ACI-matrices all of whose completions have the same rank*

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Abstract

We characterize the ACI-matrices all of whose completions have the same rank, determine the largest number of indeterminates in such partial matrices of a given size, and determine the partial matrices that attain this largest number.

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1 Introduction

A *partial matrix* over a set Ω is a matrix in which some entries all from Ω are specified and the other entries are free to be chosen from Ω. A *completion* of a partial matrix is a specific choice of values from Ω for its unspecified entries. A completion may also mean a completed matrix of a partial matrix. We call the unspecified entries *indeterminates* since they are free to range over Ω.

Let $M_{m,n}(\Omega)$ be the set of $m \times n$ matrices whose entries are from a given set Ω, and let $P_{m,n}(\Omega)$ be the set of $m \times n$ partial matrices over Ω. If $m = n$, then we use the abbreviations $M_n(\Omega)$ and $P_n(\Omega)$ respectively. We call elements in Ω *constants* and call matrices in $M_{m,n}(\Omega)$ *constant matrices*, in contrast to indeterminates and partial matrices respectively.

To study partial matrices, it is more convenient to consider a larger class of matrices for technical reasons. Let $\mathbf{F}[x_1, \ldots, x_k]$ be the ring of polynomials in the indeterminates $x_1, \ldots, x_k$

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with coefficients from a field $\mathbf{F}$. We call a matrix A over $\mathbf{F}[x_1, \ldots, x_k]$ an *affine column independent* (abbreviated as ACI) matrix if each entry of A is a polynomial of degree at most one and no indeterminate appears in two distinct columns of A. Obviously, every submatrix of an ACI-matrix is also an ACI-matrix. Given a partial matrix, we may label its unspecified entries with distinct indeterminates. Thus partial matrices are ACI-matrices. By a *completion* of a matrix over $\mathbf{F}[x_1, \ldots, x_k]$ we mean an assignment of values in $\mathbf{F}$ to the indeterminates $x_1, \ldots, x_k$. A completion may also mean a completed polynomial matrix.

The ACI-matrices all of whose completions are nonsingular and the ACI-matrices all of whose completions are singular are characterized in [1]. The following problem was also posed in [1, Problem 5]:

Problem 1 *Let $\mathbf{F}$ be a field. Characterize the ACI-matrices over $\mathbf{F}[x_1, \ldots, x_k]$ all of whose completions have the same rank.*

We will solve this problem under a minor condition on the field $\mathbf{F}$ and determine the maximum number of indeterminates in such partial matrices as well as the matrices attaining this maximum number.

2 Main Results

First we give some lemmas which will be used to prove our main results. $\mathbf{F}$ is a field throughout.

Lemma 1 ([**?**]) *Let A be an $m \times n$ ACI-matrix over $\mathbf{F}[x_1, \ldots, x_k]$. If $T \in M_m(\mathbf{F})$ is a constant matrix and $P \in M_n(\mathbf{F})$ is a permutation matrix, then TAP is an ACI-matrix over $\mathbf{F}[x_1, \ldots, x_k]$.*

A *proper ACI-matrix* is an ACI-matrix containing at least one indeterminate, i.e., it is not a constant matrix.

Lemma 2 *Let A be an $m \times n$ proper ACI-matrix over $\mathbf{F}[x_1, \ldots, x_k]$. Then there exists a nonsingular constant matrix $T \in M_m(\mathbf{F})$ and a permutation matrix $Q \in M_n(\mathbf{F})$ such that*

$$TAQ = \begin{bmatrix} b_1 & * & * & \cdots & * & * \\ c_1^{(1)} & b_2 & * & \cdots & * & * \\ c_1^{(2)} & c_2^{(1)} & b_3 & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_1^{(s-1)} & c_2^{(s-2)} & c_3^{(s-3)} & \cdots & b_s & * \\ c_1^{(s)} & c_2^{(s-1)} & c_3^{(s-2)} & \cdots & c_s^{(1)} & B \end{bmatrix} \qquad (1)$$

where for $j = 1, \ldots, s$, b_j is a column vector each of whose components is a polynomial of degree 1 in which there is an indeterminate that appears nowhere else in TAQ, $c_j^{(i)}$ are constant column *vectors for* $1 \le j \le s$, $1 \le i \le s-j+1$, *and* B *is a constant matrix.*

Proof. We use induction on n. The case for $n = 1$ is easy to check. Assume that the result holds for all proper ACI-matrices with $n-1$ columns and let A be an $m \times n$ proper ACI-matrix.

Suppose A has an entry in the position (r, t) that contains an indeterminate, say, x_1. We interchange rows 1 and r, and then interchange columns 1 and t to get a matrix $A_1 = P_0 A Q_0 = (\tilde{a}_{ij})$, where

$$\tilde{a}_{ij} = \tilde{a}_{ij}^{(0)} + \sum_{u=1}^{k} \tilde{a}_{ij}^{(u)} x_u,$$

$\tilde{a}_{11}^{(1)} \neq 0$, $P_0 \in M_m(\mathbf{F})$ and $Q_0 \in M_n(\mathbf{F})$ are permutation matrices. Adding $-\tilde{a}_{i1}^{(1)}/\tilde{a}_{11}^{(1)}$ times the first row to the i-th row in A_1 for $i = 2, \ldots, m$ successively we get a matrix $A_2 = T_1 A_1$, where $T_1 \in M_m(\mathbf{F})$ is the nonsingular matrix corresponding to these elementary row operations. Now x_1 appears only in the $(1, 1)$ position of A_2.

