Publ. Math. Debrecen
103/3-4 (2023), 339–384
DOI: 10.5486/PMD.2023.9471

Bases which admit exactly two expansions

By YI CAI (Shanghai) and WENXIA LI (Shanghai)

Abstract. Given a positive integer m, let $\Omega_m = \{0, 1, \dots, m\}$, and let $\mathcal{B}_2(m)$ denote the set of bases $q \in (1, m+1]$ in which there exist numbers having precisely two q-expansions over the alphabet Ω_m. Sidorov [23] firstly studied the set $\mathcal{B}_2(1)$ and raised some questions. Komornik and Kong [15] further investigated the set $\mathcal{B}_2(1)$ and partially answered Sidorov's questions. In the present paper, we consider the set $\mathcal{B}_2(m)$ for general positive integer m, and generalise the results obtained by Komornik and Kong.

1. Introduction

Given a positive integer m and a real number $q \in (1, m+1]$, sequence $(c_i) \in \Omega_m^{\mathbb{N}}$ is called a *q-expansion* with respect to the alphabet $\Omega_m := \{0, 1, \dots, m\}$ of a real number $x \in I_q := [0, \frac{m}{q-1}]$ if

$$x = \sum_{i=1}^{\infty} \frac{c_i}{q^i} := (c_i)_q.$$

A sequence $(c_i) \in \Omega_m^{\mathbb{N}}$ is said to be *infinite* if either all $c_i = 0$ or $c_i \neq 0$ for infinite many i.

Many works were devoted to the sets

$$\mathcal{U}_q := \{x \in I_q : x \text{ has a unique } q\text{-expansion w.r.t. } \Omega_m\} \tag{1.1}$$

Mathematics Subject Classification: Primary: 11A63; Secondary: 37F20, 37B10.
Key words and phrases: q-expansion, unique q-expansion, quasi-greedy q-expansion, generalized golden ratio.
The authors were supported by NSFC No. 12071148, the first author was also supported by NSFC No. 12301108, and the second author by NSFC No. 11971079 and the Science and Technology Commission of Shanghai Municipality (STCSM) No. 22DZ2229014.
The first author is the corresponding author.

and

$$\mathcal{V}_q := \{x \in I_q : x \text{ has at most one doubly infinite } q\text{-expansion}\},$$

where a q-expansion (a_i) w.r.t. Ω_m is called *doubly infinite* if both (a_i) and its *reflection* $\overline{(a_i)} := (m - a_i)$ are infinite. The Hausdorff dimensions of $\mathcal{U}_q$ were determined in [16] and [20].

The following two subsets of $(1, m+1]$ are quite important:

$$\mathcal{U} := \{q \in (1, m+1] : 1 \in \mathcal{U}_q\},$$
$$\mathcal{V} := \{q \in (1, m+1] : 1 \in \mathcal{V}_q\}.$$

It was known that $\mathcal{U}_q \subseteq \mathcal{V}_q$ and $\mathcal{U} \subseteq \mathcal{V}$. Two numbers, the generalized golden ratio $\mathcal{G}(m) \in (1, m+1]$ and the Komornik–Loreti constant $q_{KL} \in (1, m+1]$, play an important role in the study of q-expansions. Their definitions are given in Section 2. When $m = 1$, we have $\mathcal{G}(1) = \frac{1+\sqrt{5}}{2} =: \mathcal{G}$ (the golden ratio) and $q_{KL} \approx 1.787$. It was known that $\mathcal{V} \cap (1, q_{KL}) = \{q_1 < q_2 < \cdots\}$ is countable and $q_n \uparrow q_{KL}$.

For a positive integer k, let

$$\mathcal{B}_k(m) := \{q \in (1, m+1] : \exists\, x \in I_q \text{ has exactly } k \text{ different } q\text{-expansions w.r.t. } \Omega_m\}. \tag{1.2}$$

For the case $m = 1$, there are lots of results have been obtained. Glendinning and Sidorov [14] showed that $\mathcal{U}_q$ is countable for $q \in (\mathcal{G}, q_{KL})$, and uncountable with positive Hausdorff dimension for $q \in (q_{KL}, 2]$. Komornik and Loreti [19], and de Vries, Komornik and Loreti [10] proved that $\mathcal{U}$ is closed from above but not from below, and its closure $\overline{\mathcal{U}}$ is a Cantor set. Komornik and Loreti ([17], [18]) found the smallest number of $\mathcal{U}$ is q_{KL}. Erdős *et al.* ([11], [12], [13]) proved that $\mathcal{B}_k(1) \neq \emptyset$ for each $k \in \mathbb{N} \cup \{\aleph_0\}$ and $\min \mathcal{B}_{\aleph_0}(1) = \mathcal{G}$. Later, it was shown in [4] that $\min \mathcal{B}_k(1) \approx 1.75488$ for any $k \geq 3$. However, one knows very few about $\mathcal{B}_k(1)$ for $k \geq 3$. Fortunately, $\mathcal{B}_2(1)$ is well understood. Sidorov in [23] showed the following:

Theorem A. *Let* $m = 1$, $\mathcal{B}_2(1)$, *and* $\mathcal{U}_q$ *be defined by* (1.2) *and* (1.1). *Then:*

(i) $q \in \mathcal{B}_2(1) \iff 1 \in \mathcal{U}_q - \mathcal{U}_q$;

(ii) $\mathcal{U} \subseteq \mathcal{B}_2(1)$;

(iii) $[T, 2] \subseteq \mathcal{B}_2(1)$ *where* $T \approx 1.83929$ *denotes the Tribonacci number, i.e., the root of* $q^3 - q^2 - q - 1 = 0$;

(iv) *the smallest two elements of* $\mathcal{B}_2(1)$ *are* $q_s \approx 1.71064$, *the root of*

$$q^4 - 2q^2 - q - 1 = 0,$$

and $q_f \approx 1.75488$, *the root of*

$$q^3 - 2q^2 + q - 1 = 0.$$

Now, for given $m \in \mathbb{N}$ and $i \in \mathbb{N} \cup \{0\}$, let

$$\mathcal{B}_2^{(i+1)}(m) := \left\{ q \in (1, m+1] : q \text{ is an accumulation point of } B_2^{(i)}(m) \right\},$$

where $\mathcal{B}_2^{(0)}(m) = \mathcal{B}_2(m)$. Thus, the sequence of sets $B_2^{(i)}(m)$, $i = 1, 2, \ldots$, is decreasing. We denote by $\mathcal{B}_2^{(\infty)}(m)$ the limit of $B_2^{(i)}(m)$, $i = 1, 2, \ldots$, i.e., $\mathcal{B}_2^{(\infty)}(m) = \lim_{i\to\infty} \mathcal{B}_2^{(i)}(m)$. The next results are from [15, Theorems 1.3, 1.5. 1.7, 1.9] for the case $m = 1$.

(I) The following conditions are equivalent: (A) $q \in \mathcal{B}_2(1)$; (B) $1 \in \mathcal{U}_q - \mathcal{U}_q$; (C) $1 \in \overline{\mathcal{U}_q} - \overline{\mathcal{U}_q}$; (D) $1 \in \mathcal{V}_q - \mathcal{V}_q$ and $q \neq \mathcal{G}$.

(II) $\overline{\mathcal{U}} \subset \mathcal{B}_2^{(\infty)}(1)$, $\mathcal{V} \setminus \{\mathcal{G}\} \subset \mathcal{B}_2^{(2)}(1)$.

(III) $\mathcal{B}_2^{(i)}(1)$ is compact for all $i \geq 0$, and $\min \mathcal{B}_2^{(1)}(1) = \min \mathcal{B}_2^{(2)}(1) = q_f \approx 1.75488$.

(IV) Every set $\mathcal{B}_2^{(i)}(1)$ has infinitely many accumulation points in each connected component (q_0, q_0^*) of $(1, 2] \setminus \overline{\mathcal{U}}$.

(V) $\mathcal{B}_2(1) \cap (1, q_{KL})$ contains only algebraic integers, and hence it is countable, where q_{KL} is the Komornik–Loreti constant.

(VI) If $\mathcal{V} \cap (1, q_{KL}) = \{q_n : n = 1, 2, \ldots\}$ where $q_n \uparrow q_{KL}$, then

$$q_{j+1} < \min \mathcal{B}_2^{(2j)}(1) < q_{2j+1}$$

for all $j \geq 1$, and hence, $\min \mathcal{B}_2^{(j)}(1) \nearrow \min \mathcal{B}_2^{(\infty)}(1) = q_{KL}$ as $j \to \infty$.

(VII) For each $j = 0, 1, \ldots$, $\mathcal{B}_2^{(j)}(1) \cap (1, q_{KL})$ has infinitely many isolated points, and they are dense in $\mathcal{B}_2^{(j)}(1) \cap (1, q_{KL})$.

(VIII) For any $q \in \mathcal{B}_2(1)$, we have

$$\lim_{\delta\to 0} \dim_H(\mathcal{B}_2(1) \cap (q - \delta, q + \delta)) \leq 2 \dim_H \mathcal{U}_q.$$

The conclusions of (V), (VII) and (VIII) keep true for the case of the general alphabet Ω_m, since their proofs are independent of the choice of m. Thus, we have:

(V') $\mathcal{B}_2(m)\cap(1,q_{KL})$ contains only algebraic integers, and hence it is countable, where q_{KL}[1] is the Komornik–Loreti constant.

(VII') For each $j=0,1,\dots$, $\mathcal{B}_2^{(j)}(m)\cap(1,q_{KL})$ has infinitely many isolated points, and they are dense in $\mathcal{B}_2^{(j)}(m)\cap(1,q_{KL})$.

(VIII') For any $q\in\mathcal{B}_2(m)$, we have

$$\lim_{\delta\to 0}\dim_H(\mathcal{B}_2(m)\cap(q-\delta,q+\delta))\le 2\dim_H\mathcal{U}_q.$$

As to conclusion (IV), its proof is independent of the choice of m when $i\ge 3$ and need to be reproved for $i=0,1,2$. We shall show that $\mathcal{B}_2^{(i)}(m)$ are compact (see Theorem 1.2 (i)), and so, $\mathcal{B}_2^{(i)}(m)$, $i\ge 0$ is a decreasing sequence. Thus, (IV) keeps true for the general alphabet Ω_m:

(IV') Every set $\mathcal{B}_2^{(i)}(m)$ has infinitely many accumulation points in each connected component (q_0,q_0^*) of $(1,m+1]\setminus\overline{\mathcal{U}}$.

About conclusion (VI), we shall point out that $q_{j+1}\le\min\mathcal{B}_2^{(2j)}(m)$ may not hold for general $m>1$. A counter-example will be given in the remark after Corollary 5.6. However, $q_j\le\min\mathcal{B}_2^{(2j)}(m)$ still holds. Thus, we have the following (IV') with a minor modification:

(VI') If $\mathcal{V}\cap(1,q_{KL})=\{q_n:n=1,2,\dots\}$ where $q_n\uparrow q_{KL}$, then $q_j\le\min\mathcal{B}_2^{(2j)}(m)<q_{2j+1}$ for all $j\ge 1$, and hence, $\min\mathcal{B}_2^{(j)}(m)\nearrow\min\mathcal{B}_2^{(\infty)}(m)=q_{KL}$ as $j\to\infty$.

In this paper, we focus on generalising the conclusions (I), (II) and (III) of [15] by Komornik and Kong into the case of the general alphabet Ω_m. Corresponding to (I), we have

Theorem 1.1. *The following conditions are equivalent:*

(i) $q\in\mathcal{B}_2(m)$;

(ii) $1\in\mathcal{U}_q-\mathcal{U}_q$;

(iii) $1\in\overline{\mathcal{U}_q}-\overline{\mathcal{U}_q}$;

(iv) $1\in\mathcal{V}_q-\mathcal{V}_q$, $q\ne\mathcal{G}(m)$, *where* $\mathcal{G}(m)$ *is the generalized golden ratio.*

---

[1]To simplify notation, we write q_{KL} instead of $q_{KL}(m)$, there is no any confusion once m is being considered at a given moment.

Corresponding to (II) and (III), we have

Theorem 1.2.

(i) $\mathcal{B}_2^{(i)}(m)$ *is compact for all* $i \geq 0$*;*

(ii) $\overline{\mathcal{U}} \subset \mathcal{B}_2^{(\infty)}(m)$*;*

(iii) $\mathcal{V} \setminus \{\mathcal{G}(m)\} \subset \mathcal{B}_2^{(2)}(m)$*;*

(iv) $\min \mathcal{B}_2^{(1)}(m) = \min \mathcal{B}_2^{(2)}(m) = q_f(m)$*,* $q_f(m)$ *is the largest real root of*

$$q^3 - (k+2)q^2 + q - k - 1 = 0, \qquad \text{if } m = 2k+1,$$

and

$$q^2 - (k+1)q - k = 0, \qquad \text{if } m = 2k.$$

The rest of the paper is organized as follows. In Section 2, we recall some basic results on q-expansions. The technical steps will be arranged in Section 3, while in order to present the key results clearly, we put the tedious computations in the Appendix. In Section 4, we prove the main theorems. The final section is devoted to the detailed description of unique expansions.

2. Preliminaries

In this section, we introduce some notations and list some important results. The *greedy* q-expansion of $x \in I_q$ is the largest q-expansion in lexicographical order. The *quasi-greedy* q-expansion of $x \in I_q$ is the largest *infinite* q-expansion in lexicographical order. In the whole paper, denote by $\alpha(q) = (\alpha_i)$ and $\beta(q) = (\beta_i)$ the quasi-greedy and greedy q-expansions of 1, respectively. For a finite word $a_1 \cdots a_n \in \Omega_m^n$, define

$$a_1 \cdots a_{n-1} a_n^+ := a_1 \cdots a_{n-1}(a_n + 1), \qquad \text{if } a_n < m,$$
$$a_1 \cdots a_{n-1} a_n^- := a_1 \cdots a_{n-1}(a_n - 1), \qquad \text{if } a_n > 0.$$

Recall that $\mathcal{U}$ and $\mathcal{V}$ denote the set of univoque bases $q \in (1, m+1]$ and the set of bases $q \in (1, m+1]$ for which there is a unique doubly infinite q-expansion, respectively. Moreover, $(1, m+1] \setminus \overline{\mathcal{U}} = \bigcup (p_0, p_0^*)$, where p_0 runs over $\{1\} \cup (\overline{\mathcal{U}} \setminus \mathcal{U})$ and p_0^* runs over a proper subset of $\overline{\mathcal{U}}$, and $\mathcal{V} \cap (p_0, p_0^*) = \{q_\ell : \ell = 1, 2 \ldots\}$ is a strictly increasing sequence converging to p_0^*. Especially,

$\mathcal{V} \cap (1, q_{KL})$ has a smallest element $\mathcal{G}(m)$ called *generalized golden ratio*, which was given by BAKER [3]. More precisely,

$$\mathcal{G}(m) = \begin{cases} k+1, & \text{if } m = 2k, k = 1, 2, \dots, \\ \frac{k+1+\sqrt{k^2+6k+5}}{2}, & \text{if } m = 2k+1, k = 0, 1, \dots. \end{cases} \tag{2.1}$$

Let $(\tau_i)_{i=0}^{\infty}$ be the classical Thue–Morse sequence. For each positive integer i, let

$$(d_i) = \begin{cases} (k + \tau_i - \tau_{i-1}), & \text{if } m = 2k, k = 1, 2, \dots, \\ (k + \tau_i), & \text{if } m = 2k+1, k = 0, 1, \dots. \end{cases} \tag{2.2}$$

The *Komornik–Loreti constant* q_{KL} is given by $\alpha(q_{KL}) = (d_i)$, i.e., the sequence (d_i) is just the quasi-greedy q_{KL}-expansion of 1. In fact, (d_i) is the unique q_{KL}-expansion of 1. We emphasize the fact that $1 < \mathcal{G}(m) < q_{KL} < m + 1$.

This generalized golden ratio $\mathcal{G}(m)$ plays an important part in the sense that for $q \in (1, \mathcal{G}(m))$, every $x \in (0, \frac{m}{q-1})$ has uncountable q-expansions, and each x has at least countably many q-expansions if $q = \mathcal{G}(m)$. Thus, $\mathcal{U}_q = \left\{0, \frac{m}{q-1}\right\}$ for all $q \in (1, \mathcal{G}(m)]$ (see, e.g., [13], [24]).

