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Algebraically positive matrices ☆

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We introduce the concept of algebraically positive matrices and investigate some basic properties, including a characterization, the index of algebraic positivity, and sign patterns that allow or require this property. We also pose two open problems.

1. Introduction

A *positive (nonnegative) matrix* is a matrix all of whose entries are positive (nonnegative) real numbers. The notation $A > 0$ means that A is a positive matrix. We introduce the following concept and study its basic properties.

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Definition. A square real matrix A is said to be *algebraically positive* if there exists a real polynomial f such that $f(A)$ is a positive matrix.

The motivation for this concept is twofold. First, we would like to extend the concept of primitive nonnegative matrices to algebraically positive matrices for general real matrices; second, we wish to let the negative entries and positive entries in a real matrix have equal status. Also, the action of polynomials on matrices is very common in matrix analysis ([1, Chapter V], [5, Chapter 6]) and even in graph theory [2, Section 5.3]. All the matrices in this paper are square unless otherwise specified, so we will omit the word "square".

Recall that if an irreducible nonnegative matrix A has exactly k eigenvalues of modulus equal to the spectral radius of A, then k is called the *index of imprimitivity* of A. If $k = 1$, A is said to be *primitive;* otherwise, A is *imprimitive*. It is known (see, e.g., [8, p. 134]) that a nonnegative matrix A of order at least 2 is primitive if and only if there exists a positive integer m such that A^m is a positive matrix. Thus a primitive matrix is algebraically positive.

There is another related concept. A real matrix A is said to be *eventually positive* if there exists a positive integer p such that A^k is a positive matrix for all integers $k \geq p$. See [7] and the references therein for recent development and the history of this topic. Clearly, every eventually positive matrix is algebraically positive. Note that eventually positive matrices are a proper subclass of algebraically positive matrices. For example, the matrix $A = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ is algebraically positive since $I - A > 0$, but A is not eventually positive, since it is idempotent and has some negative entries.

Obviously, if a real matrix A is algebraically positive, then so are $-A$ and P^TAP for any permutation matrix P.

We remark that the possible corresponding concepts of *algebraically nonnegative matrices* and *algebraically positive definite matrices* are not so compelling, since every real matrix is algebraically nonnegative by the Cayley–Hamilton theorem (see, e.g., [8, p. 24]) while it is straightforward to see that any real symmetric matrix is algebraically positive definite.

We will study some basic properties of algebraically positive matrices, including a characterization of algebraic positivity, the index of algebraic positivity, and sign patterns. At the end we pose two open problems.

2. Main results

Let A be a real matrix of order n with characteristic polynomial $h(x)$ and let $f(x)$ be any real polynomial. By the division algorithm there exist real polynomials $q(x)$ and $r(x)$ with $\deg r(x) \leq n - 1$ such that $f(x) = h(x)q(x) + r(x)$. The Cayley–Hamilton theorem asserts that $h(A) = 0$. Thus $f(A) = r(A)$. It follows that to check whether a given real matrix of order n is algebraically positive we need only consider polynomials of degree less than or equal to $n-1$.

Throughout we denote by I the identity matrix whose order will be clear from the context and by A^T the transpose of a matrix A. We write the elements of $\mathbb{R}^n$ as column vectors. A *simple eigenvalue* is an eigenvalue of algebraic multiplicity 1.

Theorem 1. *A real matrix is algebraically positive if and only if it has a simple real eigenvalue and corresponding left and right positive eigenvectors.*

Proof. Let A be an algebraically positive matrix and let f be a real polynomial such that $f(A) > 0$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. The spectral mapping theorem [8, p. 8] asserts that the eigenvalues of $f(A)$ are $f(\lambda_1), \ldots, f(\lambda_n)$. Since $f(A)$ is a positive matrix, the Perron–Frobenius theorem [8, p. 123] ensures that the spectral radius of $f(A)$ is a simple eigenvalue (Perron root). Suppose $f(\lambda_j)$ is the Perron root of $f(A)$. Then λ_j is a simple eigenvalue of A, for otherwise $f(\lambda_j)$ would be a multiple eigenvalue of $f(A)$. We claim that λ_j is a real number. To see the claim, suppose to the contrary that λ_j is nonreal. Since the nonreal eigenvalues of any real matrix occur in conjugate pairs, $\overline{\lambda_j}$ is also an eigenvalue of A and $\overline{\lambda_j} \neq \lambda_j$, where the bar notation means the complex conjugate. Since $f(\lambda_j)$ is a real number and f is a real polynomial, $f(\lambda_j) = \overline{f(\lambda_j)} = f(\overline{\lambda_j})$ which shows that $f(\lambda_j)$ is a multiple eigenvalue of $f(A)$, a contradiction. Thus λ_j is a simple real eigenvalue of A.