If there is another indeterminate in the first column of A_2 but not in the $(1, 1)$ position, say, x_2 in the $(i_1, 1)$ position, then interchange row i_1 and row 2 we get a new matrix $A_3 = PA_2$ where P is a permutation matrix. Suppose the coefficient of x_2 in position $(i, 1)$ of A_3 is u_{i1}, $i = 1, 2, \ldots, m$. Adding $-u_{i1}/u_{21}$ times the second row to the i-th row for $i = 1, 3, 4, \ldots, m$ we get a new matrix $A_4 = T_2 A_3$ where T_2 is a nonsingular constant matrix. Now x_1 and x_2 appear only in the $(1, 1)$ position and the $(2, 1)$ position of A_4 respectively. If there is an indeterminate in the last $m - 2$ components of the first column of A_4, continue in this way until we get

$$A_5 = T_3 A_4 = \begin{bmatrix} \tilde{b}_1 & B_1 \\ c_1 & B_2 \end{bmatrix}$$

where c_1 is a constant column vector and $\tilde{b}_1$ is a column vector each of whose components is a polynomial of degree 1 in which there is an indeterminate that appears nowhere else in A_5.

If c_1 is void in A_5 or B_2 is a constant matrix, we have already finished the proof since A_5 has the form in (1). Otherwise let the length of c_1 be $l \ge 1$ and assume that B_2 contains at least one indeterminate. Note that B_2 is an ACI-matrix by Lemma 1. Use the induction we know that there exists a nonsingular constant matrix $T_4 \in M_l(\mathbf{F})$ and a permutation matrix $Q_1 \in M_{n-1}(\mathbf{F})$ such that $T_4 B_2 Q_1$ has the form in (1), i.e.,

$$T_4 B_2 Q_1 = \begin{bmatrix} b_2 & * & \cdots & * & * \\ c_2^{(1)} & b_3 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2^{(s-2)} & c_3^{(s-3)} & \cdots & b_s & * \\ c_2^{(s-1)} & c_3^{(s-2)} & \cdots & c_s^{(1)} & B \end{bmatrix}$$

where $b_2, \ldots, b_s$ are column vectors each of whose components is a polynomial of degree 1 in which there is an indeterminate that appears nowhere else in $T_4B_2Q_1$, $c_j^{(i)}$ are constant column vectors for $2 \le j \le s$, $1 \le i \le s-j+1$, and B is a constant matrix.

Denote by I_t the identity matrix of order t. Let

$$A_6 = (I_{m-l} \oplus T_4)A_5(1 \oplus Q_1) = \begin{bmatrix} \tilde{b}_1 & B_1Q_1 \\ T_4c_1 & T_4B_2Q_1 \end{bmatrix}.$$

If some entries of B_1Q_1 contain the same indeterminates as those in $b_2, \ldots, b_s$ that appear only once in $T_4B_2Q_1$ mentioned above, by using elementary row operations on A_6 we can make these indeterminates vanish in B_1Q_1, i.e., there exists a nonsingular matrix $T_5 \in M_m(\mathbf{F})$ such that T_5A_6 has form (1).

Set $T = T_5(I_{m-l} \oplus T_4)T_3T_2PT_1P_0$ and $Q = Q_0(1 \oplus Q_1)$. Then

$$TAQ = T_5(I_{m-l} \oplus T_4)T_3T_2PT_1P_0AQ_0(1 \oplus Q_1) = T_5A_6$$

has form (1). □

We use $|S|$ to denote the cardinality of a set S.

Lemma 3 *Let $m \ge n$ be positive integers, let $\mathbf{F}$ be a field with $|\mathbf{F}| \ge m$ and let A be an $m \times n$ ACI-matrix over $\mathbf{F}[x_1, \ldots, x_k]$. If all the completions of A have rank n, then there exists a nonsingular constant matrix $T \in M_m(\mathbf{F})$ and a permutation matrix $Q \in M_n(\mathbf{F})$ such that*

$$TAQ = \begin{bmatrix} * \\ U \end{bmatrix}$$

where U is an $n \times n$ upper triangular ACI-matrix with nonzero constant diagonal entries.

Proof. We prove the lemma in the equivalent form:

There exists a nonsingular constant matrix $T \in M_m(\mathbf{F})$ and a permutation matrix Q such that

$$TAQ = \begin{bmatrix} L \\ * \end{bmatrix} \tag{2}$$

where L is an $n \times n$ lower triangular ACI-matrix with nonzero constant diagonal entries.

We use induction on n to prove this equivalent version. For $n = 1$ the result follows from Lemma 2. Assume the result holds for all ACI-matrices over $\mathbf{F}[x_1, \ldots, x_k]$ with $n-1$ columns and let A be an $m \times n$ ACI-matrix all of whose completions have rank n.

