Algebraically positive matrices

Steve Kirkland\textsuperscript{a}, Pu Qiao\textsuperscript{b}, Xingzhi Zhan\textsuperscript{b,∗}

\textsuperscript{a} Department of Mathematics, University of Manitoba, Winnipeg, MB, Canada
\textsuperscript{b} Department of Mathematics, East China Normal University, Shanghai 200241, China

\textbf{A R T I C L E A B S T R A C T}

We introduce the concept of algebraically positive matrices and investigate some basic properties, including a characterization, the index of algebraic positivity, and sign patterns that allow or require this property. We also pose two open problems.

© 2016 Elsevier Inc. All rights reserved.

\textbf{1. Introduction}

A positive (nonnegative) matrix is a matrix all of whose entries are positive (nonnegative) real numbers. The notation $A > 0$ means that $A$ is a positive matrix. We introduce the following concept and study its basic properties.

\@doi{10.1016/j.laa.2016.03.049}

0024-3795/© 2016 Elsevier Inc. All rights reserved.
**Definition.** A square real matrix $A$ is said to be *algebraically positive* if there exists a real polynomial $f$ such that $f(A)$ is a positive matrix.

The motivation for this concept is twofold. First, we would like to extend the concept of primitive nonnegative matrices to algebraically positive matrices for general real matrices; second, we wish to let the negative entries and positive entries in a real matrix have equal status. Also, the action of polynomials on matrices is very common in matrix analysis ([1, Chapter V], [5, Chapter 6]) and even in graph theory [2, Section 5.3]. All the matrices in this paper are square unless otherwise specified, so we will omit the word “square”.

Recall that if an irreducible nonnegative matrix $A$ has exactly $k$ eigenvalues of modulus equal to the spectral radius of $A$, then $k$ is called the *index of imprimitivity* of $A$. If $k = 1$, $A$ is said to be *primitive*; otherwise, $A$ is *imprimitive*. It is known (see, e.g., [8, p. 134]) that a nonnegative matrix $A$ of order at least 2 is primitive if and only if there exists a positive integer $m$ such that $A^m$ is a positive matrix. Thus a primitive matrix is algebraically positive.

There is another related concept. A real matrix $A$ is said to be *eventually positive* if there exists a positive integer $p$ such that $A^k$ is a positive matrix for all integers $k \geq p$. See [7] and the references therein for recent development and the history of this topic. Clearly, every eventually positive matrix is algebraically positive. Note that eventually positive matrices are a proper subclass of algebraically positive matrices. For example, the matrix $A = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ is algebraically positive since $I - A > 0$, but $A$ is not eventually positive, since it is idempotent and has some negative entries.

Obviously, if a real matrix $A$ is algebraically positive, then so are $-A$ and $P^TAP$ for any permutation matrix $P$.

We remark that the possible corresponding concepts of *algebraically nonnegative matrices* and *algebraically positive definite matrices* are not so compelling, since every real matrix is algebraically nonnegative by the Cayley–Hamilton theorem (see, e.g., [8, p. 24]) while it is straightforward to see that any real symmetric matrix is algebraically positive definite.

We will study some basic properties of algebraically positive matrices, including a characterization of algebraic positivity, the index of algebraic positivity, and sign patterns. At the end we pose two open problems.

**2. Main results**

Let $A$ be a real matrix of order $n$ with characteristic polynomial $h(x)$ and let $f(x)$ be any real polynomial. By the division algorithm there exist real polynomials $q(x)$ and $r(x)$ with $\deg r(x) \leq n - 1$ such that $f(x) = h(x)q(x) + r(x)$. The Cayley–Hamilton theorem asserts that $h(A) = 0$. Thus $f(A) = r(A)$. It follows that to check whether a
given real matrix of order $n$ is algebraically positive we need only consider polynomials of degree less than or equal to $n − 1$.

Throughout we denote by $I$ the identity matrix whose order will be clear from the context and by $A^T$ the transpose of a matrix $A$. We write the elements of $\mathbb{R}^n$ as column vectors. A \textit{simple eigenvalue} is an eigenvalue of algebraic multiplicity $1$.