Every left or right eigenvector of A corresponding to λ_j is also a left or right eigenvector of $f(A)$ corresponding to the Perron root $f(\lambda_j)$. Since $f(\lambda_j)$ is simple, it has geometric multiplicity 1. Hence every left or right eigenvector of A corresponding to λ_j is a real scalar multiple of a left or right Perron vector of $f(A)$. It follows that A has positive left and right eigenvectors corresponding to the simple real eigenvalue λ_j.

Conversely, let A be a real matrix of order n and suppose that A has a simple real eigenvalue r with corresponding left and right positive eigenvectors v^T and u respectively. Without loss of generality we normalize u, v so that $v^T u = 1$. Let q be the minimal polynomial of A, and let $p(x) = q(x)/(x-r)$. Then $0 = q(A) = p(A)(A - rI)$. Applying Sylvester's inequality on the rank of a matrix product and using the condition that $\operatorname{rank}(A - rI) = n-1$, we have $\operatorname{rank} p(A) \leq 1$. Since q is the minimal polynomial and $\deg p = (\deg q) - 1$, we have $p(A) \neq 0$. Hence $\operatorname{rank} p(A) = 1$. Since r is a simple eigenvalue, $p(r) \neq 0$. Now it is easy to show that the conditions

$$\operatorname{rank} p(A) = 1, \;\; p(A)u = p(r)u, \;\; v^T p(A) = p(r)v^T, \;\; p(r) \neq 0, \;\; v^T u = 1$$

imply that $p(A) = p(r)uv^T$. Since u and v are positive vectors, we have

$$(p(A))^2 = (p(r))^2 uv^T > 0.$$

This proves that A is algebraically positive. □

From the above proof of Theorem 1 we see that a real matrix A is algebraically positive if and only if there is a real polynomial f such that $f(A)$ is irreducible and nonnegative. This fact also follows from the result [8, p. 121] that if B is an irreducible nonnegative matrix of order n, then $(I+B)^{n-1} > 0$.

Proposition 2. *Algebraically positive matrices have the following properties.*

(i) Every algebraically positive matrix is irreducible.
(ii) If A is a real matrix and there is a positive integer k such that A^k is algebraically positive, then so is A.

Proof. (i) If A is a reducible real matrix, then there is a permutation matrix P such that $A = P^T \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix} P$ where A_1 and A_3 are square and nonvoid. For any real polynomial f we have $f(A) = P^T \begin{bmatrix} f(A_1) & 0 \\ \star & f(A_3) \end{bmatrix} P$. Thus $f(A)$ has zero entries and cannot be positive.

(ii) Obvious by definition. □

It is straightforward to see from the definition that if a matrix A is algebraically positive, then for each $t \in \mathbb{R}, A + tI$ is also algebraically positive. Hence, shifting the entire diagonal of an algebraically positive matrix will preserve that property. However, as the following example shows, perturbing a single diagonal entry may not preserve algebraic positivity.

Example 3. Consider the matrices

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & 1 \\ 2 & -2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 2 & 1 \\ 2 & -1 & 1 \\ 2 & -2 & -1 \end{bmatrix}, \tag{1}$$

which differ only in the first diagonal entry. Since

$$2I + 2A + A^2 = \begin{bmatrix} 16 & 4 & 5 \\ 8 & 3 & 2 \\ 2 & 4 & 1 \end{bmatrix} > 0,$$

we see that A is algebraically positive. Let $f(x) = a + bx + cx^2$ be a real polynomial. Consider the entries in the positions $(1,2)$ and $(3,2)$ of

$$f(B) = aI + b \begin{bmatrix} -2 & 2 & 1 \\ 2 & -1 & 1 \\ 2 & -2 & -1 \end{bmatrix} + c \begin{bmatrix} 10 & -8 & -1 \\ -4 & 3 & 0 \\ -10 & 8 & 1 \end{bmatrix}.$$

To make the $(1, 2)$ entry positive we must have $b > 4c$. But to make the $(3, 2)$ entry positive we must have $b < 4c$. Thus no a, b, c can make $f(B)$ positive. This shows that B is not algebraically positive.