Case 1. A has a constant column, say, the j-th column which has a nonzero entry, say, the t-th entry, since A cannot have zero columns. Without loss of generality, $A_0 = P_1AQ_1 = (\tilde{a}_{ij})$

with $\tilde{a}_{11} \neq 0$ where P_1, Q_1 are permutation matrices of orders m and n respectively, and the entries in the first column of A_0 are constants. Adding $-\tilde{a}_{i1}/\tilde{a}_{11}$ times the first row to the i-th row of A_0 for $i = 2, \ldots, m$ successively we get a matrix $A_1 = T_1 A_0$, where $T_1 \in M_m(\mathbf{F})$ is the nonsingular matrix corresponding to these elementary row operations. Partition A_1 as

$$A_1 = T_1 A_0 = \begin{bmatrix} \tilde{a}_{11} & u^T \\ 0 & H \end{bmatrix}. \tag{3}$$

By Lemma 1, A_1 and hence H is an ACI-matrix. Now all completions of A_1 have rank n. Thus all completions of H have rank $n-1$. By the induction hypothesis on H, there exists a nonsingular constant matrix $T_2 \in M_{m-1}(\mathbf{F})$ and a permutation matrix $Q_2 \in M_{n-1}(\mathbf{F})$ such that $T_2 H Q_2$ is of form (2).

Denote

$$Q_0 = \begin{bmatrix} 0 & I_{n-1} \\ 1 & 0 \end{bmatrix},$$

the basic circulant permutation matrix. Set $T = (Q_0 \oplus I_{m-n})(1 \oplus T_2) T_1 P_1$ and $Q = Q_1 (1 \oplus Q_2) Q_0^T$. Then $T \in M_m(\mathbf{F})$ is a nonsingular constant matrix, Q is a permutation matrix and TAQ is of form (2).

Case 2. A has no constant column. By Lemma 2 there exists a nonsingular constant matrix $T_3 \in M_m(\mathbf{F})$ and a permutation matrix Q_3 such that $T_3 A Q_3$ has form (1). Set

$$A_2 \equiv T_3 A Q_3 = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

where $B_2 = (c_1^{(s)}, c_2^{(s-1)}, \ldots, c_s^{(1)}, B)$. We claim that B_2 is nonvoid and $B_2 \neq 0$. Otherwise, in A_2 adding the j-th column to the first column for $2 \leq j \leq s$ and choosing suitable values for the indeterminates successively we can make the sum of the first s columns be a zero vector by the property of $b_1, \ldots, b_s$ stated in Lemma 2. This contradicts the fact that all completions of A and hence all completions of A_2 have rank n.

Let B_2 be $p \times n$ and let the rank of B_2 be $r \geq 1$. Then there exists a nonsingular constant matrix $T_4 \in M_p(\mathbf{F})$ and a permutation matrix $Q_4 \in M_n(\mathbf{F})$ such that

$$T_4 B_2 Q_4 = \begin{bmatrix} D & E \\ 0 & 0 \end{bmatrix}$$

where $D = \mathrm{diag}(d_1, \ldots, d_r)$ is a diagonal matrix with each d_i nonzero. Let $T_5 = I_{m-p} \oplus T_4$ and

$$A_3 \equiv T_5 A_2 Q_4 = T_5 T_3 A Q_3 Q_4 = \begin{bmatrix} B_1 Q_4 \\ T_4 B_2 Q_4 \end{bmatrix}.$$

If $r = n$, i.e., E is void, then by considering row permutations of A_3 we easily see that the conclusion holds. Next suppose $r < n$. Now we prove that E has at least one zero row. To the contrary suppose that E has no zero row.

Let $t = m - p$, $g = n - r$. Partition $B_1 Q_4 = (C_1, C_2)$ where C_1 is $t \times r$ and C_2 is $t \times g$. Let

$$R_1 = \begin{bmatrix} I_r & -D^{-1}E \\ 0 & I_{n-r} \end{bmatrix}, \quad A_4 \equiv A_3 R_1 = \begin{bmatrix} C_1 & Z \\ D & 0 \\ 0 & 0 \end{bmatrix}$$

where $Z = -C_1 D^{-1} E + C_2$. By Lemma 2, each row of $B_1 Q_4 = (C_1, C_2)$ contains an indeterminate that appears only in one position of A_2. Without loss of generality we may suppose that $x_1, \ldots, x_t$ are these indeterminates and x_i appears in the i-th row of (C_1, C_2), $i = 1, \ldots, t$. Since x_i appears only in one position of (C_1, C_2) and $D^{-1}E$ has no zero row, x_i appears in the i-th row of $Z = -C_1 D^{-1} E + C_2$. Note that Z is $t \times g$.

Let

$$Z = \begin{bmatrix} w_{11}x_1 + \cdots & w_{12}x_1 + \cdots & \cdots & w_{1g}x_1 + \cdots \\ w_{21}x_2 + \cdots & w_{22}x_2 + \cdots & \cdots & w_{2g}x_2 + \cdots \\ \vdots & \vdots & \ddots & \vdots \\ w_{t1}x_t + \cdots & w_{t2}x_t + \cdots & \cdots & w_{tg}x_t + \cdots \end{bmatrix}$$

where $w_{ij} \in \mathbf{F}$ and for each $1 \le i \le t$, $w_{ij}, j = 1, \ldots, g$ are not all zero. We show that there exist $k_j \in \mathbf{F}, j = 1, \ldots, g-1$ such that

$$w_{ig} + k_{g-1} w_{i,g-1} + \cdots + k_1 w_{i1} \neq 0 \quad \text{for } i = 1, \ldots, t. \tag{4}$$