\textbf{Theorem 1.} A real matrix is algebraically positive if and only if it has a simple real eigenvalue and corresponding left and right positive eigenvectors.

\textbf{Proof.} Let $A$ be an algebraically positive matrix and let $f$ be a real polynomial such that $f(A) > 0$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A$. The spectral mapping theorem [8, p. 8] asserts that the eigenvalues of $f(A)$ are $f(\lambda_1), \ldots, f(\lambda_n)$. Since $f(A)$ is a positive matrix, the Perron–Frobenius theorem [8, p. 123] ensures that the spectral radius of $f(A)$ is a simple eigenvalue (Perron root). Suppose $f(\lambda_j)$ is the Perron root of $f(A)$. Then $\lambda_j$ is a simple eigenvalue of $A$, for otherwise $f(\lambda_j)$ would be a multiple eigenvalue of $f(A)$. We claim that $\lambda_j$ is a real number. To see the claim, suppose to the contrary that $\lambda_j$ is nonreal. Since the nonreal eigenvalues of any real matrix occur in conjugate pairs, $\overline{\lambda}_j$ is also an eigenvalue of $A$ and $\overline{\lambda}_j \neq \lambda_j$, where the bar notation means the complex conjugate. Since $f(\lambda_j)$ is a real number and $f$ is a real polynomial, $f(\lambda_j) = \overline{f(\lambda_j)} = f(\overline{\lambda}_j)$ which shows that $f(\lambda_j)$ is a multiple eigenvalue of $f(A)$, a contradiction. Thus $\lambda_j$ is a simple real eigenvalue of $A$.

Every left or right eigenvector of $A$ corresponding to $\lambda_j$ is also a left or right eigenvector of $f(A)$ corresponding to the Perron root $f(\lambda_j)$. Since $f(\lambda_j)$ is simple, it has geometric multiplicity $1$. Hence every left or right eigenvector of $A$ corresponding to $\lambda_j$ is a real scalar multiple of a left or right Perron vector of $f(A)$. It follows that $A$ has positive left and right eigenvectors corresponding to the simple real eigenvalue $\lambda_j$.

Conversely, let $A$ be a real matrix of order $n$ and suppose that $A$ has a simple real eigenvalue $\rho$ with corresponding left and right positive eigenvectors $\nu^T$ and $u$ respectively. Without loss of generality we normalize $u, v$ so that $v^Tu = 1$. Let $q$ be the minimal polynomial of $A$, and let $p(x) = q(x)/(x - \rho)$. Then $0 = q(A) = p(A)(A - \rho I)$. Applying Sylvester’s inequality on the rank of a matrix product and using the condition that $\text{rank}(A - \rho I) = n - 1$, we have $\text{rank} p(A) \leq 1$. Since $q$ is the minimal polynomial and $\deg p = (\deg q) - 1$, we have $p(A) \neq 0$. Hence $\text{rank} p(A) = 1$. Since $\rho$ is a simple eigenvalue, $p(\rho) \neq 0$. Now it is easy to show that the conditions

\[ \text{rank } p(A) = 1, \quad p(A)u = p(\rho)u, \quad v^Tp(A) = p(\rho)v^T, \quad p(\rho) \neq 0, \quad v^Tu = 1 \]

imply that $p(A) = p(\rho)uv^T$. Since $u$ and $v$ are positive vectors, we have

\[ (p(A))^2 = (p(\rho))^2uv^T > 0. \]

This proves that $A$ is algebraically positive. $\square$
From the above proof of Theorem 1 we see that a real matrix \( A \) is algebraically positive if and only if there is a real polynomial \( f \) such that \( f(A) \) is irreducible and nonnegative. This fact also follows from the result [8, p. 121] that if \( B \) is an irreducible nonnegative matrix of order \( n \), then \((I + B)^{n-1} > 0\).

**Proposition 2.** Algebraically positive matrices have the following properties.

(i) Every algebraically positive matrix is irreducible.

(ii) If \( A \) is a real matrix and there is a positive integer \( k \) such that \( A^k \) is algebraically positive, then so is \( A \).