We remark that irreducibility and having a simple real eigenvalue are not sufficient for algebraic positivity. The matrix B in Example 3 is irreducible and has three simple real eigenvalues 0, $-2 + \sqrt{3}$, $-2 - \sqrt{3}$.

Recall that a matrix A is said to be *skew-symmetric* if $A^T = -A$.

Corollary 4. *No nonsingular skew-symmetric real matrix is algebraically positive.*

Proof. Since all the eigenvalues of any skew-symmetric real matrix are pure imaginary, the only possible real eigenvalue is 0. Hence a nonsingular skew-symmetric real matrix has no real eigenvalue and is not algebraically positive by Theorem 1. □

Note that the nonsingularity condition in Corollary 4 cannot be dropped. For example,

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad 3I + A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} > 0$$

showing that the skew-symmetric real matrix A is algebraically positive. Of course A is singular.

Theorem 5. *If A is an irreducible real matrix all of whose off-diagonal entries are nonnegative (or nonpositive), then A is algebraically positive.*

Proof. It suffices to consider the nonnegative case by considering $-A$ if necessary. Choose a positive number d such that d is larger than the maximum absolute value of the diagonal entries of A. Then $dI + A$ is an irreducible nonnegative matrix. It follows [8, p. 121] that $(I + (dI + A))^{n-1} > 0$ where n is the order of A; i.e., $((d + 1)I + A)^{n-1} > 0$. Thus A is algebraically positive. □

Corollary 6. *A nonnegative matrix is algebraically positive if and only if it is irreducible.*

Proof. This follows immediately from Proposition 2(i) and Theorem 5. □

The next example shows that similarity transformations may not preserve algebraic positivity.

Example 7. Consider the matrix A given by

$$A = -\begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

and note that A is algebraically positive by Theorem 5. Let

$$B = T^{-1}AT = \begin{bmatrix} -2 & 2 & 1 \\ 2 & -1 & 1 \\ 2 & -2 & -1 \end{bmatrix} \text{ where } T = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 2 & 0 & 1 \end{bmatrix}.$$

Then B is similar to A. Note that this matrix B is the same B as in Example 3 where we have proved that it is not algebraically positive.

Definition. If a real matrix A is algebraically positive, we define the *index of algebraic positivity* of A to be the least degree of a real polynomial f such that $f(A)$ is positive.

Clearly, this index of an algebraically positive matrix of order $n \geq 2$ is an integer between 1 and $n-1$. Every real matrix of order 1 is algebraically positive and has index of algebraic positivity 0.

Recall that for a primitive nonnegative matrix A, the *exponent* of A is the smallest positive integer m such that $A^m > 0$ (see [2, p. 78]). For a nonnegative matrix of order n, the exponent must lie in the interval $[1, n^2 - 2n + 2]$, and it is well-known ([2, pp. 84–85], [9]) that the exponents of primitive matrices of a given order ($n \geq 4$) have gaps – i.e. there are integers $k \in [1, n^2 - 2n + 2]$ that cannot be realized as the exponent of any primitive $n \times n$ matrix. The following result shows that the indices of algebraically positive matrices of a given order have no gaps.

Theorem 8. *Let n, k be integers such that $1 \leq k \leq n-1$. Then there exists an algebraically positive matrix of order n whose index of algebraic positivity is equal to k.*

Proof. The assumption $1 \leq k \leq n-1$ implies that $n \geq 2$. Any positive matrix of order at least 2 has index of algebraic positivity 1. Next suppose $k \geq 2$. Let A be the adjacency matrix of the graph T which is a tree with vertices $1, 2, \ldots, n$ consisting of the $(1, k)$-path with vertices $1, 2, \ldots, k$ and the edges ki, $i = k+1, k+2, \ldots, n$. Then A is of order n. Note that there is a loop at every vertex of the directed graph of the matrix $I + A$. Since the diameter of T is k, it follows that $(I + A)^k > 0$ and for any real polynomial f of degree $\leq k-1$, $f(A)$ has zero entries. In fact, the entry of $f(A)$ in the position $(1, n)$ is zero, since in the directed graph of A there is no walk from vertex 1 to vertex n of length $\leq k-1$. Hence the index of algebraic positivity of A is equal to k. $\square$

If A is an irreducible nonnegative matrix with index of imprimitivity $k \geq 2$, then A is permutationally similar to a matrix of the form

$$\begin{bmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{k-1,k} \\ A_{k1} & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{2}$$

where every zero block on the diagonal is a square matrix [8, p. 136]. The matrix in (2) is called the *cyclic canonical form* of A.