In fact we may successively choose k_j such that if $w_{i_0,j} \neq 0$ for some i_0, j, then

$$w_{i_0,g} + k_{g-1} w_{i_0,g-1} + \cdots + k_j w_{i_0,j} \neq 0. \tag{5}$$

(4) will follow from (5) since w_{ij}, $j = 1, \ldots, g$ are not all zero for each $1 \le i \le t$. If $w_{i,g-1} = 0$ for all $1 \le i \le t$, we choose $k_{g-1} = 0$. Otherwise let $w_{i_1,g-1}, \ldots, w_{i_v,g-1}$ be the nonzero elements among $w_{1,g-1}, \ldots, w_{t,g-1}$. For every $q = 1, \ldots, v$, the equation $w_{i_q,g} + y w_{i_q,g-1} = 0$ has only one solution, i.e., $y = -w_{i_q,g-1}^{-1} w_{i_q,g}$. Since $v \le t < m$ and $|\mathbf{F}| \ge m$, there exists $k_{g-1} \in \mathbf{F}$ such that $w_{i_q,g} + k_{g-1} w_{i_q,g-1} \neq 0$ holds for all $q = 1, \ldots, v$. Next if $w_{i,g-2} = 0$ for all $1 \le i \le t$, we choose $k_{g-2} = 0$. Otherwise, as above there exists $k_{g-2} \in \mathbf{F}$ such that $w_{ig} + k_{g-1} w_{i,g-1} + k_{g-2} w_{i,g-2} \neq 0$ holds for all those i for which $w_{i,g-2} \neq 0$. Continuing in this way we can find all the $k_{g-1}, k_{g-2}, \ldots, k_1$ satisfying (5).

In Z adding k_j times column j to column g for $1 \le j \le g-1$ we get a new matrix $G = Z R_2$, where $R_2 \in M_g(\mathbf{F})$ is the nonsingular matrix corresponding to these elementary column operations. Since x_i appears with a nonzero coefficient in the i-th component of the last column of G for $i = 1, \ldots, t$, we can choose suitable values for $x_1, \ldots, x_t$ successively to make the last column of G be a zero column, which means that there is at least one completion of G which has rank $\leq g - 1 = n - r - 1$. But on the other hand, all completions of $A_4 = T_5T_3AQ_3Q_4R_1$ have rank n. Therefore all completions of Z and hence all completions of $G = ZR_2$ have rank $n - r$, which is a contradiction.

So E has a zero row, say, the f-th row being zero. In A_3 interchanging the $(t + f)$-th row and the first row, and then interchanging the f-th column and the first column, we get a new ACI-matrix

$$A_5 = P_2A_3Q_5 = P_2T_5T_3AQ_3Q_4Q_5 = \begin{bmatrix} d_f & 0 \\ * & H_2 \end{bmatrix}$$

where $P_2 \in M_m(\mathbf{F})$ and $Q_5 \in M_n(\mathbf{F})$ are permutation matrices corresponding to these elementary row and column operations respectively.

Since all completions of A_5 have rank n, all completions of H_2 have rank $n - 1$. By Lemma 1, H_2 is an ACI-matrix. So using the induction hypothesis on H_2, there exists a nonsingular constant matrix $T_6 \in M_{m-1}(\mathbf{F})$ and a permutation matrix $Q_6 \in M_{n-1}(\mathbf{F})$ such that $T_6H_2Q_6$ is of form (2).

Set $T = (1 \oplus T_6)P_2T_5T_3$ and $Q = Q_3Q_4Q_5(1 \oplus Q_6)$. Then $T \in M_m(\mathbf{F})$ is nonsingular, $Q \in M_n(\mathbf{F})$ is a permutation matrix and TAQ is of form (2). This completes the proof. $\square$

For a matrix G, we denote by $G(i, j)$ the entry of G in the position (i, j).

Lemma 4 *Let $\mathbf{F}$ be a field with $|\mathbf{F}| \geq n+1$ and A be an $m \times n$ ACI-matrix over $\mathbf{F}[x_1, \ldots, x_k]$. If all completions of A have the same rank $r < n$, then A has at least one column with only constant entries.*

Proof. To the contrary suppose each column of A contains at least one indeterminate, which implies that the number of indeterminates in A is at least n. Without loss of generality we assume that x_j appears in the j-th column of A for $j = 1, \ldots, n$. Let $A = (a_{ij})$ with

$$a_{ij} = a_{ij}^{(0)} + b_{ij}x_j + \sum_{u \neq j} a_{ij}^{(u)} x_u,$$

where for each j, b_{ij}, $i = 1, \ldots, m$, are not all zero. We show that there exist $t_i \in \mathbf{F}, i = 2, \ldots, m$ such that

$$b_{1j} + t_2b_{2j} + \cdots + t_mb_{mj} \neq 0 \text{ for } j = 1, \ldots, n. \tag{6}$$