**Proof.** (i) If \( A \) is a reducible real matrix, then there is a permutation matrix \( P \) such that 
\[ A = P^T \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix} P \] 
where \( A_1 \) and \( A_3 \) are square and nonvoid. For any real polynomial \( f \) we have 
\[ f(A) = P^T \begin{bmatrix} f(A_1) & 0 \\ \ast & f(A_3) \end{bmatrix} P. \]
Thus \( f(A) \) has zero entries and cannot be positive.

(ii) Obvious by definition. \( \square \)

It is straightforward to see from the definition that if a matrix \( A \) is algebraically positive, then for each \( t \in \mathbb{R} \), \( A + tI \) is also algebraically positive. Hence, shifting the entire diagonal of an algebraically positive matrix will preserve that property. However, as the following example shows, perturbing a single diagonal entry may not preserve algebraic positivity.

**Example 3.** Consider the matrices
\[
A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & 1 \\ 2 & -2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 2 & 1 \\ 2 & -1 & 1 \\ 2 & -2 & -1 \end{bmatrix},
\]
which differ only in the first diagonal entry. Since
\[
2I + 2A + A^2 = \begin{bmatrix} 16 & 4 & 5 \\ 8 & 3 & 2 \\ 2 & 4 & 1 \end{bmatrix} > 0,
\]
we see that \( A \) is algebraically positive. Let \( f(x) = a + bx + cx^2 \) be a real polynomial. Consider the entries in the positions (1,2) and (3,2) of
\[
f(B) = aI + b \begin{bmatrix} -2 & 2 & 1 \\ 2 & -1 & 1 \\ 2 & -2 & -1 \end{bmatrix} + c \begin{bmatrix} 10 & -8 & -1 \\ -4 & 3 & 0 \\ -10 & 8 & 1 \end{bmatrix}.
\]
To make the (1, 2) entry positive we must have $b > 4c$. But to make the (3, 2) entry positive we must have $b < 4c$. Thus no $a, b, c$ can make $f(B)$ positive. This shows that $B$ is not algebraically positive.

We remark that irreducibility and having a simple real eigenvalue are not sufficient for algebraic positivity. The matrix $B$ in Example 3 is irreducible and has three simple real eigenvalues $0$, $-2 + \sqrt{3}$, $-2 - \sqrt{3}$.

Recall that a matrix $A$ is said to be skew-symmetric if $A^T = -A$.

**Corollary 4.** No nonsingular skew-symmetric real matrix is algebraically positive.

**Proof.** Since all the eigenvalues of any skew-symmetric real matrix are pure imaginary, the only possible real eigenvalue is $0$. Hence a nonsingular skew-symmetric real matrix has no real eigenvalue and is not algebraically positive by Theorem 1. □

Note that the nonsingularity condition in Corollary 4 cannot be dropped. For example,

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad 3I + A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} > 0$$

showing that the skew-symmetric real matrix $A$ is algebraically positive. Of course $A$ is singular.

**Theorem 5.** If $A$ is an irreducible real matrix all of whose off-diagonal entries are nonnegative (or nonpositive), then $A$ is algebraically positive.

**Proof.** It suffices to consider the nonnegative case by considering $-A$ if necessary. Choose a positive number $d$ such that $d$ is larger than the maximum absolute value of the diagonal entries of $A$. Then $dI + A$ is an irreducible nonnegative matrix. It follows [8, p. 121] that $(I + (dI + A))^{n-1} > 0$ where $n$ is the order of $A$; i.e., $((d + 1)I + A)^{n-1} > 0$. Thus $A$ is algebraically positive. □

**Corollary 6.** A nonnegative matrix is algebraically positive if and only if it is irreducible.

**Proof.** This follows immediately from Proposition 2(i) and Theorem 5. □

The next example shows that similarity transformations may not preserve algebraic positivity.

