On the metric subgraphs of a graph[*]

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Abstract

The three subgraphs of a connected graph induced by the center, annulus and periphery are called its metric subgraphs. The main results are as follows. (1) There exists a graph of order n whose metric subgraphs are all paths if and only if $n \geq 13$ and the smallest size of such a graph of order 13 is 22; (2) there exists a graph of order n whose metric subgraphs are all cycles if and only if $n \geq 15$, and there are exactly three such graphs of order 15; (3) for every integer $k \geq 3$, we determine the possible orders for the existence of a graph whose metric subgraphs are all connected k-regular graphs; (4) there exists a graph of order n whose metric subgraphs are connected and pairwise isomorphic if and only if $n \geq 24$ and n is divisible by 3. An unsolved problem is posed.

Key words. Center; annulus; periphery; metric subgraphs; path; cycle

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1   Introduction

We consider finite simple graphs. For terminology and notations we follow the books [3] and [11]. The *order* of a graph G, denoted $|G|$, is its number of vertices, and the *size* is its number of edges. We denote by $V(G)$ and $E(G)$ the vertex set and edge set of a graph G respectively. Denote by $d_G(u, v)$ the distance between two vertices u and v in G. If the graph G is clear from the context, we simply write $d(u, v)$. The *eccentricity*, denoted

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by $e(v)$, of a vertex v in a graph G is the distance to a vertex farthest from v. Thus $e(v) = \max\{d(v, u) | u \in V(G)\}$. If $e(v) = d(v, x)$, then the vertex x is called an *eccentric vertex* of v. The *radius* of a graph G, denoted rad(G), is the minimum eccentricity of all the vertices in $V(G)$, whereas the *diameter* of G, denoted diam(G), is the maximum eccentricity. For graphs we will use equality up to isomorphism, so $G_1 = G_2$ means that G_1 and G_2 are isomorphic. A graph is called *null* if it has order 0; otherwise it is *non-null.*

Let G be a connected graph. A vertex u is a *central vertex* of G if $e(u) = $ rad(G). The *center* of G, denoted $C(G)$, is the set of all central vertices of G. A vertex v is a *peripheral vertex* of G if $e(v) = $ diam(G). The *periphery* of G, denoted $P(G)$, is the set of all peripheral vertices. A vertex w is an *annular vertex* of G if rad$(G) < e(w) < $ diam(G). The *annulus* of G, denoted $A(G)$, is the set of all annular vertices.

Definition 1. Let G be a connected graph. The subgraph of G induced by its center is called the *central subgraph* of G; the subgraph of G induced by its annulus is called the *annular subgraph* of G; the subgraph of G induced by its periphery is called the *peripheral subgraph* of G. These three subgraphs are called the *metric subgraphs* of G.

A self-centered graph has empty annulus, and hence its annular graph is the null graph. There are much work studying the central subgraph (e.g. [2], [3, Chapter 2], [7], [8]), some studying the peripheral subgraph ([1], [3], [4]) and little studying the annular subgraph [5]. In this paper, we consider these three subgraphs as a whole.

We will determine possible orders for the existence of a graph whose metric subgraphs are either all paths, or all cycles, or all connected k-regular graphs with $k \geq 3$, or connected and pairwise isomorphic graphs, and we also consider the smallest size problem. At the end we pose an unsolved problem.

2 Main results

For two graphs G and H, $G \vee H$ denotes the *join* of G and H, which is obtained from the disjoint union $G + H$ by adding edges joining every vertex of G to every vertex of H. Given two vertex subsets S and T of a graph, we denote by $[S, T]$ the set of edges having one endpoint in S and the other in T. P_n, C_n and K_n denote the path of order n, the cycle of order n and the complete graph of order n respectively. As usual, qK_2 denotes the graph consisting of q pairwise vertex-disjoint edges, and deg(v) denotes the degree of a vertex v. We denote by $\overline{G}$ the complement of a graph G.

Definition 2. Given a sequence of graphs $H_1, H_2, \ldots, H_p$, their *circular join* is defined to be the graph obtained from the disjoint union $H_1 + H_2 + \ldots + H_p$ by adding edges joining each vertex of H_i to each vertex of H_{i+1} for every $i = 1, 2, \ldots, p$ where H_{p+1} means H_1.

Definition 3. Let G and H be two graphs with $|G| \leq |H|$. Labeling the vertices of G and H by $x_1, \ldots, x_s$ and $y_1, \ldots, y_t$ respectively, the *nice connection* of G and H with respect to this vertex labeling is the graph obtained from the disjoint union $G + H$ by adding the edges $x_i y_i$, $i = 1, 2, \ldots, s$ and the edges $x_1 y_j$, $j = s + 1, \ldots, t$.

Given two graphs, there are possibly many nice connections of them, depending on the vertex labeling. For our purposes below, any nice connection works.

We will need the following lemmas.

Lemma 1 (Lesniak [10]). *Let G be a connected graph of order n. Then for every integer k with $\mathrm{rad}(G) < k \leq \mathrm{diam}(G)$, there exist at least two vertices in G of eccentricity k.*

Lemma 2. *If G is a graph with a nonempty annulus, then $\mathrm{rad}(G) \geq 2$, $\mathrm{diam}(G) \geq 4$, and $|A(G)| \geq 2$.*

Proof. Since $\mathrm{diam}(G) \leq 2\,\mathrm{rad}(G)$ [11, p.78], if $\mathrm{rad}(G) = 1$ then $\mathrm{diam}(G) \leq 2$, implying that the annulus of G is empty, a contradiction. Hence $\mathrm{rad}(G) \geq 2$. Consequently $\mathrm{diam}(G) \geq \mathrm{rad}(G) + 2 \geq 4$.