In fact we may successively choose $t_2, \ldots, t_n$ such that if $b_{i,j_0} \neq 0$ for some i, j_0, then

$$b_{1,j_0} + t_2b_{2,j_0} + \cdots + t_ib_{i,j_0} \neq 0. \tag{7}$$

(6) will follow from (7) since the matrix $B \equiv (b_{ij})_{m\times n}$ has no zero column. If the second row of B is a zero row, we choose $t_2 = 0$. Otherwise let $b_{2,j_1}, \ldots, b_{2,j_s}$ be the nonzero entries in the second row of B. For every $p = 1, \ldots, s$, the equation $b_{1,j_p} + yb_{2,j_p} = 0$ has only one solution, i.e., $y = -b_{2,j_p}^{-1} b_{1,j_p}$. Since $s \le n$ and $|\mathbf{F}| \ge n+1$, there exists $t_2 \in \mathbf{F}$ such that $b_{1,j_p} + t_2 b_{2,j_p} \ne 0$ holds for all $p = 1, \ldots, s$. Next if the third row of B is a zero row, choose $t_3 = 0$. Otherwise, as above there exists $t_3 \in \mathbf{F}$ such that $b_{1,j} + t_2 b_{2,j} + t_3 b_{3,j} \ne 0$ holds for all those j for which $b_{3,j} \ne 0$. Continuing in this way we can find all the $t_2, t_3, \ldots, t_n$ satisfying (7). Now in A adding t_i times the i-th row to the first row for $i = 2, \ldots, m$ we get a matrix A_1 all of whose completions have the same rank r. By Lemma 1, A_1 is an ACI-matrix. By the condition (6), x_j appears in $A_1(1, j)$ with a nonzero coefficient.

Obviously there is a choice of values for the indeterminates such that the first row of A_1 becomes a zero row. For this completion of A_1, without loss of generality suppose the first r columns are linearly independent. Now in A_1 change the value of x_{r+1} such that $A_1(1, r+1) \ne 0$ and keep the values of other indeterminates unchanged. Now for the second choice of values for the indeterminates, the rank of the completed matrix is $r+1$, which is a contradiction. □

Now we are ready to characterize the ACI-matrices all of whose completions have the same rank. In a block matrix, the condition that some block B is $s \times t$ with $s = 0$ means that the rows in which B lies are void, i.e., they do not appear, and the condition that some block B is $s \times t$ with $t = 0$ means that the columns in which B lies are void.

Theorem 5 *Let m, n be positive integers, $\mathbf{F}$ be a field with $|\mathbf{F}| \ge \max\{m, n+1\}$ and A be an $m \times n$ ACI-matrix over $\mathbf{F}[x_1, \ldots, x_k]$. Then all completions of A have the same rank r if and only if there exists a nonsingular constant matrix $T \in M_m(\mathbf{F})$ and a permutation matrix $Q \in M_n(\mathbf{F})$ such that*

$$TAQ = \begin{bmatrix} U_1 & * & * \\ 0 & 0 & * \\ 0 & 0 & U_2 \end{bmatrix} \tag{8}$$

where U_1 and U_2 are square upper triangular ACI-matrices with nonzero constant diagonal entries and the sum of their orders equals r.

Proof. The sufficiency of the condition is obvious. To prove the necessity we use induction on n. For $n = 1$ the conclusion is easy to verify by using Lemma 2. Now let $n \ge 2$ and assume that the result holds for all ACI-matrices with $n-1$ columns. Let A be an $m \times n$ ACI-matrix all of whose completions have rank r.

If $r = n$, which implies $m \ge n$, then the result follows from Lemma 3 (with the first block row and the first two block columns in (8) void). If A has a zero column, say, column j, interchanging column 1 and j we get a matrix $A_0 = AQ_0 = (0, B)$, where $Q_0 \in M_n(\mathbf{F})$ is a permutation matrix and B is an $m \times (n-1)$ ACI-matrix all of whose completions have rank r. Using the induction hypothesis on B, there exists a nonsingular matrix $T_0 \in M_m(\mathbf{F})$ and a permutation matrix $Q_1 \in M_{n-1}(\mathbf{F})$ such that

$$T_0 B Q_1 = \begin{bmatrix} U_1 & * & * \\ 0 & 0 & * \\ 0 & 0 & U_2 \end{bmatrix}$$

where U_1 and U_2 are $r_1 \times r_1$ and $r_2 \times r_2$ upper triangular ACI-matrices with nonzero constant diagonal entries respectively, $r_1 + r_2 = r$.

We have

$$A_1 \equiv T_0 A_0 (1 \oplus Q_1) = (0, T_0 B Q_1) = \begin{bmatrix} 0 & U_1 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & U_2 \end{bmatrix}.$$

In A_1 interchange column 1 and the second block column, and denote the corresponding permutation matrix by Q_2. Set $T = T_0$ and $Q = Q_0(1 \oplus Q_1)Q_2$. Then

$$TAQ = T_0 A Q_0 (1 \oplus Q_1) Q_2 = A_1 Q_2$$

has form (8).

Next we consider the case that $r < n$ and A has no zero column. By Lemma 4, A has a column with only constant entries, which are not all zero, say, the j-th column with the i-th entry nonzero. Interchanging rows 1 and i, and then interchanging columns 1 and j we get a new matrix $A_2 = P_1 A Q_3 = (\tilde{a}_{ij})$ with $\tilde{a}_{11} \neq 0$ and $\tilde{a}_{i1} \in \mathbf{F}$ for $1 \leq i \leq m$, where P_1, Q_3 are permutation matrices. In A_2 adding $-\tilde{a}_{i,1}/\tilde{a}_{11}$ times the first row to the i-th row for $i = 2, \ldots, m$ successively we get a matrix $A_3 = T_1 A_2$, where $T_1 \in M_m(\mathbf{F})$ is the nonsingular matrix corresponding to these elementary row operations. Partition A_3 as