**Example 7.** Consider the matrix $A$ given by

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$
and note that $A$ is algebraically positive by Theorem 5. Let

$$B = T^{-1}AT = \begin{bmatrix} -2 & 2 & 1 \\ 2 & -1 & 1 \\ 2 & -2 & -1 \end{bmatrix} \text{ where } T = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 2 & 0 & 1 \end{bmatrix}.$$  

Then $B$ is similar to $A$. Note that this matrix $B$ is the same $B$ as in Example 3 where we have proved that it is not algebraically positive.

**Definition.** If a real matrix $A$ is algebraically positive, we define the *index of algebraic positivity* of $A$ to be the least degree of a real polynomial $f$ such that $f(A)$ is positive.

Clearly, this index of an algebraically positive matrix of order $n \geq 2$ is an integer between 1 and $n-1$. Every real matrix of order 1 is algebraically positive and has index of algebraic positivity 0.

Recall that for a primitive nonnegative matrix $A$, the *exponent* of $A$ is the smallest positive integer $m$ such that $A^m > 0$ (see [2, p. 78]). For a nonnegative matrix of order $n$, the exponent must lie in the interval $[1, n^2 - 2n + 2]$, and it is well-known ([2, pp. 84–85], [9]) that the exponents of primitive matrices of a given order $(n \geq 4)$ have gaps – i.e. there are integers $k \in [1, n^2 - 2n + 2]$ that cannot be realized as the exponent of any primitive $n \times n$ matrix. The following result shows that the indices of algebraically positive matrices of a given order have no gaps.

**Theorem 8.** Let $n, k$ be integers such that $1 \leq k \leq n-1$. Then there exists an algebraically positive matrix of order $n$ whose index of algebraic positivity is equal to $k$.

**Proof.** The assumption $1 \leq k \leq n-1$ implies that $n \geq 2$. Any positive matrix of order at least 2 has index of algebraic positivity 1. Next suppose $k \geq 2$. Let $A$ be the adjacency matrix of the graph $T$ which is a tree with vertices $1, 2, \ldots, n$ consisting of the $(1, k)$-path with vertices $1, 2, \ldots, k$ and the edges $ki, i = k+1, k+2, \ldots, n$. Then $A$ is of order $n$. Note that there is a loop at every vertex of the directed graph of the matrix $I + A$. Since the diameter of $T$ is $k$, it follows that $(I + A)^k > 0$ and for any real polynomial $f$ of degree $\leq k - 1$, $f(A)$ has zero entries. In fact, the entry of $f(A)$ in the position $(1, n)$ is zero, since in the directed graph of $A$ there is no walk from vertex 1 to vertex $n$ of length $\leq k - 1$. Hence the index of algebraic positivity of $A$ is equal to $k$. □

If $A$ is an irreducible nonnegative matrix with index of imprimitivity $k \geq 2$, then $A$ is permutationally similar to a matrix of the form
Proof. Since then Lemma. The matrix in (2) is called the cyclic canonical form of $A$.

Following [4], given a consecutive ordered partition $P = (\pi_1, \ldots, \pi_k)$ of $\{1, \ldots, n\}$ into $k$ nonempty subsets, the cyclic characteristic matrix, denoted by $C_P$, is the $n \times n$ matrix whose $(i, j)$-entry is 1 if there is an element $t \in \{1, \ldots, k\}$ such that $i \in \pi_t$ and $j \in \pi_{t+1}$, and 0 otherwise, where the index $k + 1$ is interpreted as 1. Denote by $\Gamma(G)$ the directed graph of a matrix $G$. Note that

1. $C_P$ is $k$-cyclic and $\Gamma(C_P)$ contains every arc $(i, j)$ for $i \in \pi_t$ and $j \in \pi_{t+1}$; and
2. $A \in M_n(\mathbb{C})$ is $k$-cyclic with ordered partition $P$ if and only if $\Gamma(A) \subseteq \Gamma(C_P)$.

**Lemma.** If $P = (\pi_1, \ldots, \pi_k)$ $(k \geq 2)$ is a consecutive ordered partition of $\{1, \ldots, n\}$ and $C_P$ is its corresponding cyclic characteristic matrix, then, for each $m = 1, 2, \ldots, k - 1$, the block-diagonal of $(C_P)^m$ has only zero blocks.

