{smallmatrix}\right]Q$. Then $D \in Z_F(A)$ is nonsingular. Since $A \in IT_n(F)$,

$$\left(D^{-1}\right)^T = P\begin{bmatrix} (D_1^{-1})^T & -(D_1^{-1})^T D_2^T (D_3^{-1})^T \\ 0 & (D_3^{-1})^T \end{bmatrix} Q \in Z_F(A).$$

Thus $(D_1^{-1})^T \in Z_F(A_1)$, $A_2 = 0$, and $(D_3^{-1})^T \in Z_F(A_3)$. Then $A = P\left[\begin{smallmatrix} A_1 & 0 \\ 0 & A_3 \end{smallmatrix}\right]Q$, where $A_1 \in IT_r(F)$ and $A_3 \in IT_{n-r}(F)$.

We use induction on the order n. Note that the assertion already holds for $n=1$ and $n=2$. Now let $n \geqslant 3$ and assume that the assertion holds for every order less than n. By what we have proved above, A is permutation equivalent to a pattern of the form $\operatorname{diag}(A_1, A_3)$ with $A_1 \in IT_r(F)$ and $A_3 \in IT_{n-r}(F)$. Applying the induction hypothesis to A_1 and A_3, we conclude that A is permutation equivalent to a block diagonal matrix with each diagonal block being $[*]$ or $\left[\begin{smallmatrix} * & * \\ * & * \end{smallmatrix}\right]$. This completes the proof. $\square$

With an easy further analysis, Theorem 2.8 of [2] on sign patterns can be deduced from Theorem 4.

We remark that Theorem 4 does not hold for the field F_2 with only two elements. When $n \geqslant 4$ is even, consider the following zero–nonzero pattern of order n:

$$A = \begin{bmatrix} 0 & * & \cdots & * \\ * & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ * & \cdots & * & 0 \end{bmatrix}.$$

A simple calculation shows that the only matrix B in $Z_{F_2}(A)$ satisfies $(B^{-1})^T = B$. Thus $A \in IT_n(F_2)$. But A cannot be permutation equivalent to a block diagonal matrix with each diagonal block being $[*]$ or $\left[\begin{smallmatrix} * & * \\ * & * \end{smallmatrix}\right]$. This can be seen by considering the number of zero entries.

Next we investigate the structure of irreducible zero–nonzero patterns in $SI_n(F)$. We need several lemmas.

Lemma 5. *Let $A \in SI_n(F)$ and let $\alpha \subseteq \{1,2,\ldots,n\}$. If $A[\alpha]$ is potentially nonsingular over F, then $A(\alpha)$ is potentially nonsingular over F.*

Proof. By Lemma 3, there is a nonsingular matrix $B \in Z_F(A)$ such that $B[\alpha]$ is also nonsingular. Then by Jacobi's determinant identity ([4, p. 24] or [5, p. 183]), $\det B^{-1}(\alpha) = \det B[\alpha]/\det B \neq 0$. Since $A \in SI_n(F)$, $B^{-1}(\alpha) \in Z_F(A(\alpha))$. Thus $A(\alpha)$ is potentially nonsingular over F. $\square$

Let $A=(a_{ij})$ be a matrix of order n. A *simple cycle* of length k in A is a formal product of the form $a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ki_1}$, where each of the entries is nonzero and the index set $\{i_1, i_2, \ldots, i_k\}$ consists of distinct indices. Throughout this paper, a cycle means a simple cycle. If $A=(a_{ij})$ has a cycle $\gamma = a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ki_1}$ and the entries $a_{i_1i_k}, a_{i_ki_{k-1}}, \ldots, a_{i_2i_1}$ are all nonzero, then we denote the reverse cycle of γ by $\overline{\gamma} = a_{i_1i_k}a_{i_ki_{k-1}}\cdots a_{i_2i_1}$.

Lemma 6. *Let F be a field with $|F| \geqslant 3$ and let $A \in SI_n(F)$. If A has a cycle γ, then A also has the reverse cycle $\overline{\gamma}$.*

Proof. Suppose $A = (a_{ij})$ has a cycle $\gamma = a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ki_1}$ of length k. For every $a_{i_si_t} \in \gamma$, consider $A(i_s|i_t)$. Let $\alpha = \{i_1, i_2, \dots, i_k\}$. Then the sequence $a_{i_1i_2}, a_{i_2i_3}, \dots, a_{i_ki_1}$ is a nonzero transversal of $A[\alpha]$. By Lemma 1, $A[\alpha]$ is potentially nonsingular. Then by Lemma 5, $A(\alpha)$ is potentially nonsingular; i.e., $A(\alpha)$ has a nonzero transversal. This transversal and $\{a_{i_1i_2}, \dots, a_{i_ki_1}\} \setminus \{a_{i_si_t}\}$ constitute a nonzero transversal of $A(i_s|i_t)$. By Lemma 1, $A(i_s|i_t)$ is potentially nonsingular over F. Then by Lemma 3, there is a nonsingular matrix $B \in Z_F(A)$ such that $B(i_s|i_t)$ is also nonsingular. Thus

$$B^{-1}(i_t, i_s) = (-1)^{i_t+i_s}\frac{\det B(i_s|i_t)}{\det B} \neq 0.$$

Since $A \in SI_n(F)$, $B(i_t, i_s) \neq 0$. Hence $a_{i_ti_s} = *$ for every $a_{i_si_t} \in \gamma$. This shows that A has $\bar{\gamma}$. □