$$A_3 = T_1 A_2 = \begin{bmatrix} \tilde{a}_{11} & u^T \\ 0 & H \end{bmatrix}.$$

By Lemma 1, A_3 is an ACI-matrix all of whose completions have rank r. Therefore H is an $(m-1) \times (n-1)$ ACI-matrix all of whose completions have rank $r-1$. Using the induction hypothesis on H, we know that there exists a nonsingular constant matrix $T_2 \in M_{m-1}(\mathbf{F})$ and a permutation matrix $Q_4 \in M_{n-1}(\mathbf{F})$ such that

$$T_2 H Q_4 = \begin{bmatrix} V_1 & * & * \\ 0 & 0 & * \\ 0 & 0 & U_2 \end{bmatrix}$$

where V_1 and U_2 are $\tilde{r}_1 \times \tilde{r}_1$ and $r_2 \times r_2$ upper triangular ACI-matrices with nonzero constant diagonal entries respectively, $\tilde{r}_1 + r_2 = r - 1$.

Set $T = (1 \oplus T_2)T_1P_1$ and $Q = Q_3(1 \oplus Q_4)$. Then $T \in M_m(\mathbf{F})$ is a nonsingular constant matrix, $Q \in M_n(\mathbf{F})$ is a permutation matrix and

$$TAQ = \begin{bmatrix} \tilde{a}_{11} & * & * & * \\ 0 & V_1 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & U_2 \end{bmatrix} \equiv \begin{bmatrix} U_1 & * & * \\ 0 & 0 & * \\ 0 & 0 & U_2 \end{bmatrix}.$$

Let $r_1 = \tilde{r}_1 + 1$. Then U_1 and U_2 are $r_1 \times r_1$ and $r_2 \times r_2$ upper triangular ACI-matrices with nonzero constant diagonal entries and $r_1 + r_2 = \tilde{r}_1 + r_2 + 1 = r$. □

We remark that in the form (8) some block rows or/and block columns may be void. For example (8) includes the following forms as special cases:

$$U_1, \quad \begin{bmatrix} U_1 & * \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & * \\ 0 & U_2 \end{bmatrix}.$$

Now we study the possible numbers of indeterminates in the partial matrices of a given size all of whose completions have the same rank. Obviously it suffices to determine the largest number.

Theorem 6 *Let $m \geq n$ be positive integers, $\mathbf{F}$ be a field with $|\mathbf{F}| \geq \max\{m, n+1\}$ and A be an $m \times n$ partial matrix over $\mathbf{F}$ all of whose completions have the same rank r. Then the number of indeterminates of A is less than or equal to $mr - r(r+1)/2$. This maximum number is attained at A if and only if there exist permutation matrices $P \in M_m(\mathbf{F}), Q \in M_n(\mathbf{F})$ such that*

$$PAQ = \begin{bmatrix} V_1 & C_1 & C_2 \\ 0 & 0 & C_3 \\ 0 & 0 & V_2 \end{bmatrix}$$

where C_1, C_2, C_3 are partial matrices all of whose entries are indeterminates, V_1 and V_2 are $s \times s$ and $(r-s) \times (r-s)$ upper triangular matrices with nonzero constant diagonal entries and with all the entries above the diagonal being indeterminates, and $s = 0$ when $m > n$.

Proof. By Theorem 5 there exists a nonsingular constant matrix $T = (t_{ij}) \in M_m(\mathbf{F})$ and a permutation matrix $Q \in M_n(\mathbf{F})$ such that

$$TAQ = \begin{bmatrix} U_1 & * & * \\ 0 & 0 & * \\ 0 & 0 & U_2 \end{bmatrix} \tag{9}$$

where U_1 and U_2 are $r_1 \times r_1$ and $r_2 \times r_2$ upper triangular ACI-matrices with nonzero constant diagonal entries respectively, $r_1 + r_2 = r$. Let $\tilde{A} = AQ = (a_{ij})$ and $B = (b_{ij}) = T\tilde{A} = TAQ$.

We assert that the j-th column of $\tilde{A}$ contains at most $j-1$ indeterminates for $j=1,\ldots,r_1$, at most r_1 indeterminates for $j=r_1+1,\ldots,n-r_2$ and at most $m-n+j-1$ indeterminates for $j=n-r_2+1,\ldots,n$.

Suppose the j-th column of $\tilde{A}$ has exactly p indeterminates, say, $a_{i_1,j}, a_{i_2,j},\ldots,a_{i_p,j}$. From (9) we have

$$b_{ij}=\sum_{k=1}^{m} t_{ik}a_{kj}=\sum_{h=1}^{p} t_{i,i_h}a_{i_h,j}+d_{ij},\ \ d_{ij}\in\mathbf{F}.$$

Since for $1\le j\le r_1$ and $i\ge j$, b_{ij} are constants, we have $t_{i,i_h}=0$ for $j\le i\le m$ and $1\le h\le p$. So T has an $(m-j+1)\times p$ zero submatrix. If $p\ge j$, then $(m-j+1)+p\ge m+1$ and by the Frobenius-König theorem [**?**], $\det T=0$, which contradicts the fact that T is nonsingular. This shows that if $1\le j\le r_1$, then $p\le j-1$.

Since for $r_1+1\le j\le n-r_2$ and $i\ge r_1+1$, b_{ij} are constants, we have $t_{i,i_h}=0$ for $r_1+1\le i\le m$ and $1\le h\le p$. Thus T has an $(m-r_1)\times p$ zero submatrix. If $p\ge r_1+1$ then $m-r_1+p\ge m+1$ and hence T is singular, contradiction. This shows that if $r_1+1\le j\le n-r_2$, then $p\le r_1$.