**Proof.** Since $C_P$ is a 0-1 matrix, the $(i, j)$-entry of $(C_P)^m$ specifies the number of directed walks of length $m$ from vertex $i$ to vertex $j$ in $\Gamma(C_P)$ [2]. For any $t$ with $1 \leq t \leq k$ and $i, j \in \pi_t$, every walk from vertex $i$ to vertex $j$ in $\Gamma(C_P)$ has length at least $k$. Hence there is no walk of length $m$ from $i$ to $j$. This shows that each block on the main diagonal of the block matrix $(C_P)^m$ is a zero block. □

**Theorem 9.** Let $A$ be an irreducible nonnegative matrix of order $n$ and let $\alpha$ and $k$ be the index of algebraic positivity and the index of imprimitivity of $A$ respectively. If $k \leq n - 1$ then

$$\alpha \geq k.$$ \hfill (3)

When $k = 1$, equality in (3) holds if and only if all the off-diagonal entries of $A$ are positive. When $2 \leq k \leq n - 1$ and the cyclic canonical form of $A$ is the matrix in (2), equality in (3) holds if and only if all the matrices $A_{12}, A_{23}, \ldots, A_{k-1,k}, A_{k1}$ are positive. If $k = n$, then $\alpha = n - 1$.

**Proof.** First note that $A$ is algebraically positive by Corollary 6. The case when $k = 1$ is clear. Now suppose $2 \leq k \leq n - 1$. By applying a permutation similarity transformation if necessary, it suffices to consider the case when $A$ itself is the matrix in (2) and we make this assumption. The condition $k \leq n - 1$ implies that at least one zero block on the diagonal of $A$ has order $\geq 2$. The $p$-th powers of $A$ for $p = 1, 2, \ldots, k$ are

$$\begin{bmatrix}
0 & A_{12} & 0 & \cdots & 0 \\
0 & 0 & A_{23} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & A_{k-1,k} \\
A_{k1} & 0 & 0 & \cdots & 0 \\
\end{bmatrix}$$ \hfill (2)

where every zero block on the diagonal is a square matrix [8, p. 136]. The matrix in (2) is called the cyclic canonical form of $A$.
By the lemma preceding Theorem 9, for every $1 \leq m \leq k - 1$, the block-diagonal of $A^m$ contains only zero blocks. Thus, for any real polynomial $f$ of degree $\leq k - 1$, $f(A)$ has zero entries in some diagonal block and hence is not positive. This shows the inequality (3).

If all the matrices $A_{12}, A_{23}, \ldots, A_{k-1,k}, A_{k1}$ are positive, we have $A + A^2 + \cdots + A^k > 0$. Hence $\alpha \leq k$ which, together with $\alpha \geq k$, yields $\alpha = k$. Conversely suppose $\alpha = k$. Then there are real numbers $c_0, c_1, \ldots, c_k$ such that $c_0 I + c_1 A + \cdots + c_k A^k > 0$. Since $A_{12}, A_{23}, \ldots, A_{k-1,k}, A_{k1}$ and their cyclic products appearing in $A, A^2, \ldots, A^k$ lie in different positions, we must have that $A_{12}, A_{23}, \ldots, A_{k-1,k}, A_{k1}$ are all positive.

If $k = n$, each of the matrices $A_{12}, A_{23}, \ldots, A_{k-1,k}, A_{k1}$ must be of order 1, and hence is a positive number, since $A$ is irreducible. For any real polynomial $f$ of degree $\leq n - 2$, $f(A)$ has zero entries, showing that $\alpha \geq n - 1$. On the other hand, since the order of $A$ is $n$, we always have $\alpha \leq n - 1$. Thus $\alpha = n - 1$. \qed

**Remark.** The idea in the proof of Theorem 9 can be used to show that if the directed graph of a real matrix $A$ is a Hamilton cycle and $A$ has both positive and negative entries, then $A$ is not algebraically positive.