The following property was given in [2], which is related to PARRY's work [22] (see also [6], [7], [9]).

Lemma 2.1.

(i) *The map $q \mapsto \alpha(q)$ is a strictly increasing bijection from $(1, m+1]$ onto the set of all infinite sequences (α_i) satisfying the inequality*

$$\alpha_{n+1}\alpha_{n+2}\cdots \leq \alpha_1\alpha_2\cdots, \qquad \text{for all } n \geq 0.$$

Moreover, the map $q \mapsto \alpha(q)$ is continuous from the left.

(ii) *The map $q \mapsto \beta(q)$ is a strictly increasing bijection from $(1, m+1)$ onto the set of all sequences (β_i) satisfying the inequality*

$$\beta_{n+1}\beta_{n+2}\cdots < \beta_1\beta_2\cdots, \qquad \text{for all } n \geq 1.$$

Moreover, the map $q \mapsto \beta(q)$ is continuous from the right.

Remark that $\beta(m+1) = m^{\infty}$. Let $\mathcal{U}'_q$ be the set of the corresponding q-expansions of all elements in $\mathcal{U}_q$ defined by (1.1). We recall the following characterization of unique expansion:

Lemma 2.2 ([2])**.** *Let $q \in (1, m+1]$, then $(c_i) \in \mathcal{U}'_q$ if and only if*

$$\begin{aligned} c_{n+1}c_{n+2}\cdots &< \alpha_1(q)\alpha_2(q)\cdots, && \text{whenever } c_n < m, \\ \overline{c_{n+1}c_{n+2}\cdots} &< \alpha_1(q)\alpha_2(q)\cdots, && \text{whenever } c_n > 0. \end{aligned} \tag{2.3}$$

In fact, it is easy to check that (2.3) is equivalent to

$$\begin{aligned} c_{k+1}c_{k+2}\cdots &< \alpha_1(q)\alpha_2(q)\cdots, && \text{when } c_1\cdots c_k \neq m^k, \\ \overline{c_{k+1}c_{k+2}\cdots} &< \alpha_1(q)\alpha_2(q)\cdots, && \text{when } c_1\cdots c_k \neq 0^k. \end{aligned} \tag{2.4}$$

De Vries, Komornik and Loreti [10] investigated the sets $\mathcal{U}, \overline{\mathcal{U}}$ and $\mathcal{V}$, it was shown that $\mathcal{V}$ is closed and $\overline{\mathcal{U}}$ is a Cantor set.

Lemma 2.3 ([10])**.**

(i) *$q \in \mathcal{U} \setminus \{m+1\}$ if and only if $\alpha(q) = (\alpha_i(q))$ satisfies*

$$\overline{\alpha(q)} < \alpha_{n+1}(q)\alpha_{n+2}(q)\cdots < \alpha(q), \qquad \text{for all } n \geq 1.$$

(ii) *$q \in \overline{\mathcal{U}}$ if and only if $\alpha(q) = (\alpha_i(q))$ satisfies*

$$\overline{\alpha(q)} < \alpha_{n+1}(q)\alpha_{n+2}(q)\cdots \leq \alpha(q), \qquad \text{for all } n \geq 1.$$

(iii) *$q \in \mathcal{V}$ if and only if $\alpha(q) = (\alpha_i(q))$ satisfies*

$$\overline{\alpha(q)} \leq \alpha_{n+1}(q)\alpha_{n+2}(q)\cdots \leq \alpha(q), \qquad \text{for all } n \geq 1. \tag{2.5}$$

The authors also described the following important relations between the three sets, see [10, Theorem 1.2, Lemmas 3.11, 3.14].

Lemma 2.4.

(i) *For every $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$, there exists a sequence $(q_n) \in \mathcal{U}$ satisfying $(q_n) \nearrow q$ as $n \to \infty$.*

(ii) *For every $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$, the quasi-greedy expansion $\alpha(q)$ is periodic.*

(iii) *For every $q \in \mathcal{V} \setminus (\overline{\mathcal{U}} \cup \{\mathcal{G}(m)\})$, there exists a word $a_1 \cdots a_n$ with $n \geq 1$ such that*

$$\alpha(q) = (a_1 \cdots a_{n-1}a_n^+ \overline{a_1 \cdots a_{n-1}a_n^+})^\infty,$$

where $(a_1 \cdots a_n)^\infty$ satisfies (2.5).

Lemma 2.5 ([8, Theorems 1.4, 1.5])**.**

(i) *$\mathcal{U}_q$ is closed if and only if $q \in (1, m+1] \setminus \overline{\mathcal{U}}$.*

(ii) *$\mathcal{U}_q = \overline{\mathcal{U}_q} = \mathcal{V}_q$ if and only if $q \in (1, m+1] \setminus \mathcal{V}$.*

3. $\mathcal{B}_2(m)$ with general alphabet

In this section, we give characterizations of $\mathcal{B}_2(m)$.

Lemma 3.1. *Let $q \in (1, m+1]$. If x has exactly two different q-expansions (a_i) and (b_i) w.r.t. Ω_m satisfying*

$$a_1 \cdots a_{k-1} = b_1 \cdots b_{k-1} \qquad \text{and} \qquad b_k > a_k,$$

then $b_k = a_k + 1$.

Proof. Assume that $b_k = a_k + d, d > 1$, then we can find $1 \leq n \leq d-1$ such that

$$(a_1 \cdots a_{k-1}(a_k+n)0^\infty)_q < (b_1 b_2 \cdots)_q = x$$

and

$$(a_1 \cdots a_{k-1}(a_k+n)m^\infty)_q > (a_1 a_2 \cdots)_q = x.$$

Hence, $q^k(x - (a_1 \cdots a_{k-1}(a_k+n)0^\infty)_q) \in [0, \frac{m}{q-1}]$, and there exists a sequence $c_{k+1}c_{k+2}\cdots$ such that $q^k(x - (a_1 \cdots a_{k-1}(a_k+n)0^\infty)_q) = (c_{k+1}c_{k+2}\cdots)_q$, which is equivalent to $(a_1 \cdots a_{k-1}(a_k+n)c_{k+1}c_{k+2}\cdots)_q = x$. In other words, x has at least three q-expansions, which leads to contradiction. $\square$

Theorem 3.2. *For $q \in (1, m+1]$, $q \in \mathcal{B}_2(m)$ if and only if there exist two sequences $(c_i), (d_i) \in \mathcal{U}'_q$ satisfying the equality*

$$((n+1)(c_i))_q = (n(d_i))_q$$

for all $n = 0, 1, \ldots, m-1$.

Proof. It suffices to take $q \in (\mathcal{G}(m), m+1]$, because $\mathcal{U}_q = \left\{0, \frac{m}{q-1}\right\}$ if $1 < q \leq \mathcal{G}(m)$.

If $q \in \mathcal{B}_2(m)$, then there exists $x \in (0, \frac{m}{q-1})$ having exactly two q-expansions (a_i) and (b_i) w.r.t. Ω_m. Suppose that $a_1 \cdots a_{k-1} = b_1 \cdots b_{k-1}$ and $b_k = a_k + 1$ for some $k \geq 1$ by Lemma 3.1. Thus, the equalities $x = (a_i)_q = (b_i)_q$ imply

$$(0a_{k+1}a_{k+2}\cdots)_q = (1b_{k+1}b_{k+2}\cdots)_q,$$

and so

$$1 = (a_{k+1}a_{k+2}\cdots)_q - (b_{k+1}b_{k+2}\cdots)_q. \tag{3.1}$$

Alternatively, we can rewrite (3.1) as

$$((n+1)(c_i))_q = (n(d_i))_q$$

for all $n = 0, 1, \ldots, m-1$, where $(c_i) = b_{k+1}b_{k+2}\cdots$, $(d_i) = a_{k+1}a_{k+2}\cdots$. By assumption, we have $(c_i), (d_i) \in \mathcal{U}'_q$.

Conversely, there exist (c_i) and (d_i) belonging to $\mathcal{U}'_q$ such that

$$1 + (c_i)_q = (d_i)_q, \qquad \text{i.e., } x := (0d_1d_2\cdots)_q = (1c_1c_2\cdots)_q,$$

and then $1 \in \mathcal{U}_q - \mathcal{U}_q$. Thus, x has no more q-expansion w.r.t. Ω_m starting with 0 or 1. On the other hand, we claim that any q-expansion w.r.t. Ω_m of x ca not start with $2 \le c \le m$. Otherwise,

$$\frac{c}{q} \le x = (0d_1d_2\cdots)_q \le \frac{m}{q(q-1)},$$

which leads to $2 \le c \le \frac{m}{q-1}$. Hence, $q \le 1 + \frac{m}{2} \le \mathcal{G}(m)$ by (2.1). However, $\mathcal{U}_q = \left\{0, \frac{m}{q-1}\right\}$ for all $q \in (1, \mathcal{G}(m)]$, which contradicts $1 \in \mathcal{U}_q - \mathcal{U}_q$. □

For $q \in (1, m+1]$, we set

$$A'_q := \left\{(c_i) \in \mathcal{U}'_q : 0 \le c_1 < \alpha_1(q)\right\}.$$

According to the definition of A'_q, each sequence $(c_i) \in A'_q$ satisfies

$$c_{i+1}c_{i+2}\cdots < \alpha(q)$$

for all $i \ge 0$ by (2.4) (cf. [2]). Hence, we obtain

Lemma 3.3. $\mathcal{U}'_q = \bigcup_{\mathrm{c}\in A'_q}\{\mathrm{c}, \overline{\mathrm{c}}\}$.

Proof. Indeed, this holds for $q \in (1, \mathcal{G}(m)]$ by $\mathcal{U}_q = \{0, m/(q-1)\}$. Suppose that $q \in (\mathcal{G}(m), m+1]$. Let $(d_i) \in \mathcal{U}'_q$ with $d_1 \ge \alpha_1(q)$. Then, by Lemma 2.2, $(\overline{d_i}) \in \mathcal{U}'_q$. Furthermore, we have $\overline{d_1} < \alpha_1(q)$; because $q > \mathcal{G}(m) \ge k+1$ (no matter $m = 2k$ or $2k+1$), which implies that $\alpha_1(q) \ge k+1$, and then $2\alpha_1(q) > m$. Hence, $m - d_1 \le m - \alpha_1(q) < \alpha_1(q)$. We remark that it is possible that both $d_1 < \alpha_1(q)$ and $\overline{d_1} < \alpha_1(q)$. □

Lemma 3.4. *For $q \in (1, m+1]$, $q \in \mathcal{B}_2(m)$ if and only if q is a zero of the function*

$$f_{\mathrm{c},\mathrm{d}}(t) = (1\mathrm{c})_t + (m\mathrm{d})_t - (m^\infty)_t \tag{3.2}$$

for some $\mathrm{c}, \mathrm{d} \in A'_q$, i.e., $(1\mathrm{c})_q + (m\mathrm{d})_q = (m^\infty)_q$.

PROOF. It follows from Theorem 3.2 and Lemma 3.3 that $q \in \mathcal{B}_2(m)$ if and only if q satisfies one of the following equations for some $\mathrm{c}, \mathrm{d} \in A'_q$:

$$(1\mathrm{c})_q = (0\mathrm{d})_q, \quad (1\mathrm{c})_q = (0\overline{\mathrm{d}})_q, \quad (1\overline{\mathrm{c}})_q = (0\mathrm{d})_q \qquad \text{and} \qquad (1\overline{\mathrm{c}})_q = (0\overline{\mathrm{d}})_q. \tag{3.3}$$

We claim that q only satisfies the second equation. For $\mathrm{d} \in A'_q$, one has that $(\mathrm{d})_q < \frac{\alpha_1(q)}{q}$. Thus, for any $s \in \{0, 1, \ldots, m\}^{\mathbb{N}}$,

$$(0\mathrm{d})_q = \frac{1}{q}(\mathrm{d})_q < \frac{\alpha_1(q)}{q^2} \leq \frac{1}{q} = (10^\infty)_q \leq (1s)_q. \tag{3.4}$$

Hence, for any $\mathrm{c}, \mathrm{d} \in A'_q$, one has

$$(1\mathrm{c})_q > (0\mathrm{d})_q \qquad \text{and} \qquad (1\overline{\mathrm{c}})_q > (0\mathrm{d})_q.$$

Finally, the fourth equality $(1\overline{\mathrm{c}})_q = (0\overline{\mathrm{d}})_q$ in (3.3) is equivalent to $((m-1)\mathrm{c})_q = (m\mathrm{d})_q$, and then is equivalent to $(0\mathrm{c})_q = (1\mathrm{d})_q$. However, by (3.4), one has that $(0\mathrm{c})_q < (1\mathrm{d})_q$.

We complete the proof by the equality $(1\mathrm{c})_q - (0\overline{\mathrm{d}})_q = (1\mathrm{c})_q + (m\mathrm{d})_q - (m^\infty)_q$. □

We rewrite (3.2) as

$$f_{\mathrm{c},\mathrm{d}}(t) = ((m+1)(c_i + d_i))_t - (m^\infty)_t, \tag{3.5}$$

where $\mathrm{c} = (c_i)$ and $\mathrm{d} = (d_i)$. It is natural to observe the following properties.

Lemma 3.5. *Let $q \in (1, m+1]$ and $\mathrm{c}, \mathrm{d} \in A'_q$.*

(1) *$f_{\mathrm{c},\mathrm{d}}(t)$ is symmetric w.r.t (c, d), i.e., $f_{\mathrm{c},\mathrm{d}}(t) = f_{\mathrm{d},\mathrm{c}}(t)$.*

(2) *$f_{\mathrm{c},\mathrm{d}}(t) \in C([q, m+1])$ and $f_{\mathrm{c},\mathrm{d}}(q)$ is continuous w.r.t. $(\mathrm{c}, \mathrm{d}) \in A'_q \times A'_q$.*

(3) *If $\mathrm{c}' \in A'_q$ and $\mathrm{c}' > \mathrm{c}$, then $f_{\mathrm{c}',\mathrm{d}}(p) > f_{\mathrm{c},\mathrm{d}}(p)$ for all $p \geq q$. Similarly, if $\mathrm{d}' \in A'_q$ and $\mathrm{d}' > \mathrm{d}$, then $f_{\mathrm{c},\mathrm{d}'}(p) > f_{\mathrm{c},\mathrm{d}}(p)$ for all $p \geq q$.*

(4) *$f_{\mathrm{c},\mathrm{d}}(m+1) \geq 0$.*

PROOF. (1) It just follows from the definition (3.5) of $f_{\mathrm{c},\mathrm{d}}(q)$.

(2) Firstly, we point out that for given $\mathrm{c}, \mathrm{d} \in A'_q$, $f_{\mathrm{c},\mathrm{d}}(t)$ is well-defined for $t \in [q, m+1]$ because $\mathrm{c}, \mathrm{d} \in A'_q \subseteq A'_t$. Note that

$$f_{\mathrm{c},\mathrm{d}}(t) = ((m+1)(c_i + d_i))_t - (m^\infty)_t = \frac{m+1}{t} - \frac{m}{t-1} + \sum_{k=2}^{\infty} \frac{c_{k-1} + d_{k-1}}{t^k}.$$

Denote $S(t)=\sum_{k=2}^{\infty}\frac{c_{k-1}+d_{k-1}}{t^k}$. We show $S(t)$ is continuous in $[q,m+1]$. Note that

$$\frac{c_{k-1}+d_{k-1}}{t^k}\le\frac{2m}{q^k},\ \text{for all } t\in[q,m+1],\quad\text{and}\quad\sum_{k=2}^{\infty}\frac{2m}{q^k}<+\infty.$$

Thus, $\sum_{k=2}^{\infty}\frac{c_{k-1}+d_{k-1}}{t^k}$ converges uniformly in $[q,m+1]$. So, $S(t)\in C([q,m+1])$.