The assertion $|A(G)| \geq 2$ follows from Lemma 1 and the condition that G has a nonempty annulus. $\square$

We denote by $e_G(v)$ the eccentricity of a vertex v in G. Recall that $|G|$ denotes the order of a graph G. The following lemma is of independent interest.

Lemma 3. *Let H be the peripheral subgraph of a connected graph G. If H is connected, then $\mathrm{rad}(H) \geq \mathrm{diam}(G)$ and $|H| \geq 2\,\mathrm{diam}(G)$.*

Proof. Let v be a central vertex of H and let x be an eccentric vertex of v in G. Then $x \in P(G) = V(H)$. We have

$$\mathrm{diam}(G) = e_G(v) = d_G(v, x) \leq d_H(v, x) \leq e_H(v) = \mathrm{rad}(H),$$

showing that $\mathrm{rad}(H) \geq \mathrm{diam}(G)$. Combining this inequality with the fact that $\mathrm{rad}(H) \leq |H|/2$ we obtain $|H| \geq 2\,\mathrm{diam}(G)$. $\square$

For a graph G and $S \subseteq V(G)$, the *neighborhood* of S is defined to be $N(S) = \{x \in$

$V(G) \setminus S \,|\, x$ has a neighbor in S}. We will repeatedly use the fact that the eccentricities of two adjacent vertices differ by at most 1.

Lemma 4. *Let W be the annular subgraph of a connected graph G. If W is non-null and connected, then $\operatorname{rad}(W) \geq 2$ and consequently $|A(G)| \geq 4$.*

Proof. Clearly $N(C(G)) \subseteq A(G)$. To the contrary, suppose $\operatorname{rad}(W) = 1$. Let v be a central vertex of W. Denote $r = \operatorname{rad}(G)$. Let x be any vertex in $V(G)$. If $x \in C(G)$ then $d(v, x) \leq r$; if $x \in A(G)$ then $d(v, x) \leq 1 \leq r$. If $x \in P(G)$, choose any vertex $y \in C(G)$ and let $Q = y, ..., z, ..., x$ be a shortest (y, x)-path in G where $z \in A(G)$. Then Q has length at most r and the subpath $Q[z, x]$ has length at most $r - 1$. Again we have $d(v, x) \leq d(v, z) + d(z, x) \leq 1 + (r - 1) \leq r$. This shows that $e(v) \leq r$, a contradiction. Hence $\operatorname{rad}(W) \geq 2$.

Since a connected graph of order at most 3 has radius at most 1, we obtain $|A(G)| \geq 4$. □

Lemma 5. *If a connected graph has a connected peripheral subgraph and a non-null connected annular subgraph, then its order is at least 13.*

Proof. Suppose G is a connected graph whose peripheral subgraph is connected and whose annular subgraph is non-null and connected. Consider the diameter. By Lemma 2, $\operatorname{diam}(G) \geq 4$. Then by Lemma 3, $|P(G)| \geq 2 \operatorname{diam}(G) \geq 8$. Lemma 4 gives $|A(G)| \geq 4$. Note that every non-null graph has at least one central vertex. We obtain

$$|G| = |C(G)| + |A(G)| + |P(G)| \geq 1 + 4 + 8 = 13.$$

This completes the proof. □

Theorem 6. *There exists a connected graph of order n whose metric subgraphs are all paths if and only if $n \geq 13$, and the smallest size of such a graph of order 13 is 22.*

Proof. If G is a connected graph of order n whose metric subgraphs are all paths, then by Lemma 5, $n \geq 13$.

Conversely, for every $n \geq 13$ we will construct such a graph of order n. First, the graph G_1 in Figure 1 is a graph of order 13 whose metric subgraphs are all paths.

Fig. 1. The graph G_1

The central subgraph, the annular subgraph and the peripheral subgraph of G_1 are the paths u, $Q = v_1 v_2 v_3 v_4$, and $w_1 w_2 ... w_8$ respectively. Next for a given integer $n \geq 14$, in G_1 replacing the vertex u by the path P_{n-12} and then taking the join of P_{n-12} and the path Q we obtain a graph. This is a graph of order n whose metric subgraphs are all paths.

Finally we show that the smallest size of such a graph of order 13 is 22. Let H be a graph of order 13 whose metric subgraphs are all paths. Denote the center, annulus and periphery of H by C, A and P respectively. By the proof of Lemma 5, we have

$$|C| = 1, \ \ |A| = 4, \ \ |P| = 8, \ \ \text{diam}(H) = 4 \ \text{ and hence } \ \text{rad}(H) = 2.$$

Let $C = \{u\}$, let the annular subgraph of H be the path $v_1 v_2 v_3 v_4$, and let the peripheral subgraph of H be the path $w_1 w_2 ... w_8$. Then $e(u) = 2$, $e(v_i) = 3$, $i = 1, \ldots, 4$, and $e(w_j) = 4$, $j = 1, \ldots, 8$. Since the eccentricities of two adjacent vertices differ by at most 1, $N(u) \subseteq A$. Now the condition $e(u) = 2$ implies that each vertex in P has a neighbor in A. Hence $|[P, A]| \geq 8$. Clearly, any eccentric vertex of a vertex in P lies in P. Since $e(v_2) = 3$, every eccentric vertex of v_2 lies in P. Let w_k be an eccentric vertex of v_2. Then v_4 is the unique neighbor of w_k in A. Since $d(u, w_k) \leq 2$, u and v_4 are adjacent. Similarly, considering the vertex v_3 we deduce that u and v_1 are adjacent. We claim that $N(u) = A$. Otherwise $\deg(u) \leq 3$. Without loss of generality, suppose v_3 and u are nonadjacent. Denote $S = \{v_1, v_2\}$ and $T = \{v_4\}$. Then each vertex in P has a neighbor in $S \cup T$. We will repeatedly use this fact. Since w_1 is the only possible eccentric vertex of w_5, we have $d(w_1, w_5) = 4$. Let $x \in S \cup T$ be a neighbor of w_1 and let $y \in S \cup T$ be a neighbor of w_5. Then we have the following two possible cases.