3. **Algebraic positivity and sign patterns**

A *sign pattern* is a matrix whose entries are from the set $\{+, -, 0\}$. The *sign pattern of a real matrix $A$* is the matrix obtained from $A$ by replacing each entry by its sign. Whether a nonnegative matrix is primitive depends only on its zero–nonzero pattern [2, Section 3.4], so that primitivity is a purely combinatorial property. This is not the case for algebraic positivity: the following example shows that the sign pattern alone may not determine whether or not a real matrix is algebraically positive.
Example 10. Let

\[ A = \begin{bmatrix} 0 & 30 & -1 & -5 \\ -1 & 0 & 40 & -5 \\ 50 & -3 & 0 & 60 \\ 10 & 60 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & + & - \\ - & 0 & + \\ + & - & 0 & + \\ + & + & - \end{bmatrix}. \]

The two matrices \( A \) and \( B \) have the same sign pattern \( C \).

\[ 1000A + 20A^2 + A^3 = \begin{bmatrix} 55847 & 3945 & 11560 & 65235 \\ 62800 & 193782 & 17765 & 35960 \\ 43110 & 103680 & 199032 & 12985 \\ 122897 & 30990 & 59440 & 134235 \end{bmatrix} > 0 \]

shows that \( A \) is algebraically positive. Since \( B \) is a nonsingular skew-symmetric real matrix \((\det B = 1)\), \( B \) is not algebraically positive by Corollary 4.

Given a sign pattern \( A \), the pattern class of \( A \), denoted \( Q(A) \), is the set of real matrices whose sign patterns are \( A \). Let \( P \) be a property about real matrices. A sign pattern \( A \) is said to require \( P \) if every matrix in \( Q(A) \) has property \( P \); \( A \) is said to allow \( P \) if there exists a matrix in \( Q(A) \) that has property \( P \). For example, let

\[ A = \begin{bmatrix} 0 & + & - \\ - & 0 & + \\ + & - & 0 \end{bmatrix}, \quad B = \begin{bmatrix} + & + & + \\ + & - & + \\ + & + & - \end{bmatrix}, \quad C = \begin{bmatrix} 0 & + & 0 \\ - & 0 & + \\ 0 & - & 0 \end{bmatrix}. \]

Then \( A \) requires algebraic positivity (since for any \( M \in Q(A) \), \( M^2 \) has all positive off-diagonal entries), \( B \) allows algebraic positivity (since \( \begin{bmatrix} 3 & 3 & 3 \\ 3 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix} \in Q(B) \) and has a positive square), and \( C \) does not allow algebraic positivity (since no matrix in \( Q(C) \) admits a positive right eigenvector).

A sign pattern \( A \) of order \( n \) is said to be spectrally arbitrary if given any monic real polynomial \( f \) of degree \( n \), there is a matrix in \( Q(A) \) whose characteristic polynomial is \( f \) [3]. We have the following (negative) result.

Theorem 11. No spectrally arbitrary sign pattern requires algebraic positivity.

Proof. It is clear that any sign pattern of order 1 is not spectrally arbitrary. Let \( A \) be a spectrally arbitrary sign pattern of order \( n \geq 2 \). Then there is a real matrix \( B \) in \( Q(A) \) whose characteristic polynomial is \( f(x) = (x - 1)^n \). Obviously \( B \) has no simple eigenvalue. By Theorem 1, \( B \) is not algebraically positive. This shows that \( A \) does not require algebraic positivity. \( \square \)

Next we give a necessary condition for a sign pattern to allow algebraic positivity.
Theorem 12. If a sign pattern allows algebraic positivity, then every row and column contains a $+$, or every row and column contains a $-$. 