Following [4], given a consecutive ordered partition $P = (\pi_1, \ldots, \pi_k)$ of $\{1, \ldots, n\}$ into k nonempty subsets, the *cyclic characteristic matrix,* denoted by C_P, is the $n \times n$ matrix whose (i, j)-entry is 1 if there is an element $t \in \{1, \ldots k\}$ such that $i \in \pi_t$ and $j \in \pi_{t+1}$, and 0 otherwise, where the index $k + 1$ is interpreted as 1. Denote by $\Gamma(G)$ the directed graph of a matrix G. Note that

1, C_P is k-cyclic and $\Gamma(C_P)$ contains every arc (i, j) for $i \in \pi_t$ and $j \in \pi_{t+1}$; and
2, $A \in M_n(\mathbb{C})$ is k-cyclic with ordered partition P if and only if $\Gamma(A) \subseteq \Gamma(C_P)$.

Lemma. *If $P = (\pi_1, \ldots, \pi_k)$ $(k \geq 2)$ is a consecutive ordered partition of $\{1, \ldots, n\}$ and C_P is its corresponding cyclic characteristic matrix, then, for each $m = 1, 2, \ldots, k - 1$, the block-diagonal of $(C_P)^m$ has only zero blocks.*

Proof. Since C_P is a 0-1 matrix, the (i, j)-entry of $(C_P)^m$ specifies the number of directed walks of length m from vertex i to vertex j in $\Gamma(C_P)$ [2]. For any t with $1 \leq t \leq k$ and $i, j \in \pi_t$, every walk from vertex i to vertex j in $\Gamma(C_P)$ has length at least k. Hence there is no walk of length m from i to j. This shows that each block on the main diagonal of the block matrix $(C_P)^m$ is a zero block. □

Theorem 9. *Let A be an irreducible nonnegative matrix of order n and let α and k be the index of algebraic positivity and the index of imprimitivity of A respectively. If $k \leq n - 1$ then*

$$\alpha \geq k. \tag{3}$$

When $k = 1$, equality in (3) holds if and only if all the off-diagonal entries of A are positive. When $2 \leq k \leq n - 1$ and the cyclic canonical form of A is the matrix in (2), equality in (3) holds if and only if all the matrices $A_{12}, A_{23}, \ldots, A_{k-1,k}, A_{k1}$ are positive. If $k = n$, then $\alpha = n - 1$.

Proof. First note that A is algebraically positive by Corollary 6. The case when $k = 1$ is clear. Now suppose $2 \leq k \leq n - 1$. By applying a permutation similarity transformation if necessary, it suffices to consider the case when A itself is the matrix in (2) and we make this assumption. The condition $k \leq n - 1$ implies that at least one zero block on the diagonal of A has order ≥ 2. The p-th powers of A for $p = 1, 2, \ldots, k$ are

$$A = \begin{bmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{k-1,k} \\ A_{k1} & 0 & 0 & \cdots & 0 \end{bmatrix}, A^2 = \begin{bmatrix} 0 & 0 & A_{12}A_{23} & 0 & \cdots & 0 \\ 0 & 0 & 0 & A_{23}A_{34} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ A_{k-1,k}A_{k1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & A_{k1}A_{12} & 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\cdots, A^k = \begin{bmatrix} A_{12}A_{23}\cdots A_{k1} & 0 & 0 & \cdots & 0 \\ 0 & A_{23}\cdots A_{k1}A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{34}\cdots A_{12}A_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{k1}A_{12}\cdots A_{k-1,k} \end{bmatrix}.$$

By the lemma preceding Theorem 9, for every $1 \leq m \leq k-1$, the block-diagonal of A^m contains only zero blocks. Thus, for any real polynomial f of degree $\leq k-1$, $f(A)$ has zero entries in some diagonal block and hence is not positive. This shows the inequality (3).

