RESUME OF MY RESEARCH

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My main interests lie in partial differential equations and its geometrical applications. We divide four parts in our research description: the geometry of solutions of elliptic PDE; the Christoffel-Minkowski problem; Weingarten curvature equations and mass transport problem. The common feature is that the **convexity** appears in all four parts.

1. THE GEOMETRY OF SOLUTIONS

The papers [Ma1]-[Ma4] present the works on the geometry properties of the solutions of elliptic equations.

The papers [Ma1] - [Ma3] concern the **P-function** in elliptic partial differential equations and its applications.

In partial differential equations the finding of auxiliary functions is one of the most important techniques, in applications it always connects to the maximum principle. From which we can get the existence, regularity of solutions and other interesting estimates.

In 1900's S. Bernstein found that the gradient module of the harmonic function u in the bounded domain in R^n attains its maximum on the boundary of domain. Now this is called Bernstein estimates for its great importance in partial differential equations.

In 1948, C.Miranda [32] considered the bi-harmonic equations in R^2 and from his theorem we know for the following torsional rigidity equation

$$\Delta u = -2, \quad \text{in} \quad \Omega \subset \mathrm{R}^2, \tag{1.1}$$

the function $P = |\nabla u|^2 + 2u$ attains its maximum on $\partial\Omega$. Then in 1979, Payne-Philippin [35] generalized this to the constant mean curvature equation

$$\sum_{i=1}^{2} D_i\left(\frac{u_i}{\sqrt{1+|Du|^2}}\right) = 2H, \quad \text{in} \quad \Omega \subset \mathrm{R}^2, \tag{1.2}$$

where H is a positive constant, they proved that the function $P = 2-2(1+|Du|^2)^{-\frac{1}{2}}-2Hu$ attains its maximum on $\partial\Omega$. These technique of finding an optimal auxiliary functions had become an expanding theory and it have many applications in geometrical analysis,

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for example the estimates of first eigenvalue of Laplace operator and the sharp estimates of the geometrical shape of solution surfaces.

In application this ask us the generalization of the P-function to the Monge-Ampère equations, and we hope know where the auxiliary functions found by Mirand [32] and Payne-Philippin [35] attain its minimums.

In the paper [Ma1], we obtained the similar P - function for the Monge-Ampère equation

$$\det(u_{ij}) = 1, \quad \text{in} \quad \Omega \subset \mathrm{R}^2, \tag{1.3}$$

i.e., the function $P = |Du|^2 - 2u$ attains its maximum on $\partial\Omega$. Then we gave some applications to the boundary value problems. These results has been generalized to higher dimensions and more general equation by Soufi and Philippin in [36]. and Urbas in [42].

In the papers [Ma2]-[Ma3], we proved that if Ω is a bounded convex domain in R^2 and suitable boundary value then the auxiliary function found by Miranda and Payne-Philippin attains its minimum on $\partial\Omega$. As applications, we got the sharp size and shape estimates for capillary free surfaces without gravity, for example the following results were proved in [Ma3].

In [Ma3], we consider the influence of boundary geometry and the contact angle θ_o $(0 \le \theta_o < \frac{\pi}{2})$ on the size and shape for the capillary free surface without gravity. Precisely, let Ω be a bounded convex domain in R^2 with smooth boundary $\partial\Omega$. Give a positive constant H, consider the following equations:

$$\sum_{i=1}^{2} D_i\left(\frac{u_i}{\sqrt{1+|Du|^2}}\right) = 2H \quad \text{in} \quad \Omega \tag{1.4}$$

$$\frac{u_n}{\sqrt{1+|Du|^2}} = \cos\theta_o \quad \text{on} \quad \partial\Omega. \tag{1.5}$$

Where $u_i, i = 1, 2$ are partial derivatives of u, n denotes the unit outer normal to $\partial\Omega$, u_n denotes the direction derivative of u along n, and θ_o $(0 \le \theta_o < \frac{\pi}{2})$ is the constant with $2H|\Omega| = \cos\theta_o|\partial\Omega|$ ($|\Omega|$ is the area of Ω and $|\partial\Omega|$ is the length of $\partial\Omega$). The graph of solution u to (1.4)- (1.5) describes a capillary free surface without gravity over the cross section Ω.