Recall that in a digraph, a sequence of successively adjacent arcs is called a *walk*. A *path* is a walk in which all the vertices are distinct. We denote by $D(A)$ the digraph of a matrix A of order n. The vertices of $D(A)$ are $1, 2, \dots, n$ and (i, j) is an arc if and only if $A(i, j) \neq 0$. As for conventional matrices, a zero–nonzero pattern A is said to be *symmetric* if $A^T = A$.

Lemma 7. *Let F be a field with $|F| \geqslant 3$ and let $A \in SI_n(F)$. If A is irreducible, then A is symmetric.*

Proof. Let $A = (a_{ij})$ be irreducible. Since the digraph of an irreducible matrix is strongly connected ([1, p. 55] or [5, p. 133]), $D(A)$ is strongly connected. Suppose $a_{ij} = *$ with $i \neq j$. Then there is a path from j to i in $D(A)$, say $j \to i_1 \to i_2 \to \cdots \to i_k \to i$. It follows that $a_{ij}a_{ji_1}\cdots a_{i_ki}$ is a cycle in A. By Lemma 6, $a_{ji} = *$. Hence $a_{ij} = a_{ji}$. This implies that A is symmetric. □

Lemma 8. *Let F be a field with $|F| \geqslant 3$ and let $A \in SI_n(F)$. If A is irreducible, then there are at most two nonzero entries in each row and each column of A.*

Proof. Since $A \in SI_n(F)$ is irreducible, by Lemma 7, A is symmetric. Then every nonsingular matrix $B \in Z_F(A)$ satisfies $(B^{-1})^T \in Z_F(A^T) = Z_F(A)$. Thus $A \in IT_n(F)$. By Theorem 4, A is permutation equivalent to a block diagonal matrix with each diagonal block being $[*]$ or $\left[\begin{smallmatrix} * & * \\ * & * \end{smallmatrix}\right]$. Thus there are at most two nonzero entries in each row and each column of A. □

Lemma 9. *Let F be a field with $|F| \geqslant 3$ and let $A \in SI_n(F)$ be irreducible. If A has a cycle of length $k \geqslant 3$, then $k = n$.*

Proof. Clearly $k \leqslant n$. To the contrary, assume that $k < n$. Let $\gamma = a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ki_1}$ be a cycle of length $k \geqslant 3$ in $A = (a_{ij})$. By Lemma 6, A has the reverse cycle $\bar{\gamma} = a_{i_1i_k}a_{i_ki_{k-1}}\cdots a_{i_2i_1}$. Let $\alpha = \{i_1, i_2, \dots, i_k\}$. Then each row of $A[\alpha]$ has at least two nonzero entries. Since A is irreducible, $D(A)$ is strongly connected. Considering the paths from α to $\{1, 2, \dots, n\} \setminus \alpha$, we see that $D(A)$ has an arc (i, j) with $i \in \alpha$ and $j \notin \alpha$. Hence $a_{ij} = *$. But then there are at least three nonzero entries in row i of A, which contradicts Lemma 8. □

Lemma 10. *Let F be a field with $|F| \geqslant 3$ and let $A \in SI_n(F)$. If A is irreducible, then A has no cycle of length* 3.

Proof. By Lemma 9, it suffices to consider the case $n = 3$. Let $A = (a_{ij})$ be of order 3. To the contrary, assume that A has a cycle of length 3, say $\gamma = a_{i_1i_2}a_{i_2i_3}a_{i_3i_1}$. Then by Lemma 6, A has the reverse cycle $\bar{\gamma} = a_{i_1i_3}a_{i_3i_2}a_{i_2i_1}$. Now consider $A(i_1)$. Since $a_{i_2i_3}, a_{i_3i_2}$ is a nonzero transversal of $A(i_1)$, by Lemma 1, $A(i_1)$ is potentially nonsingular over F. Then by Lemma 3, there is a nonsingular matrix $B \in Z_F(A)$ such that $B(i_1)$ is also nonsingular. Thus $B^{-1}(i_1, i_1) = \det B(i_1)/\det B \neq 0$. Since $A \in SI_n(F)$, $B(i_1, i_1) \neq 0$. Hence $a_{i_1i_1} = *$. But then there are three nonzero entries in row i_1 of A, which contradicts Lemma 8. Thus A has no cycle of length 3. □

If (i, j) is an arc of a digraph with $i \neq j$, then i is called an *in-neighbor* of j and j is called an *out-neighbor* of i.