Proof. Let $A$ be a sign pattern that allows algebraic positivity and suppose that $M \in Q(A)$ is an algebraically positive matrix. By Theorem 1, there is a real eigenvalue $r$, a positive right eigenvector $v$, and a positive left eigenvector $w^T$ such that $Mv = rv$ and $w^T M = rw^T$. If $r > 0$, then each row of $M$ contains a positive entry (otherwise we have a contradiction to the fact that $Mv = rv$) and similarly every column of $M$ contains a positive entry. If $r < 0$, an analogous argument shows that every row and column of $M$ must contain a negative entry. Finally, if $r = 0$, then since $Mv = 0$, each row must contain both a negative entry and a positive entry (recall that $M$ is irreducible and so contains no all zero row or column) and similarly since $w^T M = 0^T$, each column of $M$ contains both a positive entry and a negative entry. \[ \square \]

The following example shows that the necessary condition of Theorem 12 is not sufficient for a sign pattern to allow algebraic positivity.

Example 13. Consider the sign pattern $A = \begin{bmatrix} + & + \\ - & + \end{bmatrix}$. Observe that if $M \in Q(A)$, then $M$ has the form $\begin{bmatrix} a & b \\ -c & d \end{bmatrix}$ for some $a, b, c, d > 0$. Observe that for any $x, y \in \mathbb{R}, xI + yM$ has a nonpositive off-diagonal entry. Hence, $M$ cannot be algebraically positive.

In the special case of symmetric tridiagonal sign patterns, we have the following characterization of the allow problem for algebraic positivity.

Theorem 14. Suppose that $A$ is an irreducible $n \times n$ symmetric tridiagonal sign pattern matrix. Then $A$ allows algebraic positivity if and only if either every row of $A$ contains a $+$, or every row of $A$ contains a $-$. 

Proof. If $A$ allows algebraic positivity, then the condition on the rows follows immediately from Theorem 12. Suppose now for concreteness that every row of $A$ contains a $+$. For each $i = 1, \ldots, n$, denote the number of $+$ and $-$ entries in row $i$ of $A$ by $n_+(i)$ and $n_-(i)$, respectively. Construct a matrix $M \in Q(A)$ as follows: put a $-1$ in the positions of $M$ corresponding to $-$ entries in $A$, and for each $+$ entry in row $i$ of $A$, put a $\frac{1+n_-(i)}{n_+(i)}$ in the corresponding position of $M$. Since the row sums of $M$ are all 1, the all ones vector is a right eigenvector of $M$ corresponding to the eigenvalue 1. Since $M$ is irreducible, sign-symmetric and tridiagonal, every eigenvalue of $M$ is real and simple. Further, there is a diagonal matrix $D$ with positive entries on the diagonal such that $\hat{M} = DMD^{-1}$ is symmetric. Evidently $\text{diag}(D)$ is a positive right eigenvector of $\hat{M}$ corresponding to the eigenvalue 1. Hence $\hat{M}$ has 1 as a simple eigenvalue with corresponding left and right eigenvectors that are positive. Consequently $\hat{M}$ is algebraically positive by Theorem 1. \[ \square \]
We now characterize a special class of $3 \times 3$ sign patterns that require algebraic positivity.

**Theorem 15.** Suppose that $A$ is an irreducible $3 \times 3$ symmetric tridiagonal sign pattern matrix. Then $A$ requires algebraic positivity if and only if one of the following holds:

1. all nonzero off-diagonal entries of $A$ are $+$;
2. all nonzero off-diagonal entries of $A$ are $-$;
3. $A$ is permutationally similar to a matrix in the set

$$S = \left\{ \begin{bmatrix} 0 & + & 0 \\ + & 0 & - \\ 0 & - & + \end{bmatrix}, \begin{bmatrix} - & + & 0 \\ + & 0 & - \\ 0 & - & + \end{bmatrix}, \begin{bmatrix} 0 & + & 0 \\ + & 0 & - \\ 0 & - & + \end{bmatrix}, \begin{bmatrix} - & + & 0 \\ - & - & - \\ 0 & - & + \end{bmatrix} \right\};$$

4. $-A$ is permutationally similar to a matrix in $S$.