Now, let $\mathrm{c}_n=(c_{n,i})$, $\mathrm{d}_n=(d_{n,i})\in A'_q$ be such that $\mathrm{c}_n\to\mathrm{c}$ and $\mathrm{d}_n\to\mathrm{d}$. Then, for any $k\in\mathbb{N}$, there exist $\ell=\ell(k)\in\mathbb{N}$ such that $c_{n,1}c_{n,2}\cdots c_{n,k}=c_1c_2\cdots c_k$ and $d_{n,1}d_{n,2}\cdots d_{n,k}=d_1d_2\cdots d_k$ whenever $n\ge\ell$. Then, for $n\ge\ell$, we have

$$|f_{\mathrm{c}_n,\mathrm{d}_n}(q)-f_{\mathrm{c},\mathrm{d}}(q)|\le\frac{m}{q^k(q-1)}+\frac{m}{q^k(q-1)}=\frac{2m}{q^k(q-1)}.$$

(3) Note that $\mathrm{c}',\mathrm{c}\in A'_q$, and $\mathrm{c}'>\mathrm{c}$ implies $(\mathrm{c}')_q>(\mathrm{c})_q$. Since $\mathcal{U}'_q\subset\mathcal{U}'_p$ for all $p\ge q$, we have $A'_q\subset A'_p$. The desired result just follows from (3.5).

(4) We have

$$f_{\mathrm{c},\mathrm{d}}(m+1)=((m+1)(c_i+d_i))_{m+1}-(m^\infty)_{m+1}=(0(c_i+d_i))_{m+1}\ge0,$$

as desired. □

We recall that $q_f(m)$ is the largest real root of

$$q^2-(k+1)q-k=0,\qquad\text{when } m=2k,\tag{3.6}$$

and

$$q^3-(k+2)q^2+q-k-1=0,\qquad\text{when } m=2k+1.\tag{3.7}$$

We emphasize the important relations:

$$k+2>q_f(2k)=\frac{k+1+\sqrt{k^2+6k+1}}{2}>\mathcal{G}(2k)=k+1$$

and

$$k+2>q_f(2k+1)>\mathcal{G}(2k+1)=\frac{k+1+\sqrt{k^2+6k+5}}{2}>k+1.$$

Remark. Actually, we obtain $q_f(2k+1)>\mathcal{G}(2k+1)$ by comparing the quasi-greedy expansions of 1, more precisely:

$$\alpha(q_f(2k+1))=((k+1)(k+1)kk)^\infty>((k+1)k)^\infty=\alpha(\mathcal{G}(2k+1)).$$

Now, we pay our attention to the following two results, which are the key steps used afterward.

Lemma 3.6. *Let $q \in [q_f(m), m+1]$ and $\mathrm{c}, \mathrm{d} \in A'_q$ w.r.t. Ω_m. If $f_{\mathrm{c},\mathrm{d}}(q) \leq 0$, then $f_{\mathrm{c},\mathrm{d}}(t)$ is strictly increasing in $[q, m+1]$.*

For $q \in [q_f(m), m+1]$, let

$$B'_q := \left\{(\mathrm{c}, \mathrm{d}) : \mathrm{c}, \mathrm{d} \in A'_q, f_{\mathrm{c},\mathrm{d}}(q) \leq 0\right\}.$$

Obviously, $B'_q \neq \emptyset$ follows from the following calculation:

$$f_{0^\infty,0^\infty}(q) = ((m+1)0^\infty)_q - (m^\infty)_q = \frac{1}{q} - \frac{m}{q(q-1)} \leq 0.$$

Lemma 3.7. *Let $q \in [q_f(m), m+1]$.*

(1) *If $(\mathrm{c}, \mathrm{d}), (\mathrm{e}, \mathrm{d}) \in B'_q$ with $\mathrm{e} > \mathrm{c}$, then $q_{\mathrm{e},\mathrm{d}} < q_{\mathrm{c},\mathrm{d}}$.*

(2) *If $(\mathrm{c}, \mathrm{d}), (\mathrm{c}, \mathrm{e}) \in B'_q$ with $\mathrm{e} > \mathrm{d}$, then $q_{\mathrm{c},\mathrm{e}} < q_{\mathrm{c},\mathrm{d}}$.*

In the remaining part of this section, we divide the above proof of two lemmas into several steps. Since $\mathcal{B}_2(m) \cap (1, q_f(m))$ is a finite discrete set (see [21, Propositions 3.5 and 4.9]), we shall focus on $\mathcal{B}_2(m) \cap [q_f(m), m+1]$. For an infinite sequence $c = c_1c_2\cdots$, write $c|_n := c_1 \cdots c_n$ and $c|_{s,n} := c_s \cdots c_n$ for $s \leq n$. A simple fact will frequently occur in the following lemmas. We list it as a proposition without proof.

Proposition 3.8. *Let $h(x) \in C^3([a, b])$. We have $h(x) > 0$ for $x \in [a, b]$ if the following conditions hold:*

(I) $h'''(x) \geq 0, x \in [a, b]$, or $h'''(x) \leq 0, x \in [a, b]$, or there exist $a < c < b$ such that $h'''(x) \geq 0, x \in [a, c]$ and $h'''(x) \leq 0, x \in [c, b]$.

(II) $h''(a) > 0$, $h'(a) > 0$, $h(a) > 0$ and $h(b) > 0$.

We first consider the case of m being odd.

Lemma 3.9. *Let $m = 2k+1$, $q \in [q_f(m), m+1]$ and $\mathrm{c}, \mathrm{d} \in A'_q$ w.r.t. Ω_m.*

(1) *If $k = 0$ and $\mathrm{c} + \mathrm{d} \geq 00120^\infty$, then $f_{\mathrm{c},\mathrm{d}}(q) > 0$.*

(2) *If $k = 0$ and $\mathrm{c} + \mathrm{d} < 00120^\infty$, then $f_{\mathrm{c},\mathrm{d}}(t)$ is strictly increasing for $t \in [q, 2]$.*

Proof. (1) Since $q_f(1) \leq q \leq 2$, we have

$$\min_{c \geq 00120^\infty} (c)_q = \min\{(00120^\infty)_q, (010^\infty)_q, (0020^\infty)_q\} = (010^\infty)_q.$$

Thus, when $(\mathrm{c}+\mathrm{d}) \geq 00120^\infty$, for $q \in [q_f(1), 2]$, we have

$$\begin{aligned} f_{\mathrm{c},\mathrm{d}}(q) &= ((1+1)(c_i+d_i))_q - (1^\infty)_q = \frac{2}{q} + \frac{1}{q}(\mathrm{c}+\mathrm{d})_q - \frac{1}{q-1} \\ &> \frac{2}{q} + \frac{1}{q}(010^\infty)_q - \frac{1}{q-1} = \frac{2}{q} + \frac{1}{q^3} - \frac{1}{q-1} = \frac{q^3-2q^2+q-1}{q^3(q-1)} \geq 0, \end{aligned}$$

where the last inequality follows from the fact that x^3-2x^2+x-1 is strictly increasing in $[q_f(1), 2]$ and $(q_f(1))^3 - 2(q_f(1))^2 + q_f(1) - 1 = 0$.

(2) Suppose that $k=0$ and $\mathrm{c}+\mathrm{d} < 00120^\infty$. Then, $\mathrm{c}+\mathrm{d} \leq 00112^\infty$. Now, take $q_1, q_2 \in [q, 2]$ with $q_2 > q_1$. It is important to point out that both $f_{\mathrm{c},\mathrm{d}}(q_2)$ and $f_{\mathrm{c},\mathrm{d}}(q_1)$ make sense, because $\mathrm{c}, \mathrm{d} \in A'_t$ for all $t \in [q, 2]$. Then, we have

$$\begin{aligned} f_{\mathrm{c},\mathrm{d}}(q_2) - f_{\mathrm{c},\mathrm{d}}(q_1) &= (2(c_i+d_i))_{q_2} - (1^\infty)_{q_2} - [(2(c_i+d_i))_{q_1} - (1^\infty)_{q_1}] \\ &= 2\left(\frac{1}{q_2} - \frac{1}{q_1}\right) + \sum_{k=1}^{\infty}(c_k+d_k)\left(\frac{1}{q_2^{k+1}} - \frac{1}{q_1^{k+1}}\right) - (1^\infty)_{q_2} + (1^\infty)_{q_1}. \end{aligned}$$

Note that $\frac{1}{q_2^k} - \frac{1}{q_1^k} < 0$ for all $k \geq 1$. Obviously, for two sequences $(\alpha_i), (\beta_i)$ of nonnegative integers, if all $\alpha_i \leq \beta_i$, then

$$\sum_{k=1}^{\infty} \alpha_k \left(\frac{1}{q_2^k} - \frac{1}{q_1^k}\right) \geq \sum_{k=1}^{\infty} \beta_k \left(\frac{1}{q_2^k} - \frac{1}{q_1^k}\right).$$

Thus, when $(\mathrm{c}+\mathrm{d})|_4 = 0011$, we have

$$2\left(\frac{1}{q_2} - \frac{1}{q_1}\right) + \sum_{k=1}^{\infty}(c_k+d_k)\left(\frac{1}{q_2^{k+1}} - \frac{1}{q_1^{k+1}}\right) \geq (200112^\infty)_{q_2} - (200112^\infty)_{q_1}.$$

Furthermore, when $(\mathrm{c}+\mathrm{d})|_4 = 0002$, we claim

$$2\left(\frac{1}{q_2} - \frac{1}{q_1}\right) + \sum_{k=1}^{\infty}(c_k+d_k)\left(\frac{1}{q_2^{k+1}} - \frac{1}{q_1^{k+1}}\right) \geq (200112^\infty)_{q_2} - (200112^\infty)_{q_1}$$

also holds. This is because we have

$$\frac{1}{q_2^5} - \frac{1}{q_1^5} > \frac{1}{q_2^4} - \frac{1}{q_1^4}$$

for $q_2 > q_1$ with $q_1, q_2 \in [q, 2] \subseteq [q_f(1), 2]$. Indeed, $f(x) = \frac{1}{x^5} - \frac{1}{x^4}$ is strictly increasing for $x > 1.25$, and $q_f(1) \approx 1.75$. Therefore, we obtain that

$$f_{\mathrm{c},\mathrm{d}}(q_2) - f_{\mathrm{c},\mathrm{d}}(q_1) \geq ((200112^\infty)_{q_2} - (1^\infty)_{q_2}) - ((200112^\infty)_{q_1} - (1^\infty)_{q_1}).$$

Now, let us think about the function

$$\begin{aligned} g(x) &= (200112^{\infty})_x - (1^{\infty})_x = \frac{2}{x} + \frac{1}{x^4} + \frac{1}{x^5} + \frac{2}{x^5(x-1)} - \frac{1}{x-1} \\ &= \frac{x^5 - 2x^4 + x^2 + 1}{x^5(x-1)}. \end{aligned}$$

We shall prove that it is strictly increasing in $[q, 2]$. Note that

$$g'(x) = \frac{-x^6 + 4x^5 - 2x^4 - 4x^3 + 3x^2 - 6x + 5}{x^6(x-1)^2} := \frac{h(x)}{x^6(x-1)^2}.$$

What left is to verify that $h(x) > 0$ for $x \in [q_f(1), 2]$. By calculating, we put the results in Table 1, for the details, see the Appendix. $\square$

function	$[q_f(1), 2]$	monotonicity
$h^{(4)}(x)$	negative	
$h^{(3)}(x)$	negative	decreasing
$h''(x)$	positive	decreasing
$h'(x)$	positive	increasing
$h(x)$	positive	increasing

Table 1

Lemma 3.10. *Let $m = 2k+1$, $q \in [q_f(m), m+1]$ and $\mathrm{c}, \mathrm{d} \in A'_q$ w.r.t. Ω_m.*

(1) *Let $k = 1$. If $(\mathrm{c}+\mathrm{d})|_1 = 0$ and $\mathrm{c}+\mathrm{d} \geq 0450^{\infty}$, or $\mathrm{c}+\mathrm{d} \geq 120^{\infty}$, then $f_{\mathrm{c},\mathrm{d}}(q) > 0$.*

(2) *Let $k = 1$. If $\mathrm{c}+\mathrm{d} < 0450^{\infty}$, or $(\mathrm{c}+\mathrm{d})|_1 = 1$ and $\mathrm{c}+\mathrm{d} < 120^{\infty}$, then $f_{\mathrm{c},\mathrm{d}}(t)$ is strictly increasing for $t \in [q, 4]$.*

Proof. (1) For the case $(\mathrm{c}+\mathrm{d})|_1 = 0$ and $(\mathrm{c}+\mathrm{d}) \geq 0450^{\infty}$, we split the proof into two cases.

Case 1. $(\mathrm{c}+\mathrm{d})|_2 = 04$ and $\mathrm{c}+\mathrm{d} \geq 0450^{\infty}$. Then,

$$\begin{aligned} f_{\mathrm{c},\mathrm{d}}(q) &= ((3+1)(c_i + d_i))_q - (3^{\infty})_q > (40450^{\infty})_q - (3^{\infty})_q \\ &= \frac{4}{q} + \frac{4}{q^3} + \frac{5}{q^4} - \frac{3}{q-1} = \frac{q^4 - 4q^3 + 4q^2 + q - 5}{q^4(q-1)}. \end{aligned}$$

Now, we need to verify the numerator is positive for $q \in [q_f(3), 4]$. Let $g(x) = x^4 - 4x^3 + 4x^2 + x - 5$. We have

$$g'(x) = 4x^3 - 12x^2 + 8x + 1 \qquad \text{and} \qquad g''(x) = 4(3x^2 - 6x + 2).$$

Note that $q_f(3)$ is the largest real root of $x^3 - 3x^2 + x - 2 = 0$ ($q_f(3) \approx 2.893$). By calculating, $g''(q_f(3))$, $g'(q_f(3))$, $g(q_f(3))$ and $g(4)$ are all positive. Thus, we have $g(x) > 0$ for $x \in [q_f(3), 4]$ by Proposition 3.8.

Case 2. If $(\mathrm{c}+\mathrm{d})|_1 = 0$ and $\mathrm{c}+\mathrm{d} \geq 050^\infty$, then

$$\begin{aligned} f_{\mathrm{c},\mathrm{d}}(q) &= ((3+1)(c_i+d_i))_q - (3^\infty)_q > (4050^\infty)_q - (3^\infty)_q \\ &= \frac{4}{q} + \frac{5}{q^3} - \frac{3}{q-1} = \frac{q^3 - 4q^2 + 5q - 5}{q^3(q-1)}. \end{aligned}$$

We need to verify the numerator is positive for $q \in [q_f(3), 4]$. Let $g(x) = x^3 - 4x^2 + 5x - 5$. Note that $g'''(x) = 6 > 0$ for $x \in [q_f(3), 4]$; and

$$g(q_f(3)) = (q_f(3))^3 - 4(q_f(3))^2 + 5q_f(3) - 5 = -(q_f(3))^2 + 4q_f(3) - 3 > 0.$$

Also, $g''(q_f(3))$, $g'(q_f(3))$ and $g(4)$ are positive. Thus, $g(x) > 0$ for $x \in [q_f(3), 4]$ by Proposition 3.8.

Now, we discuss the case $k = 1$ and $\mathrm{c}+\mathrm{d} \geq 120^\infty$. Since $q \geq q_f(3) > 2$,

$$\min_{\alpha \geq 120^\infty} (\alpha)_q = \min\{(120^\infty)_q, (20^\infty)_q\} = (120^\infty)_q.$$

Thus, we have

$$\begin{aligned} f_{\mathrm{c},\mathrm{d}}(q) &= ((3+1)(c_i+d_i))_q - (3^\infty)_q > (4120^\infty)_q - (3^\infty)_q \\ &= \frac{4}{q} + \frac{1}{q^2} + \frac{2}{q^3} - \frac{3}{q-1} = \frac{q^3 - 3q^2 + q - 2}{q^3(q-1)} \geq 0. \end{aligned}$$

The last inequality follows from the fact that $x^3 - 3x^2 + x - 2$ is strictly increasing in $[q_f(3), 4]$ and $(q_f(3))^3 - 3(q_f(3))^2 + q_f(3) - 2 = 0$ by (3.7).