Case 1. $x \in S$ and $y \in T$. Since w_8 is the unique eccentric vertex of w_4, w_8 and v_4 are nonadjacent. Hence w_8 has a neighbor in S. Consequently, the neighbor of w_4 in $S \cup T$ must be v_4. To keep $d(w_4, w_8) = 4$, w_7 and v_4 cannot be adjacent. Thus w_7 has a neighbor

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in S. Similarly, to keep $d(w_5, w_1) = 4$, w_2 and v_4 cannot be adjacent. Thus w_2 has a neighbor in S. Now w_6 is the only eccentric vertex of w_2. Hence w_6 has no neighbor in S, implying that v_4 is the only neighbor of w_6 in $S \cup T$. Since w_3 has a neighbor in $S \cup T$, we deduce that $e(w_7) \leq 3$, a contradiction.

Case 2. $x \in T$ and $y \in S$. The condition $e(w_5) = 4$ implies that the two vertices w_4 and w_6 are nonadjacent to v_4. Thus, both w_4 and w_6 have a neighbor in S. Since w_8 is the unique eccentric vertex of w_4, w_8 and v_4 are adjacent and w_3 has a neighbor in S. If w_7 is adjacent to v_4, using the fact that w_2 has a neighbor in $S \cup T$ we deduce that $e(w_6) \leq 3$, a contradiction. Hence w_7 has a neighbor in S. But then $e(w_7) \leq 3$, a contradiction again. This shows that $\deg(u) = 4$.

We conclude that the size of H is at least $3 + 7 + 8 + 4 = 22$. Conversely, the graph G_1 in Figure 1 is a graph of order 13 and size 22 whose metric subgraphs are all paths. This completes the proof. $\square$

Remark 1. By the proof of Theorem 6, it is not difficult to check that there are exactly 64 connected graphs of order 13 and size 22 whose metric subgraphs are all paths.

We will need the following two results.

Lemma 7 (Kim, Rho, Song and Hwang [9]) *Let G be a graph of order n with radius r and minimum degree k where $r \geq 3$ and $k \geq 2$. Then $n \geq 2r(k+1)/3$.*

Lemma 8. *Let k and n be integers with $1 \leq k \leq n - 1$. Then there exists a k-regular graph of order n if and only if kn is even. If kn is even and $k \geq 2$, then there exists a hamiltonian k-regular graph of order n.*

Lemma 8 can be found in [6, pp.12-13]. Its first part is well-known, but its hamiltonian part is usually not stated.

Next we determine the possible orders for the existence of a graph whose metric subgraphs are all connected and k-regular. The answer depends on the nature of k and there are six cases.

Theorem 9. *Let $k \geq 2$ be an integer and denote $q = \lfloor k/3 \rfloor$. There exists a connected graph of order n whose metric subgraphs are all connected and k-regular if and only if (1) $n \geq 14q + 6$ when $k \equiv 0 \bmod 3$ and q is even; (2) n is even and $n \geq 14q + 8$ when $k \equiv 0 \bmod 3$ and q is odd; (3) n is even and $n \geq 14q + 12$ when $k \equiv 1 \bmod 3$ and q is even; (4) $n \geq 14q + 11$ when $k \equiv 1 \bmod 3$ and q is odd; (5) $n \geq 14q + 15$ when $k \equiv 2 \bmod 3$ and q is even; (6) n is even and $n \geq 14q + 16$ when $k \equiv 2 \bmod 3$ and q is*

odd.

Proof. Let G be a connected graph of order n whose metric subgraphs are all connected and k-regular. Denote the center, annulus and periphery of G by C, A and P respectively.

Since the central subgraph $G[C]$ is k-regular, we have $|C| \geq k + 1$.

We then estimate the cardinality of A. Since the annular subgraph $W = G[A]$ is k-regular, we have $|A| \geq k + 1$ where equality holds if and only if W is complete. By Lemma 4, $\text{rad}(W) \geq 2$, implying that W is incomplete. Hence $|A| \geq k + 2$. When k is odd, the order of W must be even since W is k-regular, and consequently $|A| \geq k + 3$. It follows that (1) $|A| \geq 3q + 2$ when $k \equiv 0 \bmod 3$ and q is even; (2) $|A| \geq 3q + 3$ when $k \equiv 0 \bmod 3$ and q is odd; (3) $|A| \geq 3q + 4$ when $k \equiv 1 \bmod 3$ and q is even; (4) $|A| \geq 3q + 3$ when $k \equiv 1 \bmod 3$ and q is odd; (5) $|A| \geq 3q + 4$ when $k \equiv 2 \bmod 3$ and q is even; (6) $|A| \geq 3q + 5$ when $k \equiv 2 \bmod 3$ and q is odd.

Next we estimate the cardinality of P. Denote by H the peripheral subgraph of G. By Lemma 2, $\text{diam}(G) \geq 4$ and by Lemma 3, $\text{rad}(H) \geq \text{diam}(G)$. Thus $r = \text{rad}(H) \geq 4$. By Lemma 7 we obtain $|P| \geq 2r(k+1)/3$, from which we deduce the following information. (i) $|P| \geq 8q + 3$ when $k \equiv 0 \bmod 3$ and q is even; (ii) $|P| \geq 8q + 4$ when $k \equiv 0 \bmod 3$ and q is odd; (iii) $|P| \geq 8q + 6$ when $k \equiv 1 \bmod 3$; (iv) $|P| \geq 8q + 8$ when $k \equiv 2 \bmod 3$.