Let $A \in \partial\Omega$ be a point corresponding to a minimum boundary value of u, $B \in \partial\Omega$ be a point corresponding to a maximum boundary value of u , $C \in \Omega$ be the unique minimal (critical) point of u and $k(x)$ be the curvature of $\partial\Omega$ at $x \in \partial\Omega$. Then we have the following estimates:

Theorem 1. *Let $u \in C^3(\bar{\Omega})$ be a solution to (1.4)-(1.5), then the following inequalities hold*

$$u(A) - u(C) \leq \frac{1-\sin\theta_o}{H}, \tag{1.6}$$

$$k(A) \leq \frac{H}{\cos\theta_o}, \tag{1.7}$$

and

$$u(B) - u(C) \geq \frac{1-\sin\theta_o}{H}, \tag{1.8}$$

$$k(B) \geq \frac{H}{\cos\theta_o}. \tag{1.9}$$

If one of the equality signs in above formulas holds then Ω is a disk of radius $\frac{\cos\theta_o}{H}$ and

$$u(x) - u(C) \equiv \frac{1-\sin\theta_o}{H} \quad on \quad \partial\Omega, \tag{1.10}$$

$$k(x) \equiv \frac{H}{\cos\theta_o} \quad on \quad \partial\Omega. \tag{1.11}$$

Conversely (1.10)- (1.11) holds on $\partial\Omega$ if Ω is a disk of radius $\frac{\cos\theta_o}{H}$.

The proof of above theorem is based on Hopf maximum principle and the following minimum principle.

Theorem 2. *Let $u \in C^3(\bar{\Omega})$ be a solution to (1.4)-(1.5), then the function*

$$P(x) = 2 - 2(1+|Du|^2)^{-\frac{1}{2}} - 2Hu$$

attains its minimum on the boundary $\partial\Omega$, unless $P(x)$ is a constant on $\bar{\Omega}$.

Let's notice in [35], Payne and Philippin proved a similar maximum principle for the above function $P(x)$ that under the same condition it also attains its maximum on $\partial\Omega$ unless $P(x)$ is a constant in Ω.

The paper of [Ma4] concern the **convexity** of solutions of elliptic partial differential equations on domain in R^n.

The issue of convexity is fundamental in the theory of partial differential equations. Gabriel [20] obtained the strict convexity of level set for the Green function in thee dimension convex bounded domain in R^3. Makar-Limanov [31] studied equation (1.1) in bounded plane convex domain Ω under the homogeneous Dirichlet problem. He considered the auxiliary function

$$2u(u_{11}u_{22} - u_{12}^2) + 2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11},$$

where $u_i = \frac{\partial u}{\partial x_i}$, $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$. He proved that $\sqrt{u}$ is concave using the maximum principle. His method has great impact on the later development related to the topic of convexity.

Brascamp-Lieb [7] used the heat equation technique to prove that $\log u$ is a concave, where u is the first eigenfunction of homogeneous Dirichlet problem:

$$\begin{aligned} \Delta u + \lambda_1 u &= 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.12}$$

where Ω is bounded and convex in $\mathbb{R}^n$. In the case of dimension two, another proof of Brascamp-Lieb's result was given in Acker-Payne-Philippin [2]. It was observed that the function $v = \log u$ satisfies the following equation,

$$\begin{aligned} \Delta v &= -(\lambda_1 + |Dv|^2) \quad \text{in} \quad \Omega, \\ v &\longrightarrow -\infty \quad \text{on} \quad \Omega. \end{aligned} \tag{1.13}$$

If we instead let $w = \sqrt{u}$ in (1.1) and consider the homogeneous Dirichlet boundary value problem, then w satisfies equation

$$\begin{aligned} w\Delta w &= -(1 + |Dw|^2) \quad \text{in} \quad \Omega, \\ w &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.14}$$

In the paper [Ma4] we consider the convexity (or concavity) for the solutions of a class elliptic equation in two dimensions. We generalize the technique of [31] and [2] to a suitable large class equations. From which we can show the negative Gauss curvature set of the solution surface always extend to the boundary, unless it is empty. In the torsional rigidity equation (1.1) we can obtain a sharp lower bound estimates of the Gauss curvature for the graph of $\sqrt{u}$ with the curvature of the $\partial\Omega$, if the domain is strictly convex and u is homogeneous on the $\partial\Omega$. Up to now the methods in [2] and [Ma4] are restricted to two dimensions.