(2) We first consider the case that $k = 1$ and $\mathrm{c}+\mathrm{d} < 0450^\infty$. We have $m = 3$ and $\mathrm{c}+\mathrm{d} \in \Omega_6^{\mathbb{N}}$. The condition $\mathrm{c}+\mathrm{d} < 0450^\infty$ implies $\mathrm{c}+\mathrm{d} \leq 0446^\infty$. Take $q_1, q_2 \in [q, 4] \subseteq [q_f(3), 4]$ with $q_2 > q_1$. We have

$$\begin{aligned} f_{\mathrm{c},\mathrm{d}}(q_2) - f_{\mathrm{c},\mathrm{d}}(q_1) &= ((4(c_i+d_i))_{q_2} - (3^\infty)_{q_2}) - ((4(c_i+d_i))_{q_1} - (3^\infty)_{q_1}) \\ &= 4\left(\frac{1}{q_2} - \frac{1}{q_1}\right) + \sum_{k=1}^{\infty}(c_k+d_k)\left(\frac{1}{q_2^{k+1}} - \frac{1}{q_1^{k+1}}\right) - (3^\infty)_{q_2} + (3^\infty)_{q_1}. \end{aligned}$$

When $(\mathrm{c}+\mathrm{d})|_3 = 044$, using the same argument as that in (4), we have

$$4\left(\frac{1}{q_2} - \frac{1}{q_1}\right) + \sum_{k=1}^{\infty}(c_k+d_k)\left(\frac{1}{q_2^{k+1}} - \frac{1}{q_1^{k+1}}\right) \geq (40446^\infty)_{q_2} - (40446^\infty)_{q_1}.$$

Furthermore, when $(\mathrm{c}+\mathrm{d})|_4 = 036$, we claim

$$4\left(\frac{1}{q_2}-\frac{1}{q_1}\right)+\sum_{k=1}^{\infty}(c_k+d_k)\left(\frac{1}{q_2^{k+1}}-\frac{1}{q_1^{k+1}}\right) \geq (40446^\infty)_{q_2} - (40446^\infty)_{q_1}$$

also holds. This is because we have

$$2\left(\frac{1}{q_2^4}-\frac{1}{q_1^4}\right) > \frac{1}{q_2^3}-\frac{1}{q_1^3}$$

for $q_2 > q_1$ with $q_1, q_2 \in [q, 4] \subseteq [q_f(3), 4]$. Indeed, $f(x) = \frac{2}{x^4} - \frac{1}{x^3}$ is strictly increasing for $x > \frac{8}{3}$, and $q_f(3) \approx 2.89$. Therefore, we obtain

$$f_{\mathrm{c,d}}(q_2) - f_{\mathrm{c,d}}(q_1) \geq ((40446^\infty)_{q_2} - (3^\infty)_{q_2}) - ((40446^\infty)_{q_1} - (3^\infty)_{q_1}).$$

Now, let us take

$$g(x) = (40446^\infty)_x - (3^\infty)_x = \frac{x^4-4x^3+4x^2+2}{x^4(x-1)},$$

and show that $g(x)$ is strictly increasing in $[q, 4] \subseteq [q_f(3), 4]$. Let

$$g'(x) = \frac{-x^5+8x^4-16x^3+8x^2-10x+8}{x^5(x-1)^2} := \frac{h(x)}{x^5(x-1)^2}.$$

By calculation and Proposition 3.8, $h(x) > 0$ for $x \in [q_f(3), 4] \supseteq [q, 4]$, see Table 2.

function	$q_f(3)$	$x = 4$
$h^{(3)}(x)$	negative	
$h''(x)$	positive	
$h'(x)$	positive	
$h(x)$	positive	positive

Table 2

Now, we turn to consider the case $k = 1$, $(\mathrm{c}+\mathrm{d})|_1 = 1$ and $\mathrm{c}+\mathrm{d} < 120^\infty$. Note that $\mathrm{c}+\mathrm{d} \in \Omega_6^{\mathbb{N}}$. Thus, $\mathrm{c}+\mathrm{d} \leq 116^\infty$. As before, for $q_1, q_2 \in [q, 4] \subseteq [q_f(3), 4]$ with $q_1 < q_2$, we have

$$\begin{aligned} f_{\mathrm{c,d}}(q_2) - f_{\mathrm{c,d}}(q_1) &= (4(c_i+d_i))_{q_2} - (3^\infty)_{q_2} - ((4(c_i+d_i))_{q_1} - (3^\infty)_{q_1}) \\ &\geq (4116^\infty)_{q_2} - (3^\infty)_{q_2} - (4116^\infty)_{q_1} + (3^\infty)_{q_1}. \end{aligned}$$

Again, let

$$g(x) = (4116^\infty)_x - (3^\infty)_x = \frac{x^3 - 3x^2 + 5}{x^3(x-1)},$$

and try to prove it is strictly increasing for $x \in [q, 4] \subseteq [q_f(3), 4]$. Let

$$g'(x) = \frac{-x^4 + 6x^3 - 3x^2 - 20x + 15}{x^4(x-1)} := \frac{h(x)}{x^4(x-1)}.$$

We conclude $h(x) > 0$ in $[q_f(3), 4]$ by Table 3, then $g(x)$ is increasing, where increasing⊕decreasing means the function increases first and then decreases, the same for positive⊕negative. □

function	$[q_f(3), 4]$	monotonicity
$h''(x)$	negative	
$h'(x)$	positive⊕negative	decreasing
$h(x)$	positive	increasing⊕decreasing

Table 3

Lemma 3.11. *Let $m = 2k+1$, $q \in [q_f(m), m+1]$ and $\mathrm{c}, \mathrm{d} \in A'_q$ w.r.t. Ω_m.*

(1) *Let $k \geq 2$. If $(\mathrm{c}+\mathrm{d})|_1 = k-1$ and $\mathrm{c}+\mathrm{d} \geq (k-1)(m+2)0^\infty$, or $\mathrm{c}+\mathrm{d} \geq k(k+1)0^\infty$, then $f_{\mathrm{c},\mathrm{d}}(q) > 0$.*

(2) *Let $k \geq 2$. If $\mathrm{c}+\mathrm{d} < (k-1)(m+2)0^\infty$, or $(\mathrm{c}+\mathrm{d})|_1 = k$ and $\mathrm{c}+\mathrm{d} < k(k+1)0^\infty$, then $f_{\mathrm{c},\mathrm{d}}(t)$ is strictly increasing for $t \in [q, m+1]$.*

Proof. (1) $k \geq 2$. When $(\mathrm{c}+\mathrm{d})|_1 = k-1$ and $(\mathrm{c}+\mathrm{d}) \geq (k-1)(m+2)0^\infty$, we have

$$\begin{aligned} f_{\mathrm{c},\mathrm{d}}(q) &= ((m+1)(c_i+d_i))_q - (m^\infty)_q > ((m+1)(k-1)(m+2)0^\infty)_q - (m^\infty)_q \\ &= \frac{m+1}{q} + \frac{k-1}{q^2} + \frac{m+2}{q^3} - \frac{m}{q-1} = \frac{q^3 - (k+3)q^2 + (k+4)q - 2k - 3}{q^3(q-1)}. \end{aligned}$$

In order to verify that the numerator is positive in $[q_f(2k+1), 2k+2]$, let

$$g(x) = x^3 - (k+3)x^2 + (k+4)x - 2k - 3.$$

Clearly, $g'''(x) = 6$ satisfies condition (I) of Proposition 3.8 in $[q_f(2k+1), 2k+2]$. Table 4 shows $g(x) > 0$ for $x \in [q_f(2k+1), 2k+2]$ by Proposition 3.8.

function	$q_f(2k+1)$	$x=2k+2$
$g'''(x)$	positive	
$g''(x)$	positive	
$g'(x)$	positive	
$g(x)$	positive	positive

Table 4

Now, we consider the case $\mathrm{c}+\mathrm{d} \geq k(k+1)0^\infty$. Note that for $q \in (q_f(2k+1), 2k+2]$ and $q_f(2k+1) > k+1$,

$$\min_{c \geq k(k+1)0^\infty} (c)_q = \min\{(k(k+1)0^\infty)_q, ((k+1)0^\infty)_q\} = (k(k+1)0^\infty)_q.$$

Thus, when $\mathrm{c}+\mathrm{d} \geq k(k+1)0^\infty$, we have

$$\begin{aligned} f_{\mathrm{c},\mathrm{d}}(q) &= ((m+1)(c_i+d_i))_q - (m^\infty)_q > ((m+1)k(k+1)0^\infty)_q - (m^\infty)_q \\ &= \frac{m+1}{q} + \frac{k}{q^2} + \frac{k+1}{q^3} - \frac{m}{q-1} = \frac{q^3-(k+2)q^2+q-k-1}{q^3(q-1)}. \end{aligned}$$

Let $g(x) = x^3-(k+2)x^2+x-k-1$. Then, $g'(x) > 0$ for $x \in (q_f(2k+1), 2k+2]$. So, we have $g(x) > 0$ for $x \in (q_f(2k+1), 2k+2]$, since $g(q_f(2k+1)) = 0$.

(2) We first consider the case that $k \geq 2$ and $\mathrm{c}+\mathrm{d} < (k-1)(m+2)0^\infty$ where $m = 2k+1$. Take $q_1, q_2 \in [q, m+1] \subseteq [q_f(2k+1), 2k+2]$ with $q_2 > q_1$. We split the proof into two cases.

Case 1. $(\mathrm{c}+\mathrm{d})|_1 < (k-1)$. Then, $\mathrm{c}+\mathrm{d} \leq (k-2)(2m)^\infty$. Hence,

$$\begin{aligned} f_{\mathrm{c},\mathrm{d}}(q_2) - f_{\mathrm{c},\mathrm{d}}(q_1) \geq{}& ((m+1)(k-2)(2m)^\infty)_{q_2} - (m^\infty)_{q_2} \\ &- \left(((m+1)(k-2)(2m)^\infty)_{q_1} - (m^\infty)_{q_1}\right). \end{aligned}$$

As before, we take

$$g(x) = ((m+1)(k-2)(2m)^\infty)_x - (m^\infty)_x = \frac{x^2-(k+4)x+3k+4}{x^2(x-1)},$$

and prove $g(x)$ is strictly increasing in $[q_f(2k+1), m+1] \supseteq [q, m+1]$. We have

$$g'(x) = \frac{-x^3+(2k+8)x^2-(10k+16)x+6k+8}{x^3(x-1)^2}.$$

Let $h(x)$ be the numerator of $g'(x)$. We show $h(x) > 0$ in $[q_f(2k+1), m+1] = [q_f(2k+1), 2k+2]$ by Tables 5 and 6.

function	$[q_f(5), 6]$	$q_f(5)$
$h'(x)$	positive	
$h(x)$	positive	positive

Table 5. Case $k = 2$.

function	$[k+1, m+1]$	$k+1$	$m+1$
$h'(x)$	positive$\oplus$negative		
$h(x)$	positive	positive	positive

Table 6. Case $k \geq 3$.

Case 2. $(\mathrm{c}+\mathrm{d})|_1 = k-1$ and $\mathrm{c}+\mathrm{d} < (k-1)(m+2)0^\infty$. In this case, we have $\mathrm{c}+\mathrm{d} \leq (k-1)(m+1)(2m)^\infty$. And so,

$$\begin{aligned} f_{\mathrm{c},\mathrm{d}}(q_2) - f_{\mathrm{c},\mathrm{d}}(q_1) \geq &((m+1)(k-1)(m+1)(2m)^\infty)_{q_2} - (m^\infty)_{q_2} \\ &- (((m+1)(k-1)(m+1)(2m)^\infty)_{q_1} - (m^\infty)_{q_1}) . \end{aligned}$$

It suffices to prove $g(x)$ is strictly increasing in $[q_f(2k+1), 2k+2]$ where

$$\begin{aligned} g(x) &= ((m+1)(k-1)(m+1)(2m)^\infty)_x - (m^\infty)_x \\ &= \frac{x^3 - (k+3)x^2 + (k+3)x + 2k}{x^3(x-1)}. \end{aligned}$$

Let

$$g'(x) = \frac{-x^4 + 2(k+3)x^3 - (4k+12)x^2 - (6k-6)x + 6k}{x^4(x-1)^2} := \frac{h(x)}{x^4(x-1)^2}.$$

We have $h(x) > 0$ for $x \in [k+1, m+1] \supseteq [q_f(2k+1), 2k+2]$ by Table 7 and Proposition 3.8.

function	$[k+1, m+1]$	$k+1$	$m+1$
$h'''(x)$	negative		
$h''(x)$	positive		
$h'(x)$	positive		
$h(x)$	positive	positive	positive

Table 7

Now, we turn to consider the case that $k \geq 2$, $(\mathrm{c}+\mathrm{d})|_1 = k$ and $\mathrm{c}+\mathrm{d} < k(k+1)0^\infty$. Then, $\mathrm{c}+\mathrm{d} \leq kk(2m)^\infty$, and so

$$\begin{aligned} f_{\mathrm{c},\mathrm{d}}(q_2) - f_{\mathrm{c},\mathrm{d}}(q_1) \geq {} & ((m+1)kk(2m)^\infty)_{q_2} - (m^\infty)_{q_2} \\ & - (((m+1)kk(2m)^\infty)_{q_1} - (m^\infty)_{q_1}) . \end{aligned}$$

Let

$$g(x) = ((m+1)kk(2m)^\infty)_x - (m^\infty)_x = \frac{x^3-(k+2)x^2+3k+2}{x^3(x-1)}$$

and

$$g'(x) = \frac{-x^4+(2k+4)x^3-(k+2)x^2-(12k+8)x+3(3k+2)}{x^4(x-1)^2} := \frac{h(x)}{x^4(x-1)^2}.$$

We verify that $g(x)$ is strictly increasing in $[q, m+1] \subseteq [q_f(2k+1), m+1]$ by Table 8. $\square$

function	$[k+1, m+1]$	$k+1$	$m+1$
$h'''(x)$	negative		
$h''(x)$	positive		
$h'(x)$	positive		
$h(x)$	positive	positive	positive

Table 8

Now, we consider the case that m is even.

Lemma 3.12. *Let* $m=2$, $q \in [q_f(2), 3]$ *and* $\mathrm{c},\mathrm{d} \in A'_q$ *w.r.t.* Ω_2.

(1) *If* $\mathrm{c}+\mathrm{d} \geq 0210^\infty$, *then* $f_{\mathrm{c},\mathrm{d}}(q) > 0$.

(2) *If* $\mathrm{c}+\mathrm{d} < 0210^\infty$, *then* $f_{\mathrm{c},\mathrm{d}}(t)$ *is strictly increasing for* $t \in [q, 3]$.

Proof. (1) We have $q_f(2) = 1+\sqrt{2}$ by (3.6). For any $q \in [q_f(2), 3]$,

$$\min_{\alpha \geq 0210^\infty} (\alpha)_q = \min\{(0210^\infty)_q, (10^\infty)_q, (030^\infty)_q\} = (0210^\infty)_q.$$

Thus, when $\mathrm{c}+\mathrm{d} \geq 0210^\infty$, we have

$$\begin{aligned} f_{\mathrm{c},\mathrm{d}}(q) &= (3(c_i+d_i))_q - (2^\infty)_q > (30210^\infty)_q - (2^\infty)_q \\ &= \frac{3}{q} + \frac{2}{q^3} + \frac{1}{q^4} - \frac{2}{q-1} = \frac{q^4-3q^3+2q^2-q-1}{q^4(q-1)}. \end{aligned}$$

We need to check $g(x) := x^4 - 3x^3 + 2x^2 - x - 1 \geq 0$ for $x \in [q_f(2), 3]$. Note that

$$g'(x) = 4x^3 - 9x^2 + 4x - 1 \qquad \text{and} \qquad g''(x) = 12x^2 - 18x + 4.$$

We have $g''(x) > 0$ for $x \in [q_f(2), 3]$. As $g'(q_f(2)) = 4 + 6\sqrt{2}$, we get $g'(x) > 0$ for $x \in [q_f(2), 3]$. Finally, it follows from $g(q_f(2)) = 0$ that $g(x) \geq 0$ for $x \in [q_f(2), 3]$.