Since for $n-r_2+1\le j\le n$ and $i\ge m-(n-j)$, b_{ij} are constants, we have $t_{i,i_h}=0$ for $m-(n-j)\le i\le m$ and $1\le h\le p$. So T has an $(n-j+1)\times p$ zero submatrix. If $p\ge m-n+j$, then $n-j+1+p\ge m+1$ and T is singular, contradiction. This shows that if $j\ge n-r_2+1$, then $p\le m-n+j-1$.

Denote by $f(A)$ the number of indeterminates of A, which is equal to that of $\tilde{A}$. Using $r_1+r_2=r$ we have

$$\begin{aligned} f(A) &\le \sum_{j=1}^{r_1}(j-1)+(n-r_2-r_1)r_1+\sum_{j=n-r_2+1}^{n}(m-n+j-1)\\ &= (m-n)r_2+nr-\frac{1}{2}r(r+1)\\ &\le (m-n)r+nr-\frac{1}{2}r(r+1)\\ &= mr-\frac{1}{2}r(r+1). \end{aligned}$$

The "if" part of the second conclusion is obvious. Now suppose that the number of indeterminates of A is equal to $mr-r(r+1)/2$. From the above argument we see that

i) if $m=n$, then the j-th column of $\tilde{A}$ has exactly $j-1$ indeterminates for $j=1,\ldots,r_1$, $n-r_2+1,\ldots,n$ and r_1 indeterminates for $j=r_1+1,\ldots,n-r_2$;

ii) if $m>n$, then $r_1=0$, $r_2=r$, the j-th column of $\tilde{A}$ has no indeterminate for $j=1,\ldots,n-r$ and has exactly $m-n+j-1$ indeterminates for $j=n-r+1,\ldots,n$.

Note that $\tilde{A}$ is a partial matrix. To complete our proof of Theorem 6, it suffices to prove the following two statements:

(S1) *Let $G \in P_n(\mathbf{F})$ be a partial matrix and $T \in M_n(\mathbf{F})$ be a nonsingular constant matrix such that*

$$TG = \begin{bmatrix} U_1 & * & * \\ 0 & 0 & * \\ 0 & 0 & U_2 \end{bmatrix} \tag{10}$$

where U_1 and U_2 are $r_1 \times r_1$ and $r_2 \times r_2$ upper triangular matrices with nonzero constant diagonal entries. If the j-th column of G has exactly $j-1$ indeterminates for $j = 1, \ldots, r_1, n-r_2+1, \ldots, n$ and r_1 indeterminates for $j = r_1+1, \ldots, n-r_2$, then there exists a permutation matrix P such that

$$PG = \begin{bmatrix} V_1 & C_1 & C_2 \\ 0 & 0 & C_3 \\ 0 & 0 & V_2 \end{bmatrix} \tag{11}$$

where C_1, C_2, C_3 are partial matrices all of whose entries are indeterminates, V_1 and V_2 are $r_1 \times r_1$ and $r_2 \times r_2$ upper triangular matrices with nonzero constant diagonal entries and with all the entries above the diagonal being indeterminates.

(S2) *Let $m > n$, let $G \in P_{m,n}(\mathbf{F})$ be a partial matrix and $T \in M_m(\mathbf{F})$ be a nonsingular constant matrix such that*

$$TG = \begin{bmatrix} 0 & * \\ 0 & U_2 \end{bmatrix} \tag{12}$$

where U_2 is an $r \times r$ upper triangular matrix with nonzero constant diagonal entries. If the j-th column of G has no indeterminate for $1 \le j \le n-r$ and has exactly $m-n+j-1$ indeterminates for $n-r+1 \le j \le n$, then there exists a permutation matrix P such that

$$PG = \begin{bmatrix} 0 & C \\ 0 & U \end{bmatrix} \tag{13}$$

where C is a partial matrix all of whose entries are indeterminates, U is an $r \times r$ upper triangular matrix with nonzero constant diagonal entries and with all the entries above the diagonal being indeterminates.

Proof of (S1). We use induction on n to prove (S1). It holds trivially for the case $n = 1$. Next let $n \ge 2$ and assume (S1) holds for all matrices of order $\le n-1$. Let G be an $n \times n$ partial matrix which satisfies the condition of (S1). The case $r_1 = n$ is just the statement (S) in [1, proof of Theorem 12]. So next we suppose $r_1 < n$.

There exists a permutation matrix $P_1 \in M_n(\mathbf{F})$ such that if we denote $P_1G = (g_{ij})$, then the following hold:

i) if $r_2 = 0$, then in the last column, $g_{1n}, g_{2n}, \dots, g_{r_1,n}$ are distinct indeterminates and all the other entries $g_{r_1+1,n}, g_{r_1+2,n}, \dots, g_{nn}$ are constants;

ii) if $r_2 \geq 1$, then in the last column, g_{nn} is a constant and all the other entries $g_{1n}, g_{2n}, \dots, g_{n-1,n}$ are indeterminates.

Let $TP_1^T = (t_{ij})$ and denote

$$V = TG = (TP_1^T)(P_1G) = (v_{ij}).$$

We consider the above two cases separately.