The more detail story on convexity described in the book of Kawohl [24]. Recently, a new approach to the convexity problem was found by Alvarez-Lasry-Lions [5] and they treated a large class fully nonlinear elliptic equations.

In a fundamental work of Singer-Wong-Yau-Yau [38] and Caffarelli-Friedman [11], they devised a new technique to deal with the convexity issue via homotopy method of deformation. Caffarelli-Friedman [11] establish the strictly convexity of level sets of solution of the following equation in two dimension:

$$\begin{aligned} \Delta u(x) &= f(u(x)), \qquad x \in \Omega, \\ u &= 0 \qquad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.15}$$

Their result was generalized by Korevaar-Lewis [26] to higher dimensions. This deformation approach (see also [25] for the earlier contribution of Yau related to this development) is very powerful, it is the main inspiration for our discussion on the convexity problem of some nonlinear elliptic equations in classical differential geometry in the next sections.

2. Christoffel-Minkowski problem

The papers [GM1] - [GM2], [HMS], [GMZ] and [GMTZ] represent my works on the Christoffel-Minkowski problem, the related topics about the PDE on S^n and the geometry of convex body. Moreover we obtain a general convexity principle of solutions for a class fully nonlinear elliptic equations in [CGM]. There were worked with Luis Caffarelli, Pengfei Guan, Changqing Hu, Chunli Shen, N.Trudinger, Fang ZHOU and Xiaohua Zhu.

The Minkowski problem is a problem of finding a convex hypersurface with the prescribed Gauss curvature on its outer normals. The general problem of finding a convex hypersurface with kth symmetric function of principal radii prescribed on its outer normals is often called Christoffel-Minkowski problem. It corresponds to finding *convex* solutions of the nonlinear elliptic Hessian equation:

$$S_k(\{u_{ij} + u\delta_{ij}\}) = \varphi \quad \text{on} \quad \mathbb{S}^n, \tag{2.1}$$

where we have let $\{e_1, e_2, ..., e_n\}$ be a orthonormal frame on $\mathbb{S}^n$, S_k is the k-th elementary symmetric function (see Definition 1).

It is known that for (2.1) to be solvable, the function $\varphi(x)$ has to satisfy

$$\int_{\mathbb{S}^n} x_i \varphi(x)\, dx = 0, \qquad i = 1, ..., n+1. \tag{2.2}$$

At one end $k = n$, this is the Minkowski problem. (2.2) is also sufficient in this case. But it is not sufficient for the cases $1 \le k < n$. The natural solution class for this of type equations is in general (for $k < n$) consisting of functions not necessary convex. Hence the major issue is to find conditions for the existence of *convex* solution of (2.1). At the other end $k = 1$, equation (2.1) is linear and it corresponds to the Christoffel problem. The necessary and sufficient conditions for the existence of a convex solution can be read off from the Green function [19]. For the intermediate cases ($2 \le k \le n-1$), (2.1) is a fully nonlinear equation. The first existence theorem was obtained by Pogorelov in [37] under certain restrictive condition on φ (see Remark 5.5 in [GM1]). In [GM1], we deal with the problem using continuity method as a deformation process together with strong minimum principle to force the *convexity*. We recall some definitions.

Definition 1. *For $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$, $S_k(\lambda)$ is defined as*

$$S_k(\lambda) = \sum \lambda_{i_1} ... \lambda_{i_k},$$

where the sum is taking over for all increasing sequences $i_1, ..., i_k$ of the indices chosen from the set $\{1, ..., n\}$. The definition can be extended to symmetric matrices A by taking $S_k(A) = S_k(\lambda(A))$, where $\lambda(A)$ the eigenvalues of A. For $1 \le k \le n$, define

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \quad S_1(\lambda) > 0, ..., S_k(\lambda) > 0\}.$$

A function $u \in C^2(\mathbb{S}^n)$ is called k-convex if the eigenvalues of $W(x) = \{u_{ij}(x) + u(x)\delta_{ij}\}$ is in Γ_k for each $x \in \mathbb{S}^n$. u is called an admissible solution of (2.1) if it is k-convex. u is simply called convex if u is n-convex.