(2) We now consider the case $\mathrm{c}+\mathrm{d} < 0210^\infty$. Take $q_1, q_2 \in [q, 3]$ with $q_2 > q_1$. Suppose that $(\mathrm{c}+\mathrm{d})|_2 \leq 01$. Then, $(\mathrm{c}+\mathrm{d}) < 014^\infty$ and

$$\begin{aligned} f_{\mathrm{c},\mathrm{d}}(q_2) - f_{\mathrm{c},\mathrm{d}}(q_1) =& (3(c_i + d_i))_{q_2} - (2^\infty)_{q_2} - (3(c_i + d_i))_{q_1} + (2^\infty)_{q_1} \\ \geq& (3014^\infty)_{q_2} - (2^\infty)_{q_2} - (3014^\infty)_{q_1} - (2^\infty)_{q_1}) \,. \end{aligned}$$

Let

$$g(x) = (3014^\infty)_x - (2^\infty)_x = \frac{x^3 - 3x^2 + x + 3}{x^3(x-1)}.$$

In the same way as in Lemma 3.9, we have $g(x)$ is strictly increasing in $[q_f(2), 3]$.

Furthermore, when $(\mathrm{c}+\mathrm{d})|_2 = 02$ and $\mathrm{c}+\mathrm{d} < 0210^\infty$, we claim

$$\begin{aligned} f_{\mathrm{c},\mathrm{d}}(q_2) - f_{\mathrm{c},\mathrm{d}}(q_1) =& (3(c_i + d_i))_{q_2} - (2^\infty)_{q_2} - (3(c_i + d_i))_{q_1} + (2^\infty)_{q_1} \\ \geq& (3014^\infty)_{q_2} - (2^\infty)_{q_2} - (3014^\infty)_{q_1} - (2^\infty)_{q_1}) \end{aligned}$$

also holds. This is because $\frac{1}{x^3} - \frac{4}{x^4}$ is strictly increasing in $[q_f(2), 3]$, and so

$$\frac{1}{q_2^3} - \frac{1}{q_1^3} > \frac{4}{q_2^4} - \frac{4}{q_1^4}$$

for $q_2 > q_1$ with $q_1, q_2 \in [q, 2] \subseteq [q_f(2), 3]$. □

Lemma 3.13. *Let $m = 2k$, $q \in [q_f(m), m+1]$ and $\mathrm{c}, \mathrm{d} \in A'_q$ over Ω_m.*

(1) *Let $k \geq 1$. If $\mathrm{c}+\mathrm{d} \geq k0^\infty$, then $f_{\mathrm{c},\mathrm{d}}(q) > 0$.*

(2) *Let $k \geq 2$. We have $f_{\mathrm{c},\mathrm{d}}(q) > 0$ if c, d satisfy one of the following conditions:*
   - (i) $(\mathrm{c}+\mathrm{d})|_1 = k-1$ *and* $\mathrm{c}+\mathrm{d} \geq (k-1)(m-1)(k+1)0^\infty$;
   - (ii) $(\mathrm{c}+\mathrm{d})|_1 = k-2$ *and* $\mathrm{c}+\mathrm{d} \geq (k-2)(2m-1)(k+1)0^\infty$.

(3) *Let $k \geq 2$. Then, $f_{\mathrm{c},\mathrm{d}}(t)$ is strictly increasing in $[q, m+1]$ if c, d satisfy one of the following conditions:*
   - (i) $(\mathrm{c}+\mathrm{d})|_1 = k-1$ *and* $\mathrm{c}+\mathrm{d} < (k-1)(m-1)(k+1)0^\infty$;
   - (ii) $\mathrm{c}+\mathrm{d} < (k-2)(2m-1)(k+1)0^\infty$.

PROOF. (1) When $(\mathrm{c}+\mathrm{d}) \geq k0^\infty$, we have

$$\begin{aligned} f_{\mathrm{c},\mathrm{d}}(q) &= ((m+1)(c_i+d_i))_q - (m^\infty)_q > ((m+1)k0^\infty)_q - (m^\infty)_q \\ &= \frac{m+1}{q} + \frac{k}{q^2} - \frac{m}{q-1} = \frac{q^2-(k+1)q-k}{q^2(q-1)} \geq 0. \end{aligned}$$

The last equality follows from the fact that $x^2-(k+1)x-k$ is strictly increasing in $[q_f(m), m+1]$ and $(q_f(2k))^2-(k+1)q_f(2k)-k=0$.

(2) We first consider the case (i): $(\mathrm{c}+\mathrm{d})|_1 = k-1$ and $\mathrm{c}+\mathrm{d} \geq (k-1)(m-1)(k+1)0^\infty$. Note that for any $q \in [q_f(2k), 2k+1]$,

$$\min_{\alpha \geq (m-1)(k+1)0^\infty} (\alpha)_q = \min\{((m-1)(k+1)0^\infty)_q, (m0^\infty)_q\} = ((m-1)(k+1)0^\infty)_q.$$

Thus, we have

$$\begin{aligned} f_{\mathrm{c},\mathrm{d}}(q) &= ((m+1)(c_i+d_i))_q - (m^\infty)_q \geq ((m+1)(k-1)(m-1)(k+1)0^\infty)_q - (m^\infty)_q \\ &= \frac{q^4-(k+2)q^3+kq^2-(k-2)q-k-1}{q^4(q-1)}. \end{aligned}$$

Let $g(x) = x^4-(k+2)x^3+kx^2-(k-2)x-k-1$, and we show it is positive for $x \in [q_f(2k), 2k+1]$ by Table 9.

function	$[q_f(2k), 2k+1]$	$k+1$	$q_f(2k)$
$g''(x)$	positive		
$g'(x)$		positive	
$g(x)$	positive		positive

Table 9

Now, we consider the case (ii): $(\mathrm{c}+\mathrm{d})|_1 = k-2$ and $\mathrm{c}+\mathrm{d} \geq (k-2)(2m-1)(k+1)0^\infty$. Note that for any $q \in [q_f(2k), 2k+1]$,

$$\min_{\alpha \geq (2m-1)(k+1)0^\infty} (\alpha)_q = \min\{((2m-1)(k+1)0^\infty)_q, (2m0^\infty)_q\} = ((2m-1)(k+1)0^\infty)_q.$$

Thus, we have

$$\begin{aligned} f_{\mathrm{c},\mathrm{d}}(q) &= ((m+1)(c_i+d_i))_q - (m^\infty)_q \geq ((m+1)(k-2)(2m-1)(k+1)0^\infty)_q - (m^\infty)_q \\ &= \frac{q^4-(k+3)q^3+(3k+1)q^2-(3k-2)q-k-1}{q^4(q-1)}. \end{aligned}$$

Let $g(x) = x^4 - (k+3)x^3 + (3k+1)x^2 - (3k-2)x - k - 1$. Table 9 shows $g(x) > 0$ for $x \in [q_f(2k), 2k+1]$.

(3) Let $q_1, q_2 \in [q, m+1] = [q, 2k+1]$ with $q_2 > q_1$.

(i) Suppose that $(\mathrm{c}+\mathrm{d})|_1 = k-1$ and $\mathrm{c}+\mathrm{d} < (k-1)(m-1)(k+1)0^\infty$. We split the proof into two cases.

Case 1. $(\mathrm{c}+\mathrm{d})|_1 = k-1$ and $(\mathrm{c}+\mathrm{d}) < (k-1)(m-1)0^\infty$. Then, $(\mathrm{c}+\mathrm{d}) \leq (k-1)(m-2)(2m)^\infty$. Thus,

$$\begin{aligned} f_{\mathrm{c},\mathrm{d}}(q_2) - f_{\mathrm{c},\mathrm{d}}(q_1) \geq {} & ((m+1)(k-1)(m-2)(2m)^\infty)_{q_2} - (m^\infty)_{q_2} \\ & - (((m+1)(k-1)(m-2)(2m)^\infty)_{q_1} - (m^\infty)_{q_1}) . \end{aligned}$$

We shall prove $g(x)$ is strictly increasing in $[q, m+1]$ where

$$g(x) = ((m+1)(k-1)(m-2)(2m)^\infty)_x - (m^\infty)_x = \frac{x^3 - (k+2)x^2 + (k-1)x + 2k + 2}{x^3(x-1)}.$$

Note that

$$g'(x) = \frac{-x^4 + (2k+4)x^3 - (4k-1)x^2 - (6k+10)x + 6(k+1)}{x^4(x-1)^2} := \frac{h(x)}{x^4(x-1)^2}.$$

Then, $h(x) > 0$ for $x \in [q_f(2k), m+1]$ follows from Tables 10, 11.

function	$[3,5]$	$x=3$	$x=5$
$h''(x)$	positive$\oplus$negative		
$h'(x)$		positive	positive
$h(x)$	positive	positive	positive

Table 10. Case $k=2$.

function	$[k+1, m+1]$	$k+1$	$m+1$
$h''(x)$	positive$\oplus$negative		
$h'(x)$		positive	negative
$h(x)$	positive	positive	positive

Table 11. Case $k \geq 3$.

Case 2. $(\mathrm{c}+\mathrm{d})|_2 = (k-1)(m-1)$ and $(\mathrm{c}+\mathrm{d}) < (k-1)(m-1)(k+1)0^\infty$. Then, $(\mathrm{c}+\mathrm{d}) \leq (k-1)(m-1)k(2m)^\infty$. Thus,

$$f_{\mathrm{c,d}}(q_2)-f_{\mathrm{c,d}}(q_1)\geq ((m+1)(k-1)(m-1)k(2m)^{\infty})_{q_2}-(m^{\infty})_{q_2}\\ -(((m+1)(k-1)(m-1)k(2m)^{\infty})_{q_1}-(m^{\infty})_{q_1}).$$

Table 12 shows that $g(x)$ is strictly increasing in $[q,m+1]$, where

$$g(x)=((m+1)(k-1)(m-1)k(2m)^{\infty})_x-(m^{\infty})_x=\frac{x^4-(k+2)x^3+kx^2-(k-1)x+3k}{x^4(x-1)},$$

and $g'(x)=\frac{h(x)}{x^5(x-1)^2}$.

function	$[k+1,m+1]$	$k+1$	$m+1$
$h'''(x)$	negative		
$h''(x)$		positive	
$h'(x)$		positive	
$h(x)$	positive	positive	positive

Table 12

(ii) Suppose that $(\mathrm{c}+\mathrm{d})<(k-2)(2m-1)(k+1)0^{\infty}$. We split the proof into three cases.

Case 1. $(\mathrm{c}+\mathrm{d})|_2=(k-2)(2m-1)$ and $(\mathrm{c}+\mathrm{d})<(k-2)(2m-1)(k+1)0^{\infty}$. Then, $(\mathrm{c}+\mathrm{d})\leq(k-2)(2m-1)k(2m)^{\infty}$. Thus,

$$f_{\mathrm{c,d}}(q_2)-f_{\mathrm{c,d}}(q_1)\geq ((m+1)(k-2)(2m-1)k(2m)^{\infty})_{q_2}-(m^{\infty})_{q_2}\\ -(((m+1)(k-2)(2m-1)k(2m)^{\infty})_{q_1}-(m^{\infty})_{q_1}).$$

As before, let $g(x)=((m+1)(k-2)(2m-1)k(2m)^{\infty})_x-(m^{\infty})_x$ and try to prove it is strictly increasing in $[q,m+1]$. Note that

$$\begin{aligned}g(x)&=((m+1)(k-2)(2m-1)k(2m)^{\infty})_x-(m^{\infty})_x\\&=\frac{x^4-(k+3)x^3+(3k+1)x^2-(3k-1)x+3k}{x^4(x-1)},\end{aligned}$$

and

$$g'(x)=\frac{-x^5+(2k+6)x^4-(10k+6)x^3+(18k-2)x^2-(24k-3)x+12k}{x^5(x-1)^2}:=\frac{h(x)}{x^5(x-1)^2}.$$

By calculation, $h'''(x) = 12(-5x^2 + 4(k+3)x - 5k - 3) < 0$ for $x \in [q_f(2k), 2k+1]$, and

$$h''(q_f(2k)) > 0, \quad h'(q_f(2k)) > 0, \quad h(q_f(2k)) > 0 \quad \text{and} \quad h(m+1) > 0.$$

From Proposition 3.8, it follows that $h(x) > 0$ for $x \in [q_f(2k), 2k+1]$.

Case 2. $(\mathrm{c}+\mathrm{d})|_1 = k-2$ and $(\mathrm{c}+\mathrm{d}) < (k-2)(2m-1)0^\infty$. Then, $(c_i + d_i) \leq (k-2)(2m-2)(2m)^\infty$, and

$$\begin{aligned} f_{\mathrm{c,d}}(q_2) - f_{\mathrm{c,d}}(q_1) \geq \; & ((m+1)(k-2)(2m-2)(2m)^\infty)_{q_2} - (m^\infty)_{q_2} \\ & - (((m+1)(k-2)(2m-2)(2m)^\infty)_{q_1} - (m^\infty)_{q_1}). \end{aligned}$$

Table 12 shows that $g(x)$ is strictly increasing in $[q_f(2k), m+1]$ where

$$g(x) = ((m+1)(k-2)(2m-2)(2m)^\infty)_x - (m^\infty)_x = \frac{x^3 - (k+3)x^2 + 3kx + 2}{x^3(x-1)}$$

and

$$g'(x) = \frac{-x^4 + (2k+6)x^3 - (10k+3)x^2 + (6k-8)x + 6}{x^4(x-1)^2} := \frac{h(x)}{x^4(x-1)^2}.$$

Case 3. $k \geq 3$ and $(\mathrm{c}+\mathrm{d}) < (k-2)0^\infty$. Then, $(\mathrm{c}+\mathrm{d}) \leq (k-3)(2m)^\infty$. So,

$$\begin{aligned} f_{\mathrm{c,d}}(q_2) - f_{\mathrm{c,d}}(q_1) \geq \; & ((m+1)(k-3)(2m)^\infty)_{q_2} - (m^\infty)_{q_2} \\ & - ((m+1)(k-3)(2m)^\infty)_{q_1} - (m^\infty)_{q_1}). \end{aligned}$$

Again, one needs to prove $g(x)$ is strictly increasing in $[q_f(2k), m+1]$ where

$$g(x) = ((m+1)(k-3)(2m)^\infty)_x - (m^\infty)_x = \frac{x^2 - (k+4)x + 3k + 3}{x^2(x-1)}.$$

Note that

$$g'(x) = \frac{-x^3 + (2k+8)x^2 - (10k+13)x + 6(k+1)}{x^3(x-1)^2} := \frac{h(x)}{x^3(x-1)^2}.$$

The roots x_1, x_2 of $h'(x) = -3x^2 + 4(k+4)x - 10k - 13 = 0$ satisfy

$$\begin{aligned} & x_1 < q_f(6) < 7 < x_2, && \text{for } k = 3, \\ & x_1 < q_f(2k) < x_2 < 2k+1, && \text{for } k \geq 4. \end{aligned}$$

Moreover, one can check that $h(q_f(2k)) > 0$ and $h(2k+1) > 0$. Thus, $h(x) > 0$ in $[q_f(2k), 2k+1]$. $\square$

Lemma 3.14. *Let* $(\mathrm{c},\mathrm{d}) \in B'_q$*, then there exists a unique* $q_{\mathrm{c},\mathrm{d}} \in \mathcal{B}_2(m)$*.*

Proof. Since $f_{\mathrm{c},\mathrm{d}}(m+1) \geq 0$ always holds for any $(\mathrm{c},\mathrm{d}) \in B'_q$, from Lemma 3.6 it follows that for any $(\mathrm{c},\mathrm{d}) \in B'_q$, there exists a unique $q_{\mathrm{c},\mathrm{d}} \in [q, m+1]$ such that

$$f_{\mathrm{c},\mathrm{d}}(q_{\mathrm{c},\mathrm{d}}) = (1\mathrm{c})_{q_{\mathrm{c},\mathrm{d}}} + (m\mathrm{d})_{q_{\mathrm{c},\mathrm{d}}} - (m^\infty)_{q_{\mathrm{c},\mathrm{d}}} = 0,$$

which means that $q_{\mathrm{c},\mathrm{d}} \in \mathcal{B}_2(m)$. $\square$

Proof of Lemma 3.6. It was established by Lemmas 3.9–3.13. $\square$

Proof of Lemma 3.7. By the symmetry, it suffices to prove (3.7). According to Lemma 3.5,

$$\begin{aligned} f_{\mathrm{c},\mathrm{d}}(q_{\mathrm{e},\mathrm{d}}) &= (1\mathrm{c})_{q_{\mathrm{e},\mathrm{d}}} + (m\mathrm{d})_{q_{\mathrm{e},\mathrm{d}}} - (m^\infty)_{q_{\mathrm{e},\mathrm{d}}} \\ &< (1\mathrm{e})_{q_{\mathrm{e},\mathrm{d}}} + (m\mathrm{d})_{q_{\mathrm{e},\mathrm{d}}} - (m^\infty)_{q_{\mathrm{e},\mathrm{d}}} = f_{\mathrm{e},\mathrm{d}}(q_{\mathrm{e},\mathrm{d}}) = 0. \end{aligned}$$

By Lemma 3.6, we have $q_{\mathrm{c},\mathrm{d}} > q_{\mathrm{e},\mathrm{d}}$. $\square$

4. Proof of Theorems 1.1 and 1.2

The following results reveal that $\overline{\mathcal{U}}, \mathcal{V}$ are subsets of $\mathcal{B}_2(m)$.