Case 1. $r_2 = 0$. Since

$$v_{in} = \sum_{k=1}^{r_1} t_{ik}g_{kn} + \sum_{k=r_1+1}^{n} t_{ik}g_{kn} = 0, \text{ for } i = r_1+1, \dots, n,$$

we have $t_{ij} = 0$ for $r_1 + 1 \leq i \leq n$ and $1 \leq j \leq r_1$. Partition

$$TP_1^T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}, \qquad P_1G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

where $T_{11} \in M_{r_1}(\mathbf{F})$ and $G_{11} \in P_{r_1}(\mathbf{F})$. Since T is nonsingular, T_{11} and T_{22} are nonsingular. Thus

$$V = (TP_1^T)(P_1G) = \begin{bmatrix} U_1 & * \\ 0 & 0 \end{bmatrix}$$

implies $G_{21} = 0$ and $G_{22} = 0$ since $T_{22}G_{21} = 0$ and $T_{22}G_{22} = 0$.

Clearly the j-th column of G_{11} has exactly $j-1$ indeterminates for $j = 1, \dots, r_1$. From

$$V = (TP_1^T)(P_1G) = \begin{bmatrix} T_{11}G_{11} & * \\ 0 & 0 \end{bmatrix}$$

we deduce that $T_{11}G_{11} = U_1$ is upper triangular with nonzero constant diagonal entries. By the induction hypothesis there exists a permutation matrix P_2 of order r_1 such that P_2G_{11} is upper triangular with nonzero constant diagonal entries and with all the entries above the diagonal being indeterminates. By assumption the j-th column of G, and hence P_1G, has r_1 indeterminates for $j = r_1+1, \dots, n$. Since $G_{22} = 0$, we deduce that all the entries of G_{12} are indeterminates. Set $P = (P_2 \oplus I_{n-r_1})P_1$. Then P is a permutation matrix and

$$PG = \begin{pmatrix} P_2G_{11} & P_2G_{12} \\ 0 & 0 \end{pmatrix}$$

has form (11).

Case 2. $r_2 \geq 1$. Since

$$v_{nn} = \sum_{k=1}^{n-1} t_{nk}g_{kn} + t_{nn}g_{nn} \in \mathbf{F},$$

we have $t_{nk} = 0$ for $1 \le k \le n-1$, and the above equality reduces to $v_{nn} = t_{nn}g_{nn}$. Then $v_{nn} \ne 0$ implies $t_{nn} \ne 0$ and $g_{nn} \ne 0$. Since U_2 is upper triangular, the first $n-1$ entries in the last row of V are zero. From

$$0 = v_{nj} = \sum_{k=1}^{n} t_{nk}g_{kj} = t_{nn}g_{nj}, \text{ for } j = 1, 2, \ldots, n-1,$$

we get $g_{nj} = 0$ for $j = 1, 2, \ldots, n-1$. Partition

$$TP_1^T = \begin{bmatrix} T_1 & v \\ 0 & t_{nn} \end{bmatrix}, \quad P_1G = \begin{bmatrix} G_1 & w \\ 0 & g_{nn} \end{bmatrix}$$

where $T_1 \in M_{n-1}(\mathbf{F})$ and $G_1 \in P_{n-1}(\mathbf{F})$. Since T is nonsingular, T_1 is nonsingular. Clearly the j-th column of G_1 has exactly $j-1$ indeterminates for $j = 1, \ldots, r_1, (n-1)-(r_2-1)+1, \ldots, n-1$ and r_1 indeterminates for $j = r_1+1, \ldots, (n-1)-(r_2-1)$. By the condition ii) all the components of w are indeterminates. From

$$V = (TP_1^T)(P_1G) = \begin{bmatrix} T_1G_1 & * \\ 0 & t_{nn}g_{nn} \end{bmatrix}$$

we deduce that T_1G_1 has form (10). By the induction hypothesis there exists a permutation matrix P_2 of order $n-1$ such that P_2G_1 has form (11). Set $P = (P_2 \oplus 1)P_1$. Then P is a permutation matrix and

$$PG = \begin{pmatrix} P_2G_1 & P_2w \\ 0 & g_{nn} \end{pmatrix}$$

has form (11). Thus we complete the proof of (S1).

Proof of (S2). If $r = 0$ then $G = 0$, and the result holds trivially. It suffices to prove the case $r \ge 1$. We can use induction on n to prove (S2) by an argument similar to that in the above proof of Case 2 of (S1). We omit the details. The starting step $n = 1$ needs some care. When $m > n = r = 1$, the condition (12) says that TG is a column vector with its last component being a nonzero constant, and the second condition in (S2) says that G has exactly one constant component. The conclusion (13) states that the only constant component of G is nonzero. To the contrary, suppose it is zero. Then each component of TG is either 0 or a polynomial of degree 1, which contradicts the condition that the last component of TG is a nonzero constant. □

For those $m \times n$ partial matrices with $m < n$, we may apply Theorem 6 by considering their transposes.

Note that the possible generalization of Theorem 6 to ACI-matrices does not make sense. Just consider the matrix

$$A = \begin{bmatrix} 1 & x_1 + x_2 + \cdots + x_k \\ 0 & 1 \end{bmatrix}.$$

References

[1] R.A. Brualdi, Z. Huang and X. Zhan, Singular, nonsingular and bounded rank completions of ACI-matrices, Linear Algebra Appl., 433(2010), 1452-1462.

[2] R.A. Brualdi and H.J. Ryser, Combinatorial Matrix Theory, Cambridge University Press, 1991.