If all the matrices $A_{12}, A_{23}, \ldots, A_{k-1,k}, A_{k1}$ are positive, we have $A+A^2+\cdots+A^k > 0$. Hence $\alpha \leq k$ which, together with $\alpha \geq k$, yields $\alpha = k$. Conversely suppose $\alpha = k$. Then there are real numbers $c_0, c_1, \ldots, c_k$ such that $c_0 I + c_1 A + \cdots + c_k A^k > 0$. Since $A_{12}, A_{23}, \ldots, A_{k-1,k}, A_{k1}$ and their cyclic products appearing in $A, A^2, \ldots, A^k$ lie in different positions, we must have that $A_{12}, A_{23}, \ldots, A_{k-1,k}, A_{k1}$ are all positive.

If $k = n$, each of the matrices $A_{12}, A_{23}, \ldots, A_{k-1,k}, A_{k1}$ must be of order 1, and hence is a positive number, since A is irreducible. For any real polynomial f of degree $\leq n-2$, $f(A)$ has zero entries, showing that $\alpha \geq n-1$. On the other hand, since the order of A is n, we always have $\alpha \leq n-1$. Thus $\alpha = n-1$. $\square$

Remark. The idea in the proof of Theorem 9 can be used to show that if the directed graph of a real matrix A is a Hamilton cycle and A has both positive and negative entries, then A is not algebraically positive.

3. Algebraic positivity and sign patterns

A *sign pattern* is a matrix whose entries are from the set $\{+, -, 0\}$. The *sign pattern of a real matrix* A is the matrix obtained from A by replacing each entry by its sign. Whether a nonnegative matrix is primitive depends only on its zero–nonzero pattern [2, Section 3.4], so that primitivity is a purely combinatorial property. This is not the case for algebraic positivity: the following example shows that the sign pattern alone may not determine whether or not a real matrix is algebraically positive.

Example 10. Let

$$A = \begin{bmatrix} 0 & 30 & -1 & -5 \\ -1 & 0 & 40 & -5 \\ 50 & -3 & 0 & 60 \\ 10 & 60 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & + & - & - \\ - & 0 & + & - \\ + & - & 0 & + \\ + & + & - & 0 \end{bmatrix}.$$

The two matrices A and B have the same sign pattern C.

$$1000A + 20A^2 + A^3 = \begin{bmatrix} 55847 & 3945 & 11560 & 65235 \\ 62800 & 193782 & 17765 & 35960 \\ 43110 & 103680 & 199032 & 12985 \\ 122897 & 30990 & 59440 & 134235 \end{bmatrix} > 0$$

shows that A is algebraically positive. Since B is a nonsingular skew-symmetric real matrix ($\det B = 1$), B is not algebraically positive by Corollary 4.

Given a sign pattern A, the *pattern class* of A, denoted $Q(A)$, is the set of real matrices whose sign patterns are A. Let P be a property about real matrices. A sign pattern A is said to *require* P if every matrix in $Q(A)$ has property P; A is said to *allow* P if there exists a matrix in $Q(A)$ that has property P. For example, let

$$A = \begin{bmatrix} 0 & + & - \\ - & 0 & + \\ + & - & 0 \end{bmatrix}, \quad B = \begin{bmatrix} + & + & + \\ + & - & + \\ + & - & - \end{bmatrix}, \quad C = \begin{bmatrix} 0 & + & 0 \\ - & 0 & + \\ 0 & - & 0 \end{bmatrix}.$$

Then A requires algebraic positivity (since for any $M \in Q(A), M^2$ has all positive off-diagonal entries), B allows algebraic positivity (since $\begin{bmatrix} 3 & 3 & 3 \\ 3 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix} \in Q(B)$ and has a positive square), and C does not allow algebraic positivity (since no matrix in $Q(C)$ admits a positive right eigenvector).

A sign pattern A of order n is said to be *spectrally arbitrary* if given any monic real polynomial f of degree n, there is a matrix in $Q(A)$ whose characteristic polynomial is f [3]. We have the following (negative) result.