Lemma 4.1. $\overline{\mathcal{U}} \subset \mathcal{B}_2(m)$*. Furthermore,* $\overline{\mathcal{U}} \subset \mathcal{B}_2^{(\infty)}(m)$*.*

Proof. Take a $q \in \mathcal{U}$ arbitrarily, 1 has a unique q-expansion, write (c_i). Then, $1 = (c_i)_q - (0^\infty)_q$, which implies $q \in \mathcal{B}_2(m)$ by Theorem 3.2. Since $\overline{\mathcal{U}}$ is a Cantor set, we have $\overline{\mathcal{U}} \subset \mathcal{B}_2^{(\infty)}(m)$ when $\overline{\mathcal{U}} \subset \mathcal{B}_2(m)$. Next, we show that $\overline{\mathcal{U}} \setminus \mathcal{U} \subset \mathcal{B}_2(m)$.

Let $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$. By Lemma 2.4 (ii), there exists a word $a_1 a_2 \cdots a_n$ such that $\alpha(q) = (a_1 a_2 \cdots a_n)^\infty$ where n is the smallest period of $\alpha(q)$. If $n = 1$, then $\alpha(q) = (\alpha_1(q))^\infty$, which implies that $q = \alpha_1(q) + 1$. Otherwise, q is a noninteger. Therefore, we distinguish two cases.

Case I. q is a noninteger. In this case, $n \geq 2$ and $\beta(q) = a_1 \cdots a_n^+ 0^\infty$. From Lemmas 2.1 and 2.3 (ii), we know that

$$\overline{a_1 \cdots a_{n-i}} \leq a_{i+1} \cdots a_n < a_{i+1} \cdots a_n^+ \leq a_1 \cdots a_{n-i} \tag{4.1}$$

for all $0 < i < n$. Since $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$, Lemma 2.4 (i) tells us that there exists a $p \in \mathcal{U} \cap (1, q)$ such that

$$\alpha_1(p) \cdots \alpha_n(p) = a_1 a_2 \cdots a_n.$$

Let

$$\mathrm{c} = \overline{a_1 \cdots a_n^+}\alpha(p) \qquad \text{and} \qquad \mathrm{d} = 0^n\overline{\alpha(p)}.$$

It remains to prove that $\mathrm{c}, \mathrm{d} \in A'_q$ and $q = q_{\mathrm{c},\mathrm{d}}$.

First, we show that $\mathrm{c}, \mathrm{d} \in A'_q$. Since $p \in \mathcal{U} \cap (1, q)$, by Lemmas 2.1 and 2.3 (i), we have

$$\overline{\alpha(q)} < \overline{\alpha(p)} < \sigma^i(\alpha(p)) < \alpha(p) < \alpha(q)$$

for all $i > 0$. Moreover, $a_1 \geq k+1$ implies that $a_1 > \overline{a_1}$. Then, $\mathrm{d} \in A'_q$. On the other hand, for $0 < i < n$, by (4.1) we have $\overline{a_{i+1} \cdots a_n^+} \geq \overline{a_1 \cdots a_{n-i}}$ and $a_1 \cdots a_i \geq \overline{a_{n-i+1} \cdots a_n}$, then

$$\begin{aligned}\overline{a_{i+1} \cdots a_n^+}\alpha(p) &= \overline{a_{i+1} \cdots a_n^+} a_1 \cdots a_i \alpha_{i+1}(p) \cdots \\ &\geq \overline{a_1 \cdots a_{n-i} a_{n-i+1} \cdots a_n}\,\overline{\alpha(p)} > \overline{a_1 \cdots a_n}\,\overline{\alpha(q)} = \overline{\alpha(q)}.\end{aligned}$$

Together with $\overline{a_{i+1} \cdots a_n^+} < a_1 \cdots a_{n-i}$, we obtain $\mathrm{c} \in A'_q$. We conclude that $q = q_{\mathrm{c},\mathrm{d}} \in \mathcal{B}_2(m)$ by the following calculation:

$$\begin{aligned}f_{\mathrm{c},\mathrm{d}}(q) &= (1\mathrm{c})_q + (m\mathrm{d})_q - (m^\infty)_q = (1\overline{a_1 \cdots a_n^+}\alpha(p))_q + (m0^n\overline{\alpha(p)})_q - (m^\infty)_q \\ &= ((m+1)\overline{a_1 \cdots a_n^+}0^\infty)_q - (m^{n+1}0^\infty)_q = (10^\infty)_q - (0a_1 \cdots a_n^+0^\infty)_q = 0.\end{aligned}$$

Case II. q is an integer. Let

$$\mathrm{c} = \overline{a_1^+}\alpha(p) \qquad \text{and} \qquad \mathrm{d} = 0\overline{\alpha(p)}.$$

As was the case in the previous analysis, it can be proved similarly. □

Lemma 4.2. $\mathcal{V} \setminus \{\mathcal{G}(m)\} \subset \mathcal{B}_2(m)$.

Proof. Thanks to Lemma 4.1, it suffices to prove that $\mathcal{V} \setminus (\overline{\mathcal{U}} \cup \{\mathcal{G}(m)\}) \subset \mathcal{B}_2(m)$. Given arbitrarily $q \in \mathcal{V} \setminus (\overline{\mathcal{U}} \cup \{\mathcal{G}(m)\})$, Lemma 2.4 (iii) tells us that there exists a word $a_1 \cdots a_n$ with $n \geq 1$ such that

$$\alpha(q) = (a_1 \cdots a_n^+ \overline{a_1 \cdots a_n^+})^\infty,$$

and for all $0 < i < n$,

$$\overline{(a_1 \cdots a_n)^\infty} \leq \sigma^i((a_1 \cdots a_n)^\infty) < (a_1 \cdots a_n)^\infty.$$

Take $\mathrm{c} = (c_i) = \overline{a_1 \cdots a_n^+}(a_1 \cdots a_n)^\infty$ and $\mathrm{d} = (d_i) = 0^{2n}(\overline{a_1 \cdots a_n})^\infty$. It remains to show $\mathrm{c}, \mathrm{d} \in A'_q$ and $q = q_{\mathrm{c},\mathrm{d}} \in \mathcal{B}_2(m)$.

(i) Suppose that $n \ge 2$. Then, by the above inequalities and the definition of $\mathcal{V}$, we have for all $0 < i < n$,

$$\begin{aligned} &\overline{a_{i+1}\cdots a_n^+} < \overline{a_{i+1}\cdots a_n} \le a_1\cdots a_{n-i} \\ &a_{i+1}\cdots a_n < a_{i+1}\cdots a_n^+ \le a_1\cdots a_{n-i}. \end{aligned} \tag{4.2}$$

By Lemma 2.2, it suffices to prove for all $0 < i < 2n$,

$$\begin{aligned} &c_{i+1}c_{i+2}\cdots < \alpha(q), && \text{when } c_i < m, \\ &\overline{c_{i+1}c_{i+2}\cdots} < \alpha(q), && \text{when } c_i > 0. \end{aligned}$$

If $0 < i < n$, it follows from (4.2). If $n < i < 2n$, we use (4.2) once again. For the case $i = n$, we have $a_1 \ge k+1$ (no matter $m = 2k$ or $m = 2k+1$), it follows from $\overline{a_1} < a_1$. Hence, $\mathrm{c} \in A'_q$. Similarly, one can show $\mathrm{d} \in A'_q$.

Next, we show $q = q_{\mathrm{c},\mathrm{d}}$. Since $\beta(q) = a_1\cdots a_n^+\overline{a_1\cdots a_n}0^\infty$, we have

$$\begin{aligned} f_{\mathrm{c},\mathrm{d}}(q) &= (1\mathrm{c})_q + (m\mathrm{d})_q - (m^\infty)_q \\ &= (1\overline{a_1\cdots a_n^+}(a_1\cdots a_n)^\infty)_q + (m0^{2n}(\overline{a_1\cdots a_n})^\infty)_q - (m^\infty)_q \\ &= ((m+1)\overline{a_1\cdots a_n^+}a_1\cdots a_n0^\infty)_q - (m^{2n+1}0^\infty)_q \\ &= (10^na_1\cdots a_n0^\infty)_q - (0a_1\cdots a_n^+m^n0^\infty)_q \\ &= (10^\infty)_q - (0a_1\cdots a_n^+\overline{a_1\cdots a_n}0^\infty)_q = 0. \end{aligned}$$

(ii) Suppose that $n = 1$. Take

$$\mathrm{c} = \overline{a_1^+}a_1^\infty \qquad \text{and} \qquad \mathrm{d} = 0^2(\overline{a_1})^\infty.$$

In this case, $\alpha(q) = (a_1^+\overline{a_1^+})^\infty$, $a_1^+ \ge k+1$ ($m = 2k$) or $a_1^+ \ge k+2$ ($m = 2k+1$). Then, $\mathrm{c}, \mathrm{d} \in A'_q$ and $q = q_{\mathrm{c},\mathrm{d}}$ follow from that facts $\overline{a_1^+} < a_1 < a_1^+$ and

$$\begin{aligned} f_{\mathrm{c},\mathrm{d}}(q) &= (1\mathrm{c})_q + (m\mathrm{d})_q - (m^\infty)_q = (1\overline{a_1^+}a_1^\infty)_q + (m0^2(\overline{a_1})^\infty)_q - (m^\infty)_q \\ &= ((m+1)\overline{a_1^+}a_10^\infty)_q - (m^30^\infty)_q = (10a_10^\infty)_q - (0a_1^+m0^\infty)_q \\ &= (10^\infty)_q - (0a_1^+\overline{a_1}0^\infty)_q = 0. \end{aligned}$$

So the proof is finished. $\square$

The following result strengthens Theorem 3.2.

Proposition 4.3. *We write*

$$\mathcal{E}_2(m) := \left\{q \in (1, m+1] : 1 \in \overline{\mathcal{U}_q} - \overline{\mathcal{U}_q}\right\}$$
$$\mathcal{F}_2(m) := \left\{q \in (1, m+1] : 1 \in \mathcal{V}_q - \mathcal{V}_q\right\}.$$

Then, $\mathcal{B}_2(m) = \mathcal{E}_2(m) = \mathcal{F}_2(m) \setminus \{\mathcal{G}(m)\}$.

Proof. By Theorem 3.2, we have $\mathcal{B}_2(m) = \{q \in (1, m+1] : 1 \in \mathcal{U}_q - \mathcal{U}_q\}$. Moreover, $\mathcal{U}_q \subset \overline{\mathcal{U}_q} \subset \mathcal{V}_q$. Hence, we have $\mathcal{B}_2(m) \subset \mathcal{E}_2(m) \subset \mathcal{F}_2(m)$.

Applying Lemma 2.5 (i), we see that $\mathcal{E}_2(m) \subset \mathcal{B}_2(m) \cup \overline{\mathcal{U}}$. Thus, it follows from Lemma 4.1 that $\mathcal{E}_2(m) = \mathcal{B}_2(m)$. On the other hand, according to Lemma 2.5 (ii), we know that $\mathcal{F}_2(m) \subset \mathcal{B}_2(m) \cup \mathcal{V}$. Then, it follows from Lemma 4.2 and $\mathcal{G}(m) \notin \mathcal{B}_2(m)$ that $\mathcal{F}_2(m) = \mathcal{B}_2(m) \cup \{\mathcal{G}(m)\}$. □

Now, we give topological descriptions of $\mathcal{B}_2^{(i)}(m)$.

Lemma 4.4. $\mathcal{B}_2^{(i)}(m)$ *is compact for all* $i \geq 0$.

Proof. It suffices to prove $\mathcal{B}_2(m)$ is compact. We claim $[q_M, m+1] \setminus \mathcal{B}_2(m)$ is open, where q_M is smallest base of $\mathcal{B}_2(m)$ (cf. [21]). Take $q \in [q_M, m+1] \setminus \mathcal{B}_2(m)$ arbitrarily. So, $1 \notin \mathcal{U}_q - \mathcal{U}_q$ by Theorem 3.2, and $q \notin \overline{\mathcal{U}}$ by Lemma 4.1. Thus, $\mathcal{U}_q$ is compact, and so $\mathcal{U}_q - \mathcal{U}_q$ is compact by Lemma 2.5 (i). Then, $d_H(1, \mathcal{U}_q - \mathcal{U}_q) > 0$, where d_H denotes the Hausdorff metric. Since $\mathcal{U}_p$ continuously depends on $p \notin \overline{\mathcal{U}}$ (see [5]), we take $0 < \delta < d_H(1, \mathcal{U}_q - \mathcal{U}_q)$, and a small open set $O(\delta)$ which contains q such that $d_H(\mathcal{U}_{p_0} - \mathcal{U}_{p_0}, \mathcal{U}_q - \mathcal{U}_q) < \delta$ for all $p_0 \in O(\delta)$. Then, $d_H(1, \mathcal{U}_{p_0} - \mathcal{U}_{p_0}) > 0$, i.e., $p_0 \notin \mathcal{B}_2(m)$. □

Lemma 4.5. $\mathcal{V} \setminus \mathcal{G}(m) \subset \mathcal{B}_2^{(2)}(m)$.