Using the above estimation we can obtain a lower bound for $n = |C| + |A| + |P|$. We have (1) $n \geq 14q + 6$ when $k \equiv 0 \bmod 3$ and q is even; (2) n is even and $n \geq 14q + 8$ when $k \equiv 0 \bmod 3$ and q is odd; (3) n is even and $n \geq 14q + 12$ when $k \equiv 1 \bmod 3$ and q is even; (4) $n \geq 14q + 11$ when $k \equiv 1 \bmod 3$ and q is odd; (5) $n \geq 14q + 15$ when $k \equiv 2 \bmod 3$ and q is even; (6) n is even and $n \geq 14q + 16$ when $k \equiv 2 \bmod 3$ and q is odd.

Conversely, for every order n in the range as stated in the theorem, we construct a connected graph of order n whose metric subgraphs are all connected and k-regular.

Notation. For a positive even integer f, we denote by $K_f - PM$ the graph obtained from K_f by deleting a perfect matching; i.e., $K_f - PM = \overline{(f/2)K_2}$. For an integer $g \geq 3$, we denote by $K_g - HC$ the graph $\overline{C_g}$. For an integer $s \geq 2$, we denote by $K_s - HP$ the graph $\overline{P_s}$.

Case (1). $k \equiv 0 \bmod 3$ and q is even. Now $k = 3q$ is even. Let $n \geq 14q + 6$. We have $n - (11q + 5) \geq k + 1$. By Lemma 8, there exists a connected k-regular graph

$C^{(1)}$ of order $n - (11q + 5)$. We denote by $P^{(1)}$ the circular join of the eight graphs $H_1^{(1)}, \ldots, H_8^{(1)}$ where $H_1^{(1)} = H_4^{(1)} = H_7^{(1)} = K_{q+1}$, $H_2^{(1)} = H_3^{(1)} = H_5^{(1)} = H_6^{(1)} = K_q$ and $H_8^{(1)} = K_q - PM$. Denote $A^{(1)} = K_{3q+2} - PM$. Partition the vertex set of $A^{(1)}$ into eight subsets $V_1^{(1)}, \ldots, V_8^{(1)}$ such that $A^{(1)}[V_i^{(1)}] = K_1$ for $i = 1, 5$, $A^{(1)}[V_j^{(1)}] = K_{q/2}$ for $j = 2, 3, 4, 6, 7, 8$ and $A^{(1)}[V_1^{(1)} \cup V_5^{(1)}] = \overline{K_2}$, $A^{(1)}[V_s^{(1)} \cup V_{s+4}^{(1)}] = K_q - PM$ for $s = 2, 3, 4$. Let $R_i^{(1)} = A^{(1)}[V_i^{(1)}]$, $i = 1, \ldots, 8$. Finally let M_1 be the graph obtained from the disjoint union $C^{(1)} + A^{(1)} + P^{(1)}$ by first taking a nice connection of $R_i^{(1)}$ and $H_i^{(1)}$ for $i = 1, \ldots, 8$ and then adding edges joining every vertex of $C^{(1)}$ to every vertex of $A^{(1)}$.

Note that every vertex in $H_i^{(1)}$ has a unique neighbor in $R_i^{(1)}$ for $i = 1, \ldots, 8$. It is easy to verify that $\mathrm{rad}(M_1) = 2$ and $\mathrm{diam}(M_1) = 4$, the three graphs $C^{(1)}$, $A^{(1)}$ and $P^{(1)}$ are connected and k-regular, and they are the central subgraph, annular subgraph and peripheral subgraph of M_1 which has order n.

Case (2). $k \equiv 0 \bmod 3$ and q is odd. Now $k = 3q$ is odd. Let $n \geq 14q+8$ and n is even. We have $n-(11q+7) \geq k+1$. By Lemma 8, there exists a connected k-regular graph $C^{(2)}$ of order $n - (11q + 7)$. We denote by $P^{(2)}$ the circular join of the eight graphs $H_1^{(2)}, \ldots, H_8^{(2)}$ where $H_1^{(2)} = H_2^{(2)} = H_5^{(2)} = H_6^{(2)} = K_q$, $H_3^{(2)} = H_4^{(2)} = H_7^{(2)} = H_8^{(2)} = K_{q+1} - PM$. Denote $A^{(2)} = K_{3q+3} - HC$. We distinguish two subcases.

Subcase (2.1). $q = 1$. In this case $A^{(2)} = K_6 - HC$. Let $V(A^{(2)}) = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and $E(A^{(2)}) = \{u_i u_j \mid j \neq i + 1\}$ where $u_7 = u_1$. Let M_2 be the graph obtained from the disjoint union $C^{(2)} + A^{(2)} + P^{(2)}$ by first adding edges joining u_i to each vertex in $H_j^{(2)}$ for $(i, j) = (1, 1), (2, 2), (2, 3), (3, 6), (3, 7), (4, 8), (5, 4), (6, 5)$, then adding edges joining every vertex of $C^{(2)}$ to every vertex of $A^{(2)}$. It is easy to verify that $\mathrm{rad}(M_2) = 2$ and $\mathrm{diam}(M_2) = 4$, the three graphs $C^{(2)}$, $A^{(2)}$ and $P^{(2)}$ are connected and k-regular, and they are the central subgraph, annular subgraph and peripheral subgraph of M_2 which has order n.