Theorem 11. *No spectrally arbitrary sign pattern requires algebraic positivity.*

Proof. It is clear that any sign pattern of order 1 is not spectrally arbitrary. Let A be a spectrally arbitrary sign pattern of order $n \geq 2$. Then there is a real matrix B in $Q(A)$ whose characteristic polynomial is $f(x) = (x-1)^n$. Obviously B has no simple eigenvalue. By Theorem 1, B is not algebraically positive. This shows that A does not require algebraic positivity. $\square$

Next we give a necessary condition for a sign pattern to allow algebraic positivity.

Theorem 12. *If a sign pattern allows algebraic positivity, then every row and column contains a $+$, or every row and column contains a $-$.*

Proof. Let A be a sign pattern that allows algebraic positivity and suppose that $M \in Q(A)$ is an algebraically positive matrix. By Theorem 1, there is a real eigenvalue r, a positive right eigenvector v, and a positive left eigenvector w^T such that $Mv = rv$ and $w^T M = rw^T$. If $r > 0$, then each row of M contains a positive entry (otherwise we have a contradiction to the fact that $Mv = rv$) and similarly every column of M contains a positive entry. If $r < 0$, an analogous argument shows that every row and column of M must contain a negative entry. Finally, if $r = 0$, then since $Mv = 0$, each row must contain both a negative entry and a positive entry (recall that M is irreducible and so contains no all zero row or column) and similarly since $w^T M = 0^T$, each column of M contains both a positive entry and a negative entry. □

The following example shows that the necessary condition of Theorem 12 is not sufficient for a sign pattern to allow algebraic positivity.

Example 13. Consider the sign pattern $A = \begin{bmatrix} + & + \\ - & + \end{bmatrix}$. Observe that if $M \in Q(A)$, then M has the form $\begin{bmatrix} a & b \\ -c & d \end{bmatrix}$ for some $a, b, c, d > 0$. Observe that for any $x, y \in \mathbb{R}$, $xI + yM$ has a nonpositive off-diagonal entry. Hence, M cannot be algebraically positive.

In the special case of symmetric tridiagonal sign patterns, we have the following characterization of the allow problem for algebraic positivity.

Theorem 14. *Suppose that A is an irreducible $n \times n$ symmetric tridiagonal sign pattern matrix. Then A allows algebraic positivity if and only if either every row of A contains a $+$, or every row of A contains a $-$.*

Proof. If A allows algebraic positivity, then the condition on the rows follows immediately from Theorem 12. Suppose now for concreteness that every row of A contains a $+$. For each $i = 1, \ldots, n$, denote the number of $+$ and $-$ entries in row i of A by $n_+(i)$ and $n_-(i)$, respectively. Construct a matrix $M \in Q(A)$ as follows: put a -1 in the positions of M corresponding to $-$ entries in A, and for each $+$ entry in row i of A, put a $\frac{1+n_-(i)}{n_+(i)}$ in the corresponding position of M. Since the row sums of M are all 1, the all ones vector is a right eigenvector of M corresponding to the eigenvalue 1. Since M is irreducible, sign-symmetric and tridiagonal, every eigenvalue of M is real and simple. Further, there is a diagonal matrix D with positive entries on the diagonal such that $\hat{M} = DMD^{-1}$ is symmetric. Evidently $\mathrm{diag}(D)$ is a positive right eigenvector of $\hat{M}$ corresponding to the eigenvalue 1. Hence $\hat{M}$ has 1 as a simple eigenvalue with corresponding left and right eigenvectors that are positive. Consequently $\hat{M}$ is algebraically positive by Theorem 1. □

We now characterize a special class of 3×3 sign patterns that require algebraic positivity.

Theorem 15. *Suppose that A is an irreducible 3×3 symmetric tridiagonal sign pattern matrix. Then A requires algebraic positivity if and only if one of the following holds:*

i) all nonzero off-diagonal entries of A are $+$;
ii) all nonzero off-diagonal entries of A are $-$;
iii) A is permutationally similar to a matrix in the set

$$S = \left\{ \begin{bmatrix} 0 & + & 0 \\ + & 0 & - \\ 0 & - & + \end{bmatrix}, \begin{bmatrix} - & + & 0 \\ + & 0 & - \\ 0 & - & + \end{bmatrix}, \begin{bmatrix} 0 & + & 0 \\ + & + & - \\ 0 & - & + \end{bmatrix}, \begin{bmatrix} - & + & 0 \\ + & + & - \\ 0 & - & + \end{bmatrix}, \begin{bmatrix} 0 & + & 0 \\ + & - & - \\ 0 & - & + \end{bmatrix}, \begin{bmatrix} - & + & 0 \\ + & - & - \\ 0 & - & + \end{bmatrix} \right\};$$

iv) $-A$ is permutationally similar to a matrix in S.