Proof. It suffices to prove that $\mathcal{V} \setminus (\overline{\mathcal{U}} \cup \{\mathcal{G}(m)\}) \subset \mathcal{B}_2^{(2)}(m)$ by Lemma 4.1. Fix $q \in \mathcal{V} \setminus (\overline{\mathcal{U}} \cup \{\mathcal{G}(m)\})$ arbitrarily. By Lemma 2.4 (iii), there exists a word $a_1 \cdots a_n$ with $n \geq 1$ such that $\alpha(q) = (a_1 \cdots a_n^+ \overline{a_1 \cdots a_n^+})^\infty$, and for all $0 < i < n$,

$$\overline{(a_1 \cdots a_n)^\infty} \leq \sigma^i((a_1 \cdots a_n)^\infty) < (a_1 \cdots a_n)^\infty. \tag{4.3}$$

Set

$$\mathrm{c} = \overline{a_1 \cdots a_n^+}(a_1 \cdots a_n)^\infty, \qquad \mathrm{d} = 0^{2n}(\overline{a_1 \cdots a_n})^\infty,$$

and

$$\mathrm{d}_j = 0^{2n}(\overline{a_1 \cdots a_n})^j(\overline{a_1 \cdots a_n^+} a_1 \cdots a_n^+)^\infty.$$

(i) Suppose that $n \geq 2$. According to the proof of Lemma 4.2, one gets $\mathrm{c}, \mathrm{d} \in A'_q$ and $q = q_{\mathrm{c},\mathrm{d}}$. We will show that $\mathrm{d}_j \in A'_p$ for all $j \geq 1$ and $p \in (q, m+1]$. Since $q \in \mathcal{V}$, by Lemma 2.3 (iii), it suffices to prove

$$\overline{\alpha(p)} < \sigma^i((\overline{a_1 \cdots a_n})^j(\overline{a_1 \cdots a_n^+ a_1 \cdots a_n^+})^\infty) < \alpha(p)$$

holds for all $0 \leq i < nj$. It follows from (4.3) and $\overline{a_1} < a_1$ that

$$\overline{a_1 \cdots a_n^+} < \overline{a_1 \cdots a_n} \leq \overline{a_{i+1} \cdots a_n a_1 \cdots a_i} \leq a_1 \cdots a_n < a_1 \cdots a_n^+$$

for $0 \leq i < n$. Hence, $\mathrm{d}_j \in A'_p$ for all $j \geq 1$. Next, we prove $q_{\mathrm{c},\mathrm{d}_j} \in \mathcal{B}_2(m)$,

$$\begin{aligned} f_{\mathrm{c},\mathrm{d}_j}(q) &= (1\mathrm{c})_q + (m0^{2n}(\overline{a_1 \cdots a_n})^j(\overline{a_1 \cdots a_n^+ a_1 \cdots a_n^+})^\infty)_q - (m^\infty)_q \\ &= (1\mathrm{c})_q + (m0^{2n}(\overline{a_1 \cdots a_n})^{j+1}0^\infty)_q - (m^\infty)_q \\ &< (1\mathrm{c})_q + (m0^{2n}(\overline{a_1 \cdots a_n})^\infty)_q - (m^\infty)_q = f_{\mathrm{c},\mathrm{d}}(q) = 0. \end{aligned}$$

By Lemma 3.6, $f_{\mathrm{c},\mathrm{d}_j}(t) = 0$ has a unique root $q_{\mathrm{c},\mathrm{d}_j}$ in $(q, m+1]$ for all $j \geq 1$, i.e., $q_{\mathrm{c},\mathrm{d}_j} \in \mathcal{B}_2(m)$. Applying Lemma 3.7 and the continuity of $f_{\mathrm{c},\mathrm{d}}$ w.r.t. (c, d), we infer that $q_{\mathrm{c},\mathrm{d}_j} \searrow q$ as $j \to \infty$. Let

$$\mathrm{c}_\ell = \overline{a_1 \cdots a_n^+}(a_1 \cdots a_n)^\ell(a_1 \cdots a_n^+ \overline{a_1 \cdots a_n^+})^\infty.$$

Note that $f_{\mathrm{c}_\ell,\mathrm{d}_j}(q) < 0$ for all sufficiently large ℓ. By the same argument, we can conclude that for each fixed $j \geq 1$, we have $q_{\mathrm{c}_\ell,\mathrm{d}_j} \in \mathcal{B}_2(m)$ for all sufficiently large ℓ, and $q_{\mathrm{c}_\ell,\mathrm{d}_j} \nearrow q_{\mathrm{c},\mathrm{d}_j}$ as $\ell \to \infty$. Then, for each j, $q_{\mathrm{c},\mathrm{d}_j} \in \mathcal{B}_2^{(1)}(m)$, and then $q = q_{\mathrm{c},\mathrm{d}} \in \mathcal{B}_2^{(2)}(m)$.

(ii) Suppose that $n = 1$. Then, $\alpha(q) = (a_1^+ \overline{a_1^+})^\infty$. Let $\mathrm{c} = \overline{a_1^+} a_1^\infty$, $\mathrm{d} = 0^2 \overline{a_1}^\infty$, $\mathrm{c}_\ell = \overline{a_1^+}(a_1)^\ell(a_1^+ \overline{a_1^+})^\infty$ and $\mathrm{d}_j = 0^2(\overline{a_1})^j(\overline{a_1^+ a_1^+})^\infty$. By the same argument as in the first case, we can conclude that $q \in \mathcal{B}_2^{(2)}(m)$. □

Proof of Theorem 1.1. We get the result by Theorem 3.2 and Proposition 4.3. □

Proof of Theorem 1.2. Results (i), (ii) and (iii) follow from Lemmas 4.4, 4.1 and 4.5, respectively. Result (iv) follows from Lemma 4.5 and the fact that the set $(\mathcal{G}(m), q_f(m)) \cap \mathcal{B}_2(m)$ is finite. □

5. Some results on unique expansions

Recall that $\mathcal{U}$ and $\mathcal{V}$ denote the set of univoque bases $q \in (1, m+1]$ and the set of bases $q \in (1, m+1]$ for which there is a unique doubly infinite q-expansion, respectively. Let

$$(1, m+1] \setminus \overline{\mathcal{U}} = \cup (p_0, p_0^*),$$

where p_0 runs over $\{1\} \cup (\overline{\mathcal{U}} \setminus \mathcal{U})$ and p_0^* runs over a proper subset of $\overline{\mathcal{U}}$. It was proved in [20] that p_0 is an algebraic number, while p_0^* is a transcendental number. Now, let

$$(M, m+1] \setminus \overline{\mathcal{U}} = \cup (q_0, q_0^*), \qquad M = \left\lfloor \frac{m}{2} \right\rfloor + 1. \tag{5.1}$$

In this section, we shall give a description of $\mathcal{U}'_{q_0^*}$.

Note from [19] and [10] that for each connected component (q_0, q_0^*), there exists a finite word $a_1 \cdots a_n$ such that $\alpha(q_0) = (a_1 \cdots a_n)^\infty \in \Omega_m^{\mathbb{N}}$, where $a_1 \cdots a_n$ is assumed the smallest periodic block. The right endpoint q_0^* is the limit of sequence $\{q_\ell\}$ defined below. Let

$$c_0^- := a_1 \cdots a_n \qquad \text{and} \qquad c_{\ell+1} = c_\ell \overline{c_\ell}^+, \quad \ell = 0, 1, \ldots. \tag{5.2}$$

Then, (c_i) is a Thue–Morse type sequence generated by c_0^- (cf. [1]). From [10], it follows that for each $\ell = 1, 2, \ldots$, there exists $q_\ell \in (q_0, q_0^*)$ such that $\beta(q_\ell) = c_\ell 0^\infty$. De Vries and Komornik obtained in [19] and [10] that for each connected component (q_0, q_0^*),

$$\mathcal{V} \cap (q_0, q_0^*) = \{q_\ell; \ell \in \mathbb{N}\} \qquad \text{and} \qquad q_\ell \uparrow q_0^*.$$

We recall some standard results:

Lemma 5.1 ([20]). *Let M, c_ℓ be given in (5.1) and (5.2). Let (q_0, q_0^*) be a connected component of $(M, m+1] \setminus \overline{\mathcal{U}}$ related to c_0^-, and $(d_i) \in \mathcal{U}'_{q_0^*}$ w.r.t. Ω_m.*

(i) *If $d_j < m$ and $d_{j+1} \cdots d_{j+2^\ell n} = c_\ell$ for some $\ell \geq 0$, then*

$$d_{j+2^\ell n+1} \cdots d_{j+2^{\ell+1} n} = \overline{c_\ell} \qquad \text{or} \qquad d_{j+2^\ell n+1} \cdots d_{j+2^{\ell+1} n} = \overline{c_\ell}^+.$$

(ii) *If $d_j > 0$ and $d_{j+1} \cdots d_{j+2^\ell n} = \overline{c_\ell}$ for some $\ell \geq 0$, then*

$$d_{j+2^\ell n+1} \cdots d_{j+2^{\ell+1} n} = c_\ell \qquad \text{or} \qquad d_{j+2^\ell n+1} \cdots d_{j+2^{\ell+1} n} = c_\ell^-.$$

Lemma 5.2 ([20, Lemma 4.2]). *Let M, c_ℓ be given in (5.1) and (5.2). Let (q_0, q_0^*) be a connected component of $(M, m+1] \setminus \overline{\mathcal{U}}$ related to c_0^-. Then, for any $\ell \geq 0$, $c_\ell = a_1 \cdots a_{2^\ell n}$ satisfies*

$$\overline{a_1 \cdots a_{2^\ell n - i}} < a_{i+1} \cdots a_{2^\ell n} \leq a_1 \cdots a_{2^\ell n - i}$$

for all $0 \leq i < 2^\ell n$.

Our first result is

Theorem 5.3. *Let M, c_ℓ be given in* (5.1) *and* (5.2). *Let* $(q_0, q_0^*) = (M, q_{KL})$ *be the first connected component of* $(M, m+1] \setminus \overline{\mathcal{U}}$ *related to* c_0^-.

(I) *Suppose that* $m = 2k+1 \geq 3$. *Then,* $(b_i) \in \mathcal{U}'_{q_{KL}} \setminus \{0^\infty, m^\infty\}$ *if and only if* (b_i) *is formed by sequences of the form*

$$\omega(c_0^-)^j(c_{i_1}\overline{c_{i_1}})^{j_1}(c_{i_1}\overline{c_{i_2}})^{l_1}(c_{i_2}\overline{c_{i_2}})^{j_2}(c_{i_2}\overline{c_{i_3}})^{l_2}\cdots \tag{5.3}$$

or their reflections, where $0 \leq i_1 < i_2 < \cdots$ *are integers,* $1 \leq j \leq 2$, $0 \leq l_i \leq 1$, $0 \leq j_i \leq \infty$ *for all* $i \geq 1$, *and*

$$\omega \in \{1, \cdots, m-1\} \cup \bigcup_{N=1}^{\infty}\{0^N b : 0 < b \leq k+1\} \cup \bigcup_{N=1}^{\infty}\{m^N b : k \leq b < m\}.$$

(II) *Suppose that* $m = 2k$. *Then,* $(b_i) \in \mathcal{U}'_{q_{KL}} \setminus \{0^\infty, m^\infty\}$ *if and only if* (b_i) *is formed by sequences of the form*

$$\omega(c_0^-)^j(c_{i_1}\overline{c_{i_1}})^{j_1}(c_{i_1}\overline{c_{i_2}})^{l_1}(c_{i_2}\overline{c_{i_2}})^{j_2}(c_{i_2}\overline{c_{i_3}})^{l_2}\cdots \tag{5.4}$$

or their reflections, where $0 \leq i_1 < i_2 < \cdots$ *are integers,* $0 \leq l_i \leq 1$, $0 \leq j, j_i \leq \infty$ *for all* $i \geq 1$, *and*

$$\omega \in \{1, \cdots, m-1\} \cup \bigcup_{N=1}^{\infty}\{0^N b : 0 < b \leq k+1\} \cup \bigcup_{N=1}^{\infty}\{m^N b : k-1 \leq b < m\}.$$

Remark. The case $m = 1$ was studied in [15], which is quite different from the cases $m = 2k+1 \geq 3$. So, we assume $m = 2k+1 \geq 3$ in Theorem 5.3 (I).

Proof. We have $(q_0, q_0^*) = (M, q_{KL})$, thus $\mathcal{U}'_{q_0} = \{0^\infty, m^\infty\}$. Note that whether $m = 2k$ or $m = 2k+1$, we always have $q_0 = M = k+1$, and so, $\alpha(q_0) = k^\infty$, $c_0^- = k$.

For the sufficiency, it is not difficult to be verified by Lemma 5.2. We leave it for the readers.

In the following, we prove the necessity. Take $(b_i) \in \mathcal{U}'_{q_0^*} \setminus \mathcal{U}'_{q_0}$. Let

$$N = \min\{s : 0 < b_s < m\}.$$

Then, N is well-defined and is a positive integer.

Case I. $m = 2k+1$ with $k \geq 1$.

Note that from (2.2), $\alpha(q_{KL}) = (k+1)(k+1)k(k+1)\cdots$. By (2.4), we have for $t \geq 1$:

$$\begin{aligned} &b_{t+1}b_{t+2}\cdots < (k+1)(k+1)k(k+1)\cdots, && \text{when } b_1\cdots b_t \neq m^t, \\ &b_{t+1}b_{t+2}\cdots > kk(k+1)k\cdots, && \text{when } b_1\cdots b_t \neq 0^t. \end{aligned} \tag{5.5}$$

We now split our discussion into two steps.

Step 1. We shall show what the block $\omega = b_1\cdots b_N$ looks like. The case $N = 1$ is trivial, we only consider the case $N > 1$.

Subcase 1. Suppose that ω begins at 0. Then, by (5.5), we have $\omega = 0^{N-1}b_N$ with $0 < b_N \leq k+1$.

Subcase 2. Suppose that ω begins at m. Then, by (5.5), we have $\omega = m^{N-1}b_N$ with $k \leq b_N < m$.

Step 2. Now, let us to explore the sequence $(b_{N+i}) = (b_{N+i})_{i\geq 1}$. Note that $0 < b_N < m$ and $\alpha(q_{KL}) = (k+1)(k+1)k(k+1)\cdots$. Thus, by Lemma 2.2, we have for each $i \geq 1$,

$$kk(k+1)k\cdots < b_{N+i}b_{N+i+1}\cdots < (k+1)(k+1)k(k+1)\cdots, \tag{5.6}$$

and so,

$$k \leq b_{N+i} \leq k+1, \qquad (b_{N+i})_{i\geq 1} \text{ not ending with } k^\infty \text{ or } (k+1)^\infty.$$

From (5.6) it follows that there exists $1 \leq j \leq 2$ such that either $b_{N+1}\cdots b_{N+j} = k^j$ or $b_{N+1}\cdots b_{N+j} = (k+1)^j$. Without loss of generality, we assume that $b_{N+1}\cdots b_{N+j} = k^j$. Otherwise, we only need to consider $(\overline{b_i})_{i\geq 1}$ instead. So $b_{N+j+1} = k+1 = c_0$. In the following, we shall determine the tail $(b_{N+j+i})_{i>1}$ by means of Lemma 5.1.

Let us recall that

$$c_0 = \overline{c_0}^+ = k+1, \;\; c_0^- = \overline{c_0} = k, \qquad \text{and} \qquad c_{\ell+1} = c_\ell\overline{c_\ell}^+, \;\; \overline{c_{\ell+1}} = \overline{c_\ell}c_\ell^-. \tag{5.7}$$

Roughly speaking, Lemma 5.1 tells us that which possible blocks will follow a block c_ℓ or $\overline{c_\ell}$. This can be simply described in Figure 1.

$$c_\ell \begin{cases} \overline{c_\ell} \Rightarrow c_\ell\overline{c_\ell} \\ \overline{c_\ell}^+ \Rightarrow c_\ell\overline{c_\ell}^+ = c_{\ell+1} \end{cases} \qquad\qquad \overline{c_\ell} \begin{cases} c_\ell \Rightarrow \overline{c_\ell}c_\ell \\ c_\ell^- \Rightarrow \overline{c_\ell}c_\ell^- = \overline{c_{\ell+1}} \end{cases}$$

Figure 1. Relation induced by Lemma 5.1.

By $A \rightharpoonup B$, we denote block A followed by block B. We point out that in Figure 1 the action $c_\ell \rightharpoonup \overline{c_\ell}^+$ cannot be implemented continuously infinite times, since $(b_i) \in \mathcal{U}'_{q_{KL}}$ cannot be ended with $\alpha(q_{KL})$. Similarly, the action $\overline{c_\ell} \rightharpoonup c_\ell^-$ cannot be implemented continuously infinite times.

Now, we have $b_{N+j+1} = k+1 = c_0$ and $b_{N+j} = k < m$. Then, the following block is either $\overline{c_0}$ or $\overline{c_0}^+$ by Lemma 5.1, i.e.,

$$(b_i) = \omega(c_0^-)^j c_0 \overline{c_0}(b_{N+j+2+i})_{i\geq 1}, \tag{5.8}$$

or

$$(b_i) = \omega(c_0^-)^j c_0 \overline{c_0}^+(b_{N+j+2+i})_{i\geq 1} = \omega(c_0^-)^j c_1(b_{N+j+2+i})_{i\geq 1} \tag{5.9}$$

by (5.7). If (5.8) occurs, then

$$(b_i) = \omega(c_0^-)^j c_0 \overline{c_0} c_0 \cdots \qquad \text{or} \qquad (b_i) = \omega(c_0^-)^j c_0 \overline{c_0} c_0^- \cdots = \omega(c_0^-)^j c_0 \overline{c_1} \cdots$$

by Lemma 5.1 and (5.7). If (5.9) occurs, then

$$(b_i) = \omega(c_0^-)^j c_1 \overline{c_1} \cdots \qquad \text{or} \qquad (b_i) = \omega(c_0^-)^j c_1 \overline{c_1}^+ \cdots = \omega(c_0^-)^j c_2 \cdots$$

by Lemma 5.1 and (5.7).