Subcase (2.2). $q \geq 3$. Let $V(A^{(2)}) = \{u_1, u_2, \ldots, u_{3q+3}\}$ and $E(A^{(2)}) = \{u_i u_j \mid j \neq i+1\}$ where $u_{3q+4} = u_1$. Partition the vertex set of $A^{(2)}$ into eight subsets (four pairs)

$$V_1^{(2)} = \{u_1\}, \quad V_5^{(2)} = \{u_2\}$$
$$V_2^{(2)} = \{u_{3+2j} \mid j = 0, 1, \ldots, (q - 3)/2\}, \quad V_6^{(2)} = \{u_{4+2j} \mid j = 0, 1, \ldots, (q - 3)/2\}$$
$$V_3^{(2)} = \{u_{q+2j} \mid j = 1, \ldots, (q + 1)/2\}, \quad V_7^{(2)} = \{u_{q+1+2j} \mid j = 1, \ldots, (q + 1)/2\}$$
$$V_4^{(2)} = \{u_{2q+1+2j} \mid j = 1, \ldots, (q + 1)/2\}, \quad V_8^{(2)} = \{u_{2q+2+2j} \mid j = 1, \ldots, (q + 1)/2\}$$

such that $A^{(2)}[V_j^{(2)}] = K_1$ for $j = 1, 5$, $A^{(2)}[V_j^{(2)}] = K_{(q-1)/2}$ for $j = 2, 6$, $A^{(2)}[V_j^{(2)}] =$

$K_{(q+1)/2}$ for $j = 3, 4, 7, 8$ and $A^{(2)}[V_1^{(2)} \cup V_5^{(2)}] = \overline{K_2}$, $A^{(2)}[V_2^{(2)} \cup V_6^{(2)}] = K_{q-1} - HP$, $A^{(2)}[V_s^{(2)} \cup V_{s+4}^{(2)}] = K_{q+1} - HP$ for $s = 3, 4$. Let $R_i^{(2)} = A^{(2)}[V_i^{(2)}]$, $i = 1, \ldots, 8$. Finally let M_2 be the graph obtained from the disjoint union $C^{(2)} + A^{(2)} + P^{(2)}$ by first taking a nice connection of $R_i^{(2)}$ and $H_i^{(2)}$ for $i = 1, \ldots, 8$ and then adding edges joining every vertex of $C^{(2)}$ to every vertex of $A^{(2)}$.

Note that every vertex in $H_i^{(2)}$ has a unique neighbor in $R_i^{(2)}$ for $i = 1, \ldots, 8$. It is easy to verify that $\mathrm{rad}(M_2) = 2$ and $\mathrm{diam}(M_2) = 4$, the three graphs $C^{(2)}$, $A^{(2)}$ and $P^{(2)}$ are connected and k-regular, and they are the central subgraph, annular subgraph and peripheral subgraph of M_2 which has order n.

Case (3). $k \equiv 1 \bmod 3$ and q is even. Now $k = 3q + 1$ is odd. Let n be even and $n \geq 14q + 12$. We have $n - (11q + 10) \geq k + 1$. By Lemma 8, there exists a connected k-regular graph $C^{(3)}$ of order $n - (11q + 10)$. We denote by $P^{(3)}$ the circular join of the eight graphs $H_1^{(3)}, \ldots, H_8^{(3)}$ where $H_1^{(3)} = H_2^{(3)} = K_{q+1}$, $H_3^{(3)} = H_8^{(3)} = K_q - PM$, $H_4^{(3)} = H_7^{(3)} = K_{q+2}$ and $H_5^{(3)} = H_6^{(3)} = K_q$. Denote $A^{(3)} = K_{3q+4} - HC$. Let $V(A^{(3)}) = \{u_1, u_2, \ldots, u_{3q+4}\}$ and $E(A^{(3)}) = \{u_i u_j \mid j \neq i + 1\}$ where $u_{3q+5} = u_1$. Partition the vertex set of $A^{(3)}$ into eight subsets (four pairs)

$$
\begin{aligned}
&V_1^{(3)} = \{u_1\}, \quad V_5^{(3)} = \{u_2\} \\
&V_2^{(3)} = \{u_{3+2j} \mid j = 0, 1, \ldots, (q-2)/2\}, \quad V_6^{(3)} = \{u_{4+2j} \mid j = 0, 1, \ldots, (q-2)/2\} \\
&V_3^{(3)} = \{u_{q+1+2j} \mid j = 1, \ldots, q/2\}, \quad V_7^{(3)} = \{u_{q+2+2j} \mid j = 1, \ldots, q/2\} \\
&V_4^{(3)} = \{u_{2q+1+2j} \mid j = 1, \ldots, (q+2)/2\}, \quad V_8^{(3)} = \{u_{2q+2+2j} \mid j = 1, \ldots, (q+2)/2\}
\end{aligned}
$$

such that $A^{(3)}[V_j^{(3)}] = K_1$ for $j = 1, 5$, $A^{(3)}[V_j^{(3)}] = K_{q/2}$ for $j = 2, 3, 6, 7$, $A^{(3)}[V_j^{(3)}] = K_{(q+2)/2}$ for $j = 4, 8$ and $A^{(3)}[V_1^{(3)} \cup V_5^{(3)}] = \overline{K_2}$, $A^{(3)}[V_s^{(3)} \cup V_{s+4}^{(3)}] = K_q - HP$ for $s = 2, 3$, $A^{(3)}[V_4^{(3)} \cup V_8^{(3)}] = K_{q+2} - HP$. Let $R_i^{(3)} = A^{(3)}[V_i^{(3)}]$, $i = 1, \ldots, 8$. Finally let M_3 be the graph obtained from the disjoint union $C^{(3)} + A^{(3)} + P^{(3)}$ by first taking a nice connection of $R_i^{(3)}$ and $H_i^{(3)}$ for $i = 1, \ldots, 8$ and then adding edges joining every vertex of $C^{(3)}$ to every vertex of $A^{(3)}$.