Proof. Assume that A requires algebraic positivity. If all off-diagonal entries of A have the same sign, then either i) or ii) holds. Suppose now that A has two off-diagonal entries $+$ and two off-diagonal entries $-$. By considering $-A$ if necessary, we may assume by Theorem 12 that every row of A contains a $+$. Further, simultaneously permuting rows and columns of A if necessary, we may further assume that the $(1, 2)$ and $(2, 1)$ entries of A are $+$, and the $(2, 3)$ and $(3, 2)$ entries are $-$. Since the $(3, 1)$ entry of A is 0, we find from Theorem 12 that necessarily the $(3, 3)$ entry of A is $+$.

Let $M \in Q(A)$; without loss of generality we may take M to be symmetric, and multiplying M by a suitable positive scalar if necessary, we take M to have the form $M = \begin{bmatrix} a & 1 & 0 \\ 1 & b & -c \\ 0 & -c & d \end{bmatrix}$, where $c, d > 0$. Observe that $M^2 = \begin{bmatrix} a^2+1 & a+b & -c \\ a+b & 1+b^2+c^2 & -c(b+d) \\ -c & -c(b+d) & c^2+d^2 \end{bmatrix}$. Let $p(x) = \alpha x^2 + \beta x + \gamma$ be a polynomial such that $p(M) > 0$. By considering the $(1, 3)$ entry of $p(M)$, we deduce that $\alpha < 0$. Considering the $(1, 2)$ and $(2, 3)$ entries of $p(M)$ now yields that $\beta - |\alpha|(a + b) > 0$ and $-\beta c + |\alpha|(b + d)c > 0$. Setting $\sigma = \frac{\beta}{|\alpha|}$ and simplifying, the preceding inequalities become $\sigma > a + b$ and $b + d > \sigma$. In particular it must be the case that $d > a$. Suppose that the $(1, 1)$ entry of A is $+$. Then we may find an $M \in Q(A)$ such that $m_{1,1} = 1$ and $m_{3,3} = 1$, contradicting the fact that $d > a$ above. We thus conclude that the $(1, 1)$ entry of A must be either a 0 or a $-$. Observe that the sign patterns in the set S correspond to the six cases arising from the $(1, 1)$ entry of A being 0 or $-$, and the $(2, 2)$ entry being $0, +$ or $-$. Hence $A \in S$.

Next suppose that A satisfies one of i)–iv). If either i) or ii) is satisfied, then A requires algebraic positivity by Theorem 5. Suppose now that A satisfies iii). As above, if $M \in Q(A)$, then without loss of generality, we may take M to have the form $M = \begin{bmatrix} a & 1 & 0 \\ 1 & b & -c \\ 0 & -c & d \end{bmatrix}$, where $c, d > 0$ and $a \leq 0$. Set $\sigma = b + \frac{a+d}{2}$, and note that

$$-M^2 + \sigma M = \begin{bmatrix} \sigma a - a^2 - 1 & \sigma - a - b & c \\ \sigma - a - b & \sigma b - 1 - b^2 - c^2 & c(b+d) - \sigma c \\ c & c(b+d) - \sigma c & \sigma d - c^2 - d^2 \end{bmatrix} = \begin{bmatrix} \sigma a - a^2 - 1 & \frac{d-a}{2} & c \\ \frac{d-a}{2} & \sigma b - 1 - b^2 - c^2 & c(\frac{d-a}{2}) \\ c & c(\frac{d-a}{2}) & \sigma d - c^2 - d^2 \end{bmatrix},$$

which has positive off-diagonal entries since $d > 0 \geq a$ and $c > 0$. Adding a suitable multiple of the identity matrix now shows that $-M^2 + \sigma M + tI > 0$ for some $t \in \mathbb{R}$, so that A requires algebraic positivity.

Finally, if A satisfies iv), it immediately follows from iii) that A requires algebraic positivity. □

We close the paper with the following problems.

Problem 1. Characterize those sign patterns that require algebraic positivity.

Problem 2. Characterize those sign patterns that allow algebraic positivity.

The approach and results in [6] might be helpful in studying these problems.

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