Definition 2. *Let f be a positive $C^{1,1}$ function on $\mathbb{S}^n$ satisfies (2.2), $\forall s \in \mathbb{R}$, we say f is in $\mathcal{C}_s$ if $(f^s_{ij} + \delta_{ij} f^s)$ is semi-positive definite almost everywhere in $\mathbb{S}^n$.*

The following full rank theorem was proved in [GM1].

Theorem 3. *Suppose u is an admissible solution of equation (2.1) with semi-positive definite spherical hessian $W = \{u_{ij} + u\delta_{ij}\}$ on $\mathbb{S}^n$. If $\varphi \in C_{-\frac{1}{k}}$, then W is positive definite on $\mathbb{S}^n$.*

As a consequence, an existence result can be established for the Christoffel-Minkowski problem.

Theorem 4. *Let $\varphi(x) \in C^{1,1}(\mathbb{S}^n)$ be a positive function, suppose $\varphi \in C_{-\frac{1}{k}}$ and φ is connected to 1 in $C_{-\frac{1}{k}}$ then Christoffel-Minkowski problem (2.1) has a unique convex solution up to translations. More precisely, there exists a $C^{3,\alpha}$ ($\forall 0 < \alpha < 1$) closed strictly convex hypersurface M in $\mathbb{R}^{n+1}$ whose kth elementary symmetric function of principal radii on the outer normals is $\varphi(x)$. M is unique up to translations. Furthermore, if $\varphi(x) \in C^{l,\gamma}(\mathbb{S}^n)$ ($l \ge 2, \gamma > 0$), then M is $C^{2+l,\gamma}$.*

The proof of Theorem 3 relies on a deformation lemma for Hessian equation (2.1). This approach was motivated by works of Caffarelli-Friedman [11] and Korevaar-Lewis [26]. This type of deformation lemma enables us to apply the strong maximum principle to enforce the constant rank of $(u_{ij} + u\delta_{ij})$ on $\mathbb{S}^n$. The proof of such deformation lemma in [GM1] is delicate, since equation (2.1) is fully nonlinear.

In [GMZ], Theorem 3 was generalized to Hessian quotient equation. Moreover we establish the existence of admissible solutions for equation (2.1) and Hessian quotient equations. We use the compactness to get the C^0-estimates, the existence theorem from the degree theory.

In [CGM], we establish Theorem 3 for a general class of fully nonlinear elliptic equations. This general phenomenon follows from the ellipticity of $F(W) = -S^{-\frac{1}{k}}(W)$ and concavity of the fully nonlinear operators $G(W) = -F(W^{-1})$. Here we state a sample of this type of results for the equation in the domains of R^n. There is also a corresponding version for equations of spherical hessian on $\mathbb{S}^n$.

Theorem 5. *Let f be a C^2 symmetric function defined on a symmetric domain $\Psi \subset \Gamma_1$ in $\mathbb{R}^n$. Let $\tilde{\Psi} = \{A \in Sym(n) : \lambda(A) \in \Psi\}$, and define $F : \tilde{\Psi} \to R$ by $F(A) = f(\lambda(A))$. If $\tilde{F}(A) = -F(A^{-1})$ is concave functions on the positive definitive matrices, $f_{\lambda_i} = \frac{\partial f}{\partial \lambda_i} > 0$. Then if u is a C^4 convex solution of the following equation in a domain Ω in $\mathbb{R}^n$*

$$F(\{u_{ij}\}) = g(x) \tag{2.3}$$

and $g(x)$ is concave function in Ω. Then the Hessian u_{ij} is constant rank in Ω.

Remark 1. To our knowledge, the conditions in Theorem 5 was introduced by Alvarez-Lasry-Lions in [5]. Theorem 3 and Theorem 5 provide a positive lower bound on the eigenvalues of the corresponding hessians on S^n. In particular, Theorem 3 implies that there is a priori upper bound of principal curvatures of the convex hypersurface M satisfying (2.1).

In 1962, Firey [18] generalized the Minkowski combination to p-sums from $p = 1$ to $p \geq 1$. Later, Lutwak [27, 28] showed that Firey's p-sum also leads to a Brunn-Minkowski theory for each $p > 1$. This theory has found many geometry applications, see for example, [30] and its references. It was also shown in [27] that the classical surface area measures could be extended to the p-sum case. So it is natural to consider a generalization of the classical Christoffel-Minkowski problem for each $p > 1$. The generalized Minkowski problem has been treated in [27, 29, 23, 15].