Lemma 11. *Let F be a field with $|F| \geqslant 3$ and let $A \in SI_n(F)$ be irreducible. If A has no cycle of length 4, then $n \leqslant 2$.*

Proof. Let $A = (a_{ij})$. To the contrary, assume that $n \geqslant 3$. Since A is irreducible, $D(A)$ is strongly connected. Thus each vertex of $D(A)$ has at least one in-neighbor and one out-neighbor. By Lemma 7, $A = A^T$. Thus for each vertex of $D(A)$, its in-neighbors are just its out-neighbors and vice versa. If each vertex of $D(A)$ has exactly one in-neighbor (out-neighbor), then $D(A)$ is the disjoint union of cycles of length 2 plus possibly some loops, which contradicts its strong connectivity. Thus some vertex of $D(A)$ has at least two in-neighbors (out-neighbors); i.e., there exist pairwise distinct i, j, k such that $a_{ij} = a_{ik} = a_{ji} = a_{ki} = *$.

Now we show that for every vertex $p \neq i$, either $a_{jp} = 0$ or $a_{pk} = 0$. Otherwise there exists $p \neq i$ such that $a_{jp} = a_{pk} = *$. We distinguish two cases.

(i) $p = j$ or k. Then $a_{jk} = *$. It follows that A has a cycle $a_{ij}a_{jk}a_{ki}$ of length 3, which contradicts Lemma 10.

(ii) $p \neq j, k$. Then A has a cycle $a_{ij}a_{jp}a_{pk}a_{ki}$ of length 4, which contradicts our assumption.

Let $B \in Z_F(A)$ be nonsingular. Since $A \in SI_n(F)$, $B^{-1} \in Z_F(A)$. But then we have

$$0 = \left(B \cdot B^{-1}\right)(j,k) = \sum_{p=1}^{n} B(j,p)B^{-1}(p,k) = B(j,i)B^{-1}(i,k) \neq 0,$$

a contradiction. This proves that $n \leqslant 2$. □

The following theorem determines the irreducible zero–nonzero patterns in $SI_n(F)$.

Theorem 12. *Let F be a field with $|F| \geqslant 3$ and let A be an irreducible zero–nonzero pattern of order n. Then $A \in SI_n(F)$ if and only if A is one of the following patterns:*

$$[*], \quad \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}, \quad \begin{bmatrix} * & * \\ * & * \end{bmatrix}, \quad \begin{bmatrix} 0 & * & 0 & * \\ * & 0 & * & 0 \\ 0 & * & 0 & * \\ * & 0 & * & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & * & * & 0 \\ * & 0 & 0 & * \\ * & 0 & 0 & * \\ 0 & * & * & 0 \end{bmatrix}.$$

Proof. The sufficiency can be proved by a simple calculation. Note that the last three patterns above are pairwise permutation similar.

Next we prove the necessity. Let $A \in SI_n(F)$ be irreducible. There are two cases.

(i) A has no cycle of length 4. Then by Lemma 11, $n \leqslant 2$. When $n = 1$, clearly $A = [*]$. When $n = 2$, it is easy to verify that $A = \left[\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} * & * \\ * & * \end{smallmatrix}\right]$.

(ii) A has a cycle of length 4. Then by Lemmas 7 and 9, A is symmetric and $n = 4$. By considering the order of the vertices in the corresponding cycle of the digraph $D(A)$, it is easy to check that A must be one of the following forms:

$$\begin{bmatrix} ? & * & ? & * \\ * & ? & * & ? \\ ? & * & ? & * \\ * & ? & * & ? \end{bmatrix}, \quad \begin{bmatrix} ? & ? & * & * \\ ? & ? & * & * \\ * & * & ? & ? \\ * & * & ? & ? \end{bmatrix}, \quad \begin{bmatrix} ? & * & * & ? \\ * & ? & ? & * \\ * & ? & ? & * \\ ? & * & * & ? \end{bmatrix},$$

where ?'s denote 0 or $*$. By Lemma 8, all the ?'s must be 0. Thus the necessity is proved. □

With an easy further analysis, Theorem 1.11 of [3] on sign patterns can be deduced from Theorem 12.

We remark that Lemmas 6–11 and Theorem 12 do not hold for the field F_2 with only two elements. Consider the following zero–nonzero pattern of order 6:

$$A = \begin{bmatrix} 0 & * & * & 0 & * & 0 \\ 0 & 0 & * & * & 0 & * \\ * & 0 & 0 & * & 0 & 0 \\ 0 & * & 0 & 0 & * & 0 \\ 0 & 0 & * & 0 & 0 & * \\ * & 0 & 0 & * & * & 0 \end{bmatrix}.$$

Note that $A = (a_{ij})$ has a cycle $a_{12}a_{23}a_{34}a_{45}a_{56}a_{61}$ of length 6. This implies that $D(A)$ is strongly connected and thus A is irreducible. A direct calculation shows that the only matrix B in $Z_{F_2}(A)$ satisfies $B^2 = I$. Then $B^{-1} = B \in Z_{F_2}(A)$ and thus $A \in SI_n(F_2)$. It is easy to verify that A satisfies all the conditions except the condition on the cardinality of the field in Lemmas 6–11 and Theorem 12, but none of their conclusions holds.

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