**Proof.** Assume that $A$ requires algebraic positivity. If all off-diagonal entries of $A$ have the same sign, then either i) or ii) holds. Suppose now that $A$ has two off-diagonal entries $+$ and two off-diagonal entries $-$. By considering $-A$ if necessary, we may assume by Theorem 12 that every row of $A$ contains a $+$. Further, simultaneously permuting rows and columns of $A$ if necessary, we may further assume that the $(1,2)$ and $(2,1)$ entries of $A$ are $+$, and the $(2,3)$ and $(3,2)$ entries are $-$. Since the $(3,1)$ entry of $A$ is 0, we find from Theorem 12 that necessarily the $(3,3)$ entry of $A$ is $+.$

Let $M \in Q(A)$; without loss of generality we may take $M$ to be symmetric, and multiplying $M$ by a suitable positive scalar if necessary, we take $M$ to have the form $M = \begin{bmatrix} a & 1 & 0 \\ 1 & b & -c \\ 0 & -c & d \end{bmatrix},$ where $c,d > 0.$ Observe that $M^2 = \begin{bmatrix} a^2 + 1 & a + b & -c \\ a + b & 1 + b^2 + c^2 & -c(b + d) \\ -c & -c(b + d) & c^2 + d^2 \end{bmatrix}.$

Let $p(x) = \alpha x^2 + \beta x + \gamma$ be a polynomial such that $p(M) > 0.$ By considering the $(1,3)$ entry of $p(M),$ we deduce that $\alpha < 0.$ Considering the $(1,2)$ and $(2,3)$ entries of $p(M)$ now yields that $\beta - |\alpha|(a + b) > 0$ and $-\beta c + |\alpha|(b + d)c > 0.$ Setting $\sigma = \frac{\beta}{|\alpha|}$ and simplifying, the preceding inequalities become $\sigma > a + b$ and $b + d > \sigma.$ In particular it must be the case that $d > a.$ Suppose that the $(1,1)$ entry of $A$ is $+.$ Then we may find an $M \in Q(A)$ such that $m_{1,1} = 1$ and $m_{3,3} = 1,$ contradicting the fact that $d > a$ above. We thus conclude that the $(1,1)$ entry of $A$ must be either $0$ or $a -.$ Observe that the sign patterns in the set $S$ correspond to the six cases arising from the $(1,1)$ entry of $A$ being $0$ or $-,$ and the $(2,2)$ entry being $0, +$ or $-.$ Hence $A \in S.$

Next suppose that $A$ satisfies one of i)--iv). If either i) or ii) is satisfied, then $A$ requires algebraic positivity by Theorem 5. Suppose now that $A$ satisfies iii). As above,
if $M \in Q(A)$, then without loss of generality, we may take $M$ to have the form
\[
\begin{bmatrix}
a & 1 & 0 \\
1 & b & -c \\
0 & -c & d
\end{bmatrix},
\]
where $c, d > 0$ and $a \leq 0$. Set $\sigma = b + \frac{a+d}{2}$, and note that
\[
-M^2 + \sigma M =
\begin{bmatrix}
\sigma a - a^2 - 1 & \sigma - a - b & c \\
\sigma - a - b & \sigma b - 1 - b^2 - c^2 & c(b + d) - \sigma c \\
c & c(b + d) - \sigma c & \sigma d - c^2 - d^2
\end{bmatrix} =
\begin{bmatrix}
\sigma a - a^2 - 1 & \frac{d-a}{2} & c \\
\frac{d-a}{2} & \sigma b - 1 - b^2 - c^2 & c(d-a) \\
c & c(d-a) & \sigma d - c^2 - d^2
\end{bmatrix},
\]
which has positive off-diagonal entries since $d > 0 \geq a$ and $c > 0$. Adding a suitable multiple of the identity matrix now shows that $-M^2 + \sigma M + tI > 0$ for some $t \in \mathbb{R}$, so that $A$ requires algebraic positivity.

Finally, if $A$ satisfies iv), it immediately follows from iii) that $A$ requires algebraic positivity. \(\Box\)

We close the paper with the following problems.

**Problem 1.** Characterize those sign patterns that require algebraic positivity.

**Problem 2.** Characterize those sign patterns that allow algebraic positivity.

The approach and results in [6] might be helpful in studying these problems.

**Acknowledgements**

The authors are grateful to two anonymous referees whose comments helped to improve the presentation of the results in this paper.

**References**