Note that every vertex in $H_i^{(3)}$ has a unique neighbor in $R_i^{(3)}$ for $i = 1, \ldots, 8$. It is easy to verify that $\mathrm{rad}(M_3) = 2$ and $\mathrm{diam}(M_3) = 4$, the three graphs $C^{(3)}$, $A^{(3)}$ and $P^{(3)}$ are connected and k-regular, and they are the central subgraph, annular subgraph and peripheral subgraph of M_3 which has order n.

Case (4). $k \equiv 1 \bmod 3$ and q is odd. Now $k = 3q + 1$ is even. Let $n \geq 14q + 11$. We have $n - (11q + 9) \geq k + 1$. By Lemma 8, there exists a connected k-regular graph $C^{(4)}$ of

order $n - (11q + 9)$. We denote by $P^{(4)}$ the circular join of the eight graphs $H_1^{(4)}, \ldots, H_8^{(4)}$ where $H_1^{(4)} = H_5^{(4)} = K_q$, $H_2^{(4)} = H_4^{(4)} = H_6^{(4)} = H_8^{(4)} = K_{q+1}$, $H_3^{(4)} = H_7^{(4)} = K_{q+1} - PM$. Denote $A^{(4)} = K_{3q+3} - PM$. We distinguish two subcases.

Subcase (4.1). $q = 1$. In this case $A^{(4)} = K_6 - PM$. Let $V(A^{(4)}) = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and $E(A^{(4)}) = \{u_i u_j \mid i < j, \ (i, j) \neq (1, 4), (2, 5), (3, 6)\}$. Let M_4 be the graph obtained from the disjoint union $C^{(4)} + A^{(4)} + P^{(4)}$ by first adding edges joining u_i to each vertex in $H_j^{(4)}$ for $(i, j) = (1, 1), (2, 2), (2, 3), (3, 4), (4, 5), (5, 6), (5, 7), (6, 8)$ and then adding edges joining every vertex of $C^{(4)}$ to every vertex of $A^{(4)}$. It is easy to verify that $\mathrm{rad}(M_4) = 2$ and $\mathrm{diam}(M_4) = 4$, the three graphs $C^{(4)}$, $A^{(4)}$ and $P^{(4)}$ are connected and k-regular, and they are the central subgraph, annular subgraph and peripheral subgraph of M_4 which has order n.

Subcase (4.2). $q \geq 3$. Partition the vertex set of $A^{(4)}$ into eight subsets $V_1^{(4)}, \ldots, V_8^{(4)}$ such that $A^{(4)}[V_j^{(4)}] = K_1$ for $j = 1, 5$, $A^{(4)}[V_j^{(4)}] = K_{(q-1)/2}$ for $j = 2, 6$, $A^{(4)}[V_j^{(4)}] = K_{(q+1)/2}$ for $j = 3, 4, 7, 8$ and $A^{(4)}[V_1^{(4)} \cup V_5^{(4)}] = \overline{K_2}$, $A^{(4)}[V_2^{(4)} \cup V_6^{(4)}] = K_{q-1} - PM$, $A^{(4)}[V_s^{(4)} \cup V_{s+4}^{(4)}] = K_{q+1} - PM$ for $s = 3, 4$. Let $R_i^{(4)} = A^{(4)}[V_i^{(4)}]$, $i = 1, \ldots, 8$. Finally let M_4 be the graph obtained from the disjoint union $C^{(4)} + A^{(4)} + P^{(4)}$ by first taking a nice connection of $R_i^{(4)}$ and $H_i^{(4)}$ for $i = 1, \ldots, 8$ and then adding edges joining every vertex of $C^{(4)}$ to every vertex of $A^{(4)}$.

Note that every vertex in $H_i^{(4)}$ has a unique neighbor in $R_i^{(4)}$ for $i = 1, \ldots, 8$. It is easy to verify that $\mathrm{rad}(M_4) = 2$ and $\mathrm{diam}(M_4) = 4$, the three graphs $C^{(4)}$, $A^{(4)}$ and $P^{(4)}$ are connected and k-regular, and they are the central subgraph, annular subgraph and peripheral subgraph of M_4 which has order n.

Case (5). $k \equiv 2 \bmod 3$ and q is even. Now $k = 3q + 2$ is even. Let $n \geq 14q + 15$. We have $n - (11q + 12) \geq k + 1$. By Lemma 8, there exists a connected k-regular graph $C^{(5)}$ of order $n - (11q + 12)$. We denote by $P^{(5)}$ the circular join of the eight graphs $H_1^{(5)}, \ldots, H_8^{(5)}$ where each $H_j^{(5)} = K_{q+1}$ for $j = 1, \ldots, 8$. Denote $A^{(5)} = K_{3q+4} - PM$. We distinguish two subcases.

Subcase (5.1). $q = 0$. In this case $k = 2$ and $A^{(5)} = C_4$. Let $V(A^{(5)}) = \{u_1, u_2, u_3, u_4\}$ and $E(A^{(5)}) = \{u_i u_{i+1} \mid i = 1, 2, 3, 4\}$ where $u_5 = u_1$. Let M_5 be the graph obtained from the disjoint union $C^{(5)} + A^{(5)} + P^{(5)}$ by first adding edges joining u_i to each vertex in $H_j^{(5)}$ for $(i, j) = (1, 1), (2, 2), (2, 3), (2, 4), (3, 5), (4, 6), (4, 7), (4, 8)$ and then adding edges joining every vertex of $C^{(5)}$ to every vertex of $A^{(5)}$. It is easy to verify that $\mathrm{rad}(M_5) = 2$ and $\mathrm{diam}(M_5) = 4$, the three graphs $C^{(5)}$, $A^{(5)}$ and $P^{(5)}$ are connected and k-regular, and

they are the central subgraph, annular subgraph and peripheral subgraph of M_5 which has order n.