In [HMS], we get a corresponding existence of Christoffel-Minkowski type problem for the Firey's p-sum.

In [GMTZ], we derive a form of the Alexandrov-Fenchel inequality for appropriate class of k-convex function on S^n, this delete the convexity assumption in the classical Alexandrov-Fenchel inequality. We also prove an uniqueness theorem for Hessian equations, which generalizes the classical Alexandrov-Fenchel-Jessen theorem.

In [GM2], we review some history of convexity of solutions in elliptic partial differential equations and state some applications to classical differential geometry and convex bodies. In fact this is a conference report in "Workshop in Geometry Evolution Equations" in 2002, in NCTS, Taiwan.

3. Weingarten curvature equations

The Christoffel-Minkowski problem was deduced to a convexity problem of a spherical hessian equation on $\mathbb{S}^n$ in the last section. It can also be considered as a curvature equation on the hypersurface via inverse Gauss map. In this section, we concern some curvature equations related to problems in the classical differential geometry. We will indicate how the techniques in convexity estimates for fully nonlinear equations may help us for this type of problems.

For a compact hypersurface M in $\mathbb{R}^{n+1}$, the kth Weingarten curvature at $x \in M$ is defined as

$$W_k(x) = S_k(\kappa_1(x), \kappa_2(x), \cdots, \kappa_n(x))$$

where $\kappa = (\kappa_1, \kappa_2, ..., \kappa_n)$ the principal curvatures of M. In particular, W_1 is the mean curvature, W_2 is the scalar curvature, and W_n is the Gauss-Kronecker curvature. If the surface is starshaped about the origin, it follows that the surface can be parameterized as a graph over $\mathbb{S}^n$:

$$X = \rho(x)x, \qquad x \in \mathbb{S}^n, \tag{3.1}$$

where ρ is the radial function. In this correspondence, the Weingarten curvature can be considered as a function on $\mathbb{S}^n$ or in $\mathbb{R}^{n+1}$. The problem of prescribing curvature functions has attracted much attention. For example, given a positive function F in $\mathbb{R}^{n+1} \setminus \{0\}$, one would like to find a starshaped hypersurface M about the origin such that its kth Weingarten curvature is F. The problem is equivalent to solve the following equation

$$S_k(\kappa_1, \kappa_2, ..., \kappa_n)(X) = F(X) \quad \text{for any} \quad X \in M. \tag{3.2}$$

The uniqueness question of starshaped hypersurfaces with prescribed curvature was studied by Alexandrov [4] and Aeppli [1]. The prescribing Weingarten curvature problem and similar problems have been studied by various authors, we refer to [6, 40, 34, 12, 14, 43, 21] and references there for related works.

We will use notions of admissible solutions as in last section

Definition 3. *A C^2 surface M is called k-admissible if at every point $X \in M$, $\kappa \in \Gamma_k$.*

Under some barrier conditions and monotonicity condition of the prescribed function $F(X)$, an existence result of equation (3.2) was obtained by Bakelman-Kantor [6], Treibergs-Wei [40] for $k = 1$, by Oliker [34] for $k = n$, and by Caffarelli-Nirenberg-Spruck in [12] for general $1 \leq k \leq n$. The solution of the problem [12] in general is not *convex* if $k < n$. The question of convexity of solution in [12] was treated by Chou [14] (see also [43]) for the mean curvature case under concavity assumption on F, and by Gerhardt [21] for general Weingarten curvature case under concavity assumption on $\log F$.

The following is a general principle for the convexity proved in [GLM1].

Theorem 6. *Suppose M is a k-admissible surface of equation (3.2) in $\mathbb{R}^{n+1}$ with semi-positive definite second fundamental form $W = \{h_{ij}\}$ and $F(X) : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^+$ is a* *given smooth positive function. If $F(X)^{-\frac{1}{k}}$ is a convex function in a neighborhood of M, then $\{h_{ij}\}$ is constant rank, so M is strictly convex.*

As a consequence, we deduce the existence of convex hypersurface with prescribed Weingarten curvature in (3.2) in [GLM1]: if in addition to the barrier condition in [12], $F(X)^{-\frac{1}{k}}$ is a convex function in the region $r_1 < |X| < r_2$, then the k-admissible solution in Theorem [12] is strictly convex.

In the literature, the homogeneous Weingarten curvature problem

$$S_k(k_1, k_2, ..., k_n)(X) = \gamma f(\frac{X}{|X|})|X|^{-k}, \quad \forall X \in M, \tag{3.3}$$

also draws some attention. If M is a starshaped hypersurface about the origin in $\mathbb{R}^{n+1}$, by dilation property of the curvature function, the kth Weingarten curvature can be considered as a function of homogeneous degree $-k$ in $\mathbb{R}^{n+1} \setminus \{0\}$. If F is of homogeneous degree $-k$, then the barrier condition in [12] can not be valid unless the function is constant. Therefore equation (3.3) needs a different treatment. In fact, this problem is a nonlinear eigenvalue problem for the curvature equation. When $k = n$, then equation (3.3) can be expressed as a Monge-Ampère equation of radial function ρ on $\mathbb{S}^n$, the problem was studied by Delanoë [16]. The other special case $k = 1$ was considered by Treibergs in [39]. The difficulty for equation (3.3) is the lack of gradient estimate, such kind of estimate does not hold in general (see [39], [GLM1]). Therefore, some conditions have to be in place for f in (3.3). In [GLM1], a uniform treatment for $1 \leq k \leq n$ was given, and together with some discussion on the existence of convex solutions.

Theorem 7. *Suppose $n \geq 2, \quad 1 \leq k \leq n$ and f is a positive smooth function on $\mathbb{S}^n$. If $k < n$, assume further that f satisfies*

$$\sup_{\mathbb{S}^n} \frac{|\nabla f|}{f} < 2k, \tag{3.4}$$

Then there exist a unique constant $\gamma > 0$ with

$$\frac{C_n^k}{\max_{\mathbb{S}^n} f} \leq \gamma \leq \frac{C_n^k}{\min_{\mathbb{S}^n} f} \tag{3.5}$$

and a smooth k-admissible hypersurface M satisfying (3.3) and solution is unique up to homothetic dilations. Furthermore, for $1 \leq k < n$, if in addition $|X| f(\frac{X}{|X|})^{-\frac{1}{k}}$ is convex in $\mathbb{R}^{n+1} \setminus \{0\}$, then M is strictly convex.

Remark 2. Condition (3.4) in Theorem 7 can be weakened, we refer to [GLM1] for the precise statement. When $k = n$, the above result was proved by Delanoë [16]. In this case, the solution is convex automatically. The treatment in [GLM1] is different from [16].

When $k = 1$, the existence part of Theorem 7 was proved in [39], along with a sufficient condition (which is quite complicated) for convexity.

We now switch to a similar curvature equation arising from the problem of prescribing curvature measures in the theory of convex bodies. For a bounded convex body Ω in $\mathbb{R}^{n+1}$ with C^2 boundary M, the corresponding curvature measures of Ω can be defined according to some geometric quantities of M. The k-th curvature measure of Ω is defined as

$$\mathcal{C}_k(\Omega, \beta) := \int_{\beta \cap M} W_{n-k} dF_n,$$

for every Borel measurable set β in $\mathbb{R}^{n+1}$, where dF_n is the volume element of the induced metric of $\mathbb{R}^{n+1}$ on M. Since M is convex, M is star-shaped about some point. We may assume that the origin is inside of Ω. Since M and $\mathbb{S}^n$ is diffemorphism through radial correspondence R_M. Then the k-th curvature measure can also be defined as a measure on each Borel set β in $\mathbb{S}^n$:

$$\mathcal{C}_k(M, \beta) = \int_{R_M(\beta)} W_{n-k} dF_n.$$

Note that $\mathcal{C}_k(M, \mathbb{S}^n)$ is the k-th quermassintegral of Ω.

The problem of prescribing curvature measures is dual to the Christoffel- Minkowski problem in the previous section. The case $k = 0$ is named as the Alexandrov problem, which can be considered as a counterpart to Minkowski problem. The existence, uniqueness and the regularity of the Alexandrov problem in elliptic case was proved by Alexandrov, Pogorelov and Oliker [33](see its more references). Yet, very little is known for the existence problem of prescribing curvature measures $\mathcal{C}_{n-k}$ for $k < n$.