Subcase (5.2). $q \geq 2$. Partition the vertex set of $A^{(5)}$ into eight subsets $V_1^{(5)}, \ldots, V_8^{(5)}$ such that $A^{(5)}[V_j^{(5)}] = K_1$ for $j = 1, 5$, $A^{(5)}[V_j^{(5)}] = K_{q/2}$ for $j = 2, 3, 6, 7$, $A^{(5)}[V_j^{(5)}] = K_{(q+2)/2}$ for $j = 4, 8$ and $A^{(5)}[V_1^{(5)} \cup V_5^{(5)}] = \overline{K_2}$, $A^{(5)}[V_s^{(5)} \cup V_{s+4}^{(5)}] = K_q - PM$ for $s = 2, 3$, $A^{(5)}[V_4^{(5)} \cup V_8^{(5)}] = K_{q+2} - PM$. Let $R_i^{(5)} = A^{(5)}[V_i^{(5)}]$, $i = 1, \ldots, 8$. Finally let M_5 be the graph obtained from the disjoint union $C^{(5)} + A^{(5)} + P^{(5)}$ by first taking a nice connection of $R_i^{(5)}$ and $H_i^{(5)}$ for $i = 1, \ldots, 8$ and then adding edges joining every vertex of $C^{(5)}$ to every vertex of $A^{(5)}$.

Note that every vertex in $H_i^{(5)}$ has a unique neighbor in $R_i^{(5)}$ for $i = 1, \ldots, 8$. It is easy to verify that $\mathrm{rad}(M_5) = 2$ and $\mathrm{diam}(M_5) = 4$, the three graphs $C^{(5)}$, $A^{(5)}$ and $P^{(5)}$ are connected and k-regular, and they are the central subgraph, annular subgraph and peripheral subgraph of M_5 which has order n.

Case (6). $k \equiv 2 \bmod 3$ and q is odd. Now $k = 3q + 2$ is odd. Let n be even and $n \geq 14q + 16$. We have $n - (11q + 13) \geq k + 1$. By Lemma 8, there exists a connected k-regular graph $C^{(6)}$ of order $n - (11q + 13)$. We denote by $P^{(6)}$ the circular join of the eight graphs $H_1^{(6)}, \ldots, H_8^{(6)}$ where each $H_j^{(6)} = K_{q+1}$ for $j = 1, \ldots, 8$. Denote $A^{(6)} = K_{3q+5} - HC$. Let $V(A^{(6)}) = \{u_1, u_2, \ldots, u_{3q+5}\}$ and $E(A^{(6)}) = \{u_i u_j \mid j \neq i + 1\}$ where $u_{3q+6} = u_1$. Partition the vertex set of $A^{(6)}$ into eight subsets (four pairs)

$$V_1^{(6)} = \{u_1\}, \quad V_5^{(6)} = \{u_2\}$$
$$V_2^{(6)} = \{u_{3+2j} \mid j = 0, 1, \ldots, (q-1)/2\}, \quad V_6^{(6)} = \{u_{4+2j} \mid j = 0, 1, \ldots, (q-1)/2\}$$
$$V_3^{(6)} = \{u_{q+2j} \mid j = 2, \ldots, (q+3)/2\}, \quad V_7^{(6)} = \{u_{q+1+2j} \mid j = 2, \ldots, (q+3)/2\}$$
$$V_4^{(6)} = \{u_{2q+1+2j} \mid j = 2, \ldots, (q+3)/2\}, \quad V_8^{(6)} = \{u_{2q+2+2j} \mid j = 2, \ldots, (q+3)/2\}$$

such that $A^{(6)}[V_j^{(6)}] = K_1$ for $j = 1, 5$, $A^{(6)}[V_j^{(6)}] = K_{(q+1)/2}$ for $j = 2, 3, 4, 6, 7, 8$ and $A^{(6)}[V_1^{(6)} \cup V_5^{(6)}] = \overline{K_2}$, $A^{(6)}[V_s^{(6)} \cup V_{s+4}^{(6)}] = K_{q+1} - HP$ for $s = 2, 3, 4$. Let $R_i^{(6)} = A^{(6)}[V_i^{(6)}]$, $i = 1, \ldots, 8$. Finally let M_6 be the graph obtained from the disjoint union $C^{(6)} + A^{(6)} + P^{(6)}$ by first taking a nice connection of $R_i^{(6)}$ and $H_i^{(6)}$ for $i = 1, \ldots, 8$ and then adding edges joining every vertex of $C^{(6)}$ to every vertex of $A^{(6)}$.

Note that every vertex in $H_i^{(6)}$ has a unique neighbor in $R_i^{(6)}$ for $i = 1, \ldots, 8$. It is easy to verify that $\mathrm{rad}(M_6) = 2$ and $\mathrm{diam}(M_6) = 4$, the three graphs $C^{(6)}$, $A^{(6)}$ and $P^{(6)}$ are connected and k-regular, and they are the central subgraph, annular subgraph and peripheral subgraph of M_6 which has order n. This completes the proof. $\square$

Remark 2. In Theorem 9, the condition of being connected on metric subgraphs is essential. For example, Theorem 9 asserts that the smallest order of a connected graph whose metric subgraphs are all cubic is 22. Let Q be the graph obtained from the disconnected graph in Figure 2 by adding all the edges $x_i y_j$ for $i = 1, \ldots, 4$ and $j = 1, \ldots, 6$. Then Q is a graph of order 18 whose metric subgraphs are all cubic. The peripheral subgraph of Q is disconnected.