The problem is equivalent to solve the following curvature equation

$$S_k(\kappa_1, \kappa_2, ..., \kappa_n) = \frac{f(x)}{g(x)}, \quad 1 \le k \le n \quad \text{on} \quad \mathbb{S}^n \tag{3.6}$$

where f is the given function on $\mathbb{S}^n$ and $g(x)$ is a function involves the gradient of solution. The major difficulty around equation (3.6) is the lack of C^2 a priori estimates for admissible solutions. Though equation (3.6) is similar to the equation of prescribing Weingarten curvature equation (3.2), the function g (depending on the gradient of solution) makes the matter very delicate. Equation (3.6) was studied in an unpublished notes by Yanyan Li and Pengfei Guan. The uniqueness and C^1 estimates were established for admissible solutions in their notes. In [GLM2], we make use of some ideas in the convexity estimate for curvature equations to overcome the difficulty on C^2 estimate.

Theorem 8. *Suppose* $f(x) \in C^2(\mathbb{S}^n), \quad f > 0,\ n \geq 2,\ 1 \leq k \leq n-1$. *If* f *satisfies the condition*

$$|X|^{\frac{n+1}{k}} f(\frac{X}{|X|})^{-\frac{1}{k}} \quad \textit{is a strictly convex function in} \quad \mathbb{R}^{n+1} \setminus \{0\}, \tag{3.7}$$

then there exists a unique strictly convex hypersurface $M \in C^{3,\alpha}, \alpha \in (0,1)$ *such that it satisfies (3.6).*

When $k = 1$ or 2, the strict convex condition (3.7) can be weakened.

Theorem 9. *Suppose* $k = 1, or \quad 2$ *and* $k < n$, *and suppose* $f(x) \in C^2(\mathbb{S}^n)$ *is a positive function. If* f *satisfies*

$$|X|^{\frac{n+1}{k}} f(\frac{X}{|X|})^{-\frac{1}{k}} \quad \textit{is a convex function in} \quad \mathbb{R}^{n+1} \setminus \{0\}, \tag{3.8}$$

then there exists unique strictly convex hypersurface $M \in C^{3,\alpha}, \alpha \in (0,1)$ *such that it satisfies equation (3.6).*

Theorem 9 yields solutions to two other important measures, the mean curvature measure and scalar curvature measure under convex condition (3.8). For the existence of convex solutions, some condition on f is necessary. In the proof of Theorem 8, the novel feature is the C^2 estimates. Instead of obtaining an upper bound of the principal curvatures, we look for a lower bound of the principal curvatures (the upper bound of principal radii) by transforming (3.6) to a new equation of support function on $\mathbb{S}^n$ through Gauss map. For the proof of Theorem 9, the key part is the C^2 estimates for the case $k = 2$, which we make use of some special structure of S_2. We also establish a deformation lemma as in Theorem 3 and Theorem 6 to ensure the convexity of solutions in the process of applying the method of continuity.

4. Optimal transportation problem

The optimal transportation problem, as proposed by Monge in 1781, is to find an optimal mapping from one mass distribution to another such that a cost functional is minimized among all measure preserving mappings. It was proved [8, 22] that the optimal mapping can be determined by the potential functions, namely the maximizers of Kantorovich's dual functional.

The potential function satisfies a fully nonlinear equation of Monge-Ampère type, subject to a second boundary condition. When the cost function $c(x, y) = x \cdot y$ or $c(x, y) = |x - y|^2$, the equation can be reduced to the standard Monge-Ampère equation

$$\det D^2 u = h, \tag{4.1}$$

and the regularity of solutions has been obtained in [9, 10, 41], and [17] in dimension 2.

In [MTW], we study the regularity of potential functions for general cost functions. We will establish an a priori interior second order derivative estimate for solutions to the corresponding Monge-Ampère equation when the cost function satisfies an additional structural condition. To apply the a priori estimate to the potential functions we will introduce the notion of generalized solutions and prove that a potential function is indeed a generalized solution. The regularity of potential functions then follows as a generalized solution can be approximated by smooth ones.

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