Fig. 2. An auxiliary graph of order 18

Now we determine the smallest graphs whose metric subgraphs are all cycles.

Theorem 10. *The minimum order of a connected graph whose metric subgraphs are all cycles is* 15 *and there are exactly three such graphs of order* 15, *all of which have size* 35 *and are depicted in Figure 3.*

Proof. The case $k = 2$ of Theorem 9 asserts that the minimum order of a connected graph whose metric subgraphs are all cycles is 15. Let G be a connected graph of order 15 whose metric subgraphs are all cycles, and let C, A and P be the center, annulus and periphery of G respectively. Then $|C| + |A| + |P| = 15$. By the first four paragraphs of the proof of Theorem 9 we have $|C| \geq 3$, $|A| \geq 4$ and $|P| \geq 8$. Hence $|C| = 3$, $|A| = 4$ and $|P| = 8$. Denote by H the peripheral subgraph of G. Then $H = C_8$ and $\mathrm{rad}(H) = 4$.

By Lemma 2, $\mathrm{rad}(G) \geq 2$, $\mathrm{diam}(G) \geq 4$ and by Lemma 3, we have $4 = \mathrm{rad}(H) \geq \mathrm{diam}(G)$. Thus $\mathrm{diam}(G) = 4$ and consequently $\mathrm{rad}(G) = 2$, since A is nonempty. Combining the fact that the eccentricities of two adjacent vertices differ by at most 1 and the condition that $\mathrm{rad}(G) = 2$, we deduce that for any vertex $x \in C$ and any vertex $y \in P$, $d(x, y) = 2$. Let x, z, y be a path. Then $z \in A$. Thus every vertex in P has a neighbor in A. Note that the three metric subgraphs of G are the cycles C_3, C_4 and C_8. Let y' be the antipodal vertex of y on the even cycle $G[P]$. Then clearly y' is the unique eccentric

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vertex of y in G. The condition $d(y, y') = 4$ implies that y' has a unique neighbor z' in A which is the antipodal vertex of z on the cycle $G[A]$. Since y is the unique eccentric vertex of y', z is the unique neighbor of y in A. This shows that any vertex in P has a unique neighbor in A, implying that $|[P, A]| = 8$.

We assert that every vertex in A has a neighbor in P. To see this, choose any vertex $v \in A$. Let v' be the antipodal vertex of v in the even cycle $G[A]$. If v has no neighbor in P, then we would have $N(P) \subseteq A \setminus \{v\}$ and hence $e(v') \leq 2$, a contradiction. Next we assert that every vertex in C is adjacent to every vertex in A. To show this, given any $u \in C$ and $v \in A$ we let w be a neighbor of v in P. If u and v are nonadjacent, then $d(u, w) \geq 3$, contradicting $e(u) = 2$. Now $|[C, A]| = 12$. Also, the three metric subgraphs C_3, C_4 and C_8 contain $3 + 4 + 8 = 15$ edges. Altogether G has size $12 + 8 + 15 = 35$.

Combining the properties of G deduced above we conclude that G must be one of the three graphs depicted in Figure 3.

Fig. 3. The three graphs of order 15 whose metric subgraphs are all cycles

Conversely, the metric subgraphs of each of these three graphs are all cycles. This completes the proof. □

Theorem 11. *There exists a graph whose metric subgraphs are all isomorphic to a graph H if and only if either H is disconnected or H is connected and the radius of H is at least 4. There exists a graph of order n whose metric subgraphs are connected and pairwise isomorphic if and only if $n \geq 24$ and n is divisible by 3.*

Proof. Suppose G is a graph whose metric subgraphs are all isomorphic to a graph H. If H is connected, then by Lemma 2, $\mathrm{diam}(G) \geq 4$ and by Lemma 3, $\mathrm{rad}(H) \geq \mathrm{diam}(G)$. Hence $\mathrm{rad}(H) \geq 4$.

Conversely, let H be a given graph that is either disconnected or connected and

$\mathrm{rad}(H) \geq 4$. Recall that the eccentricity of any vertex in a disconnected graph is infinity. We will construct a graph Q whose metric subgraphs are all isomorphic to H. Let H_c, H_a and H_p be three pairwise vertex-disjoint copies of H, let $V(H_a) = \{x_1, \ldots, x_k\}$ and let f be an isomorphism from H_a to H_p. Then the graph Q is obtained from $H_c + H_a + H_p$ by first taking the join of H_c and H_a and then adding the edges $x_i f(x_i)$ for $i = 1, \ldots, k$. It is easy to verify that Q has radius 2 and diameter 4, and the central subgraph, annular subgraph and peripheral subgraph of Q are H_c, H_a and H_p respectively.

Next we prove the second assertion of Theorem 11. Suppose R is a graph of order n whose metric subgraphs are connected and pairwise isomorphic. Let W be the peripheral subgraph of R. By Lemma 2, $\mathrm{diam}(R) \geq 4$ and by Lemma 3, $|W| \geq 2\,\mathrm{diam}(R)$. Hence $|W| \geq 2 \times 4 = 8$. It follows that $n = 3|W| \geq 24$ and n is divisible by 3. Conversely, suppose $n \geq 24$ is an integer that is divisible by 3. Let H be the path $P_{n/3}$. By the first part of Theorem 11 proved above, there exists a graph whose metric subgraphs are all isomorphic to H. This completes the proof. $\square$

Lemma 5 shows that connectedness conditions on the metric subgraphs of a graph yields some restriction on the order of the graph. Finally we pose the following question.

Question. Let $k \geq 2$ be an integer. What is the smallest order of a graph whose metric subgraphs are all k-connected?

Using the information obtained in this paper, it is easy to show that the answer is 15 when $k = 2$.

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