In any cases described above, one can continue to implement the process in the same way as above. Therefore, (b_i) is of form (5.3) or its reflection.

Case II. $m = 2k$ with $k \geq 1$.

Note that $\alpha(q_{KL}) = (k+1)k(k-1)(k+1)\cdots$. By (2.4), we have for $t \geq 1$:

$$\begin{aligned} b_{t+1}b_{t+2}\cdots &< (k+1)k(k-1)(k+1)\cdots, & \text{when } b_1\cdots b_t &\neq m^t, \\ b_{t+1}b_{t+2}\cdots &> (k-1)k(k+1)(k-1)\cdots, & \text{when } b_1\cdots b_t &\neq 0^t. \end{aligned} \tag{5.10}$$

We now split our discussion into two steps.

Step 1. We shall show what the block $\omega = b_1\cdots b_N$ looks like. As in Case I, we only consider $N > 1$.

Subcase 1. Suppose that ω begins at 0. Then, by (5.10), we have $\omega = 0^{N-1}b_N$ with $0 < b_N \leq k+1$.

Subcase 2. Suppose that ω begins at m. Then, by (5.10), we have $\omega = m^{N-1}b_N$ with $k-1 \leq b_N < m$.

Step 2. Now, let us explore the sequence $(b_{N+i}) = (b_{N+i})_{i\geq 1}$. Note that $0 < b_N < m$ and $\alpha(q_{KL}) = (k+1)k(k-1)(k+1)\cdots$. Thus, by Lemma 2.2, we have for each $i \geq 1$:

$$(k-1)k(k+1)(k-1)\cdots < b_{N+i}b_{N+i+1}\cdots < (k+1)k(k-1)(k+1)\cdots. \tag{5.11}$$

From (5.11), it follows that

$$b_{N+1} \in \{k-1, k, k+1\} = \{\overline{c_0}, c_0^-, c_0\}.$$

(I) $b_{N+1} = k$. In this case, either $(b_{N+i})_{i\geq 1} = k^\infty = (c_0^-)^\infty$ or there exists a positive integer j such that $b_{N+1}\cdots b_{N+j+1} \in \{k^j(k+1), k^j(k-1)\} = \{(c_0^-)^j c_0, (c_0^-)^j \overline{c_0}\}$.

Without loss of generality, we assume that $b_{N+1}\cdots b_{N+j+1} = k^j(k+1) = (c_0^-)^j c_0$. Otherwise, we only need to consider $(\overline{b_i})_{i\geq 1}$ instead. So $b_{N+j+1} = k+1 = c_0$.

(II) $b_{N+1} \in \{k-1, k+1\} = \{\overline{c_0}, c_0\}$. Without loss of generality, we assume that $b_{N+1} = k+1 = c_0$. Otherwise, we only need to consider $(\overline{b_i})_{i\geq 1}$ instead. For the sake of uniformity, we write $b_{N+1} = k+1 = (c_0^-)^0 c_0$, corresponding to $j = 0$.

In the following, we shall determine the tail $(b_{N+j+1+i})_{i\geq 1}$ by means of Lemma 5.1, where j is a nonnegative integer.

Let us recall that

$$c_0^- = \overline{c_0}^+ = k,\quad c_0 = k+1,\quad \overline{c_0} = k-1, \qquad \text{and} \qquad c_{\ell+1} = c_\ell \overline{c_\ell}^+,\quad \overline{c_{\ell+1}} = \overline{c_\ell} c_\ell^-. \tag{5.12}$$

Roughly speaking, Lemma 5.1 tells us that which block can follow a block c_ℓ or $\overline{c_\ell}$. This can be simply described in Figure 1.

We point out that in Figure 1 the action $c_\ell \rightharpoonup \overline{c_\ell}^+$ cannot be implemented continuously infinite times, since $(b_i) \in \mathcal{U}'_{q_{KL}}$ cannot be end with $\alpha(q_{KL})$. Similarly, the action $\overline{c_\ell} \rightharpoonup c_\ell^-$ cannot be implemented continuously infinite times.

Now, we have $b_{N+j+1} = k+1 = c_0$ and $b_{N+j} < m$. Then, the following block is either $\overline{c_0}$ or $\overline{c_0}^+$ by Lemma 5.1, i.e.,

$$(b_i) = \omega (c_0^-)^j c_0 \overline{c_0} (b_{N+j+2+i})_{i\geq 1}, \tag{5.13}$$

or

$$(b_i) = \omega (c_0^-)^j c_0 \overline{c_0}^+ (b_{N+j+2+i})_{i\geq 1} = \omega (c_0^-)^j c_1 (b_{N+j+2+i})_{i\geq 1} \tag{5.14}$$

by (5.12). If (5.13) occurs, then

$$(b_i) = \omega (c_0^-)^j c_0 \overline{c_0} c_0 \cdots \qquad \text{or} \qquad (b_i) = \omega (c_0^-)^j c_0 \overline{c_0} c_0^- \cdots = \omega (c_0^-)^j c_0 \overline{c_1} \cdots$$

by Lemma 5.1 and (5.12). If (5.14) occurs, then

$$(b_i) = \omega (c_0^-)^j c_1 \overline{c_1} \cdots \qquad \text{or} \qquad (b_i) = \omega (c_0^-)^j c_1 \overline{c_1}^+ \cdots = \omega (c_0^-)^j c_2 \cdots$$

by Lemma 5.1 and (5.12).

In any case described above, one can continue to implement the process in the same way as above. Therefore, (b_i) is of form (5.4) or its reflection. $\square$

Let (q_0, q_0^*) be a connected component of $(M, m+1]\backslash\overline{\mathcal{U}}$. Suppose that $\alpha(q_0) = (a_1 \cdots a_n)^\infty = (c_0^-)^\infty$. For a word $v_1v_2\cdots v_p \in \{0,1,\cdots,m\}^p$ and $1 \le q \le p$, let

$$(v_1\cdots v_p)|_q = v_1\cdots v_q, \qquad \text{if } \sigma(v_1v_2\cdots v_p) = v_2\cdots v_p.$$

A word $u_1\cdots u_p \in \{0,1,\cdots,m\}^p$ is *matched* to $a_1\cdots a_n$ if $u_p < m$ and for any $1 \le \ell \le p$,

$$\sigma^\ell(u_1\cdots u_p a_1\cdots a_n)|_n \le a_1\cdots a_n, \qquad \text{whenever } u_1\cdots u_\ell \ne m^\ell,$$

and

$$\sigma^\ell(u_1\cdots u_p a_1\cdots a_n)|_n \ge \overline{a_1\cdots a_n}, \qquad \text{whenever } u_1\cdots u_\ell \ne 0^\ell.$$

Obviously, if $u_1\cdots u_p$ is matched to $a_1\cdots a_n$,then $u_1\cdots u_p a_1\cdots a_n$ is also matched to $a_1\cdots a_n$.

Theorem 5.4. *Let M, c_ℓ be given in* (5.1) *and* (5.2). *If (q_0, q_0^*) is a connected component of $(M, m+1]\setminus\overline{\mathcal{U}}$ related to $c_0^- = a_1\cdots a_n$ with $q_0 > M$, then $(b_i) \in \mathcal{U}'_{q_0^*}\setminus\mathcal{U}'_{q_0}$ if and only if (b_i) is formed by the sequences of the form*

$$\omega(c_0^-)^j(c_{i_1}\overline{c_{i_1}})^{j_1}(c_{i_1}\overline{c_{i_2}})^{l_1}(c_{i_2}\overline{c_{i_2}})^{j_2}(c_{i_2}\overline{c_{i_3}})^{l_2}\cdots$$

or their reflections, where

$$\omega \in \bigcup_{p=1}^{\infty}\{u_1\cdots u_p : u_1\cdots u_p \text{ is matched to } a_1\cdots a_n\},$$

$0 \le i_1 < i_2 < \cdots$ are integers, $0 \le l_i \le 1$, $j \in \{0,\infty\}$ and $0 \le j_i \le \infty$ for all $i \ge 1$.

Proof. As described in (5.2), we have

$$\alpha(q_0) = (a_1\cdots a_n)^\infty = (c_0^-)^\infty$$

and

$$\alpha(q_0^*) = c_0\overline{c_0}^+\overline{c_0}c_0\cdots = a_1\cdots a_{n-1}a_n^+\overline{a_1\cdots a_n}\cdots.$$

Now, take a $(b_i) \in \mathcal{U}'_{q_0^*}\setminus\mathcal{U}'_{q_0}$.

Case I. (b_i) ends with neither $(c_0^-)^\infty$ nor $\overline{(c_0^-)^\infty}$.

Since $(b_i) \notin \mathcal{U}'_{q_0}$, there exists a smallest positive integer η such that

$$b_1\cdots b_\eta \ne m^\eta \qquad \text{and} \qquad b_{\eta+1}\cdots b_{\eta+n} > c_0^- = a_1\cdots a_n, \tag{5.15}$$

or

$$b_1\cdots b_\eta \neq 0^\eta \qquad \text{and} \qquad b_{\eta+1}\cdots b_{\eta+n} < \overline{c_0^-} = \overline{a_1\cdots a_n}.$$

Without loss of generality, we assume that (5.15) holds. Otherwise, one can consider $\overline{(b_i)}$ instead.

On the other hand, by Lemma 2.2, we have

$$b_{\eta+1}\cdots b_{\eta+n}\cdots < c_0\overline{c_0}^+\overline{c_0}c_0\cdots.$$

Thus, combining (5.15), one can get

$$b_1\cdots b_\eta \neq m^\eta \qquad \text{and} \qquad b_{\eta+1}\cdots b_{\eta+n} = c_0 = a_1\cdots a_n^+.$$

We claim that $b_\eta < m$. This is clear when $\eta = 1$. Suppose that $\eta > 1$ and $b_\eta = m$. Then, by the minimality, we have

$$b_1\cdots b_{\eta-1} \neq m^{\eta-1} \qquad \text{and} \qquad b_\eta b_{\eta+1}\cdots b_{\eta+n-1} \leq c_0^- = a_1\cdots a_n.$$

By (5.15), this implies that $a_1\cdots a_n = m^n$, and so $q_0 = m+1$, a contradiction. Hence, we get

$$b_\eta < m \qquad \text{and} \qquad b_{\eta+1}\cdots b_{\eta+n} = c_0 = a_1\cdots a_n^+.$$

By the same argument as in Theorem 5.3, one can get $(b_{\eta+i})_{i\geq 1}$ is of the form

$$(c_{i_1}\overline{c_{i_1}})^{j_1}(c_{i_1}\overline{c_{i_2}})^{l_1}(c_{i_2}\overline{c_{i_2}})^{j_2}(c_{i_2}\overline{c_{i_3}})^{l_2}\cdots.$$

Next, let us investigate what the prefix $b_1\cdots b_\eta$ looks like.

By the definition of η, we can denote

$$b_1\cdots b_\eta = m^u b_{u+1}\cdots b_\eta,$$

where $u < \eta$ is a nonnegative integer, $b_{u+1}, b_\eta < m$. Then, by (5.15),

$$\sigma^p(b_1\cdots b_\eta a_1\cdots a_n)|_n = \sigma^p(b_1\cdots b_\eta b_{\eta+1}\cdots b_{\eta+n-1}b_{\eta+n}^-)|_n \leq a_1\cdots a_n,$$

for $u < p \leq \eta$, and

$$\sigma^p(b_1\cdots b_\eta a_1\cdots a_n)|_n = \sigma^p(b_1\cdots b_\eta b_{\eta+1}\cdots b_{\eta+n-1}b_{\eta+n}^-)|_n \geq \overline{a_1\cdots a_n},$$

for $1 \leq p \leq \eta$ with $b_1\cdots b_p \neq 0^p$. Therefore, $b_1\cdots b_\eta$ is matched to $a_1\cdots a_n$.

Case II. (b_i) ends with either $(c_0^-)^\infty$ or $\overline{(c_0^-)^\infty}$.

Without loss of generality, we assume (b_i) ends with $(c_0^-)^\infty$. Otherwise, one only needs to consider $\overline{(b_i)}$ instead. Then, (b_i) can be written as

$$(b_i) = b_1 \cdots b_p (a_1 \cdots a_n)^\infty$$

where p is a positive integer. We claim that $b_1 \cdots b_p$ is matched to $a_1 \cdots a_n$. Otherwise, there exists a smallest positive integer $1 \le \ell \le p$ such that

$$\sigma^\ell(b_1 \cdots b_p a_1 \cdots a_n)|_n = \sigma^\ell(b_1 \cdots b_p b_{p+1} \cdots b_{p+n})|_n > a_1 \cdots a_n \quad \text{and} \quad b_1 \cdots b_\ell \neq m^\ell$$

or

$$\sigma^\ell(b_1 \cdots b_p a_1 \cdots a_n)|_n = \sigma^\ell(b_1 \cdots b_p b_{p+1} \cdots b_{p+n})|_n < \overline{a_1 \cdots a_n} \quad \text{and} \quad b_1 \cdots b_\ell \neq 0^\ell.$$

Then, by the same argument as in Case I, we have that

$$b_\ell < m \quad \text{and} \quad b_{\ell+1} \cdots b_{\ell+n} = c_0 = a_1 \cdots a_{n-1} a_n^+$$

or

$$b_\ell > 0 \quad \text{and} \quad b_{\ell+1} \cdots b_{\ell+n} = \overline{c_0} = \overline{a_1 \cdots a_{n-1} a_n^+}.$$

So (b_i) cannot end with $(c_0^-)^\infty$ by the argument in Case I, which contradicts our hypothesis.

The sufficiency can be checked directly. □

Recall that for each connected component (q_0, q_0^*), we have $(q_0, q_0^*) \cap \mathcal{V} = \{q_\ell : \ell \in \mathbb{N}\}$. If $q \in (q_\ell, q_{\ell+1}]$, then $\alpha(q) \le \alpha(q_{\ell+1}) = (c_\ell \overline{c_\ell})^\infty$ by Lemma 2.1, it follows from Lemma 5.1 that the blocks wc_ℓ and $\overline{wc_\ell}$ are forbidden in each $(b_i) \in \mathcal{U}_q'$, where $w \in \{0, \ldots, m-1\}$. Otherwise, (b_i) would end with $(c_\ell \overline{c_\ell})^\infty$, which leads to contradiction by Lemma 2.2. So by Theorems 5.3 and 5.4, we obtain the following results.

Corollary 5.5. *Let M, c_ℓ be given in (5.1) and (5.2). Let $(q_0, q_0^*) = (M, q_{KL})$ be the first connected component of $(M, m+1] \setminus \overline{\mathcal{U}}$ related to c_0^-.*

(1) *Suppose that $q \in (q_\ell, q_{\ell+1}]$ for some $\ell \ge 1$ and $m = 2k+1 \ge 3$. Then, $(b_i) \in \mathcal{U}_q' \setminus \{0^\infty, m^\infty\}$ if and only if (b_i) is formed by sequences of the form*

$$\omega (c_0^-)^j (c_{i_1} \overline{c_{i_1}})^{j_1} (c_{i_1} \overline{c_{i_2}})^{l_1} (c_{i_2} \overline{c_{i_2}})^{j_2} \cdots (c_{i_{n-1}} \overline{c_{i_n}})^{l_{n-1}} (c_{i_n} \overline{c_{i_n}})^{j_n}$$

or their reflections, where $0 \le i_1 < i_2 < \cdots < i_n < \ell$ *are integers,* $1 \le j \le 2$, $0 \le l_i \le 1$, $0 \le j_i \le \infty$ *for all* $i \ge 1$, *and*

$$\omega \in \{1, \cdots, m-1\} \cup \bigcup_{N=1}^{\infty} \{0^N b : 0 < b \le k+1\} \cup \bigcup_{N=1}^{\infty} \{m^N b : k \le b < m\}.$$

(2) *Suppose that* $q \in (q_\ell, q_{\ell+1}]$ *for some* $\ell \ge 1$ *and* $m = 2k$. *Then,* $(b_i) \in \mathcal{U}'_q \setminus \{0^\infty, m^\infty\}$ *if and only if* (b_i) *is formed by sequences of the form*

$$\omega (c_0^-)^j (c_{i_1}\overline{c_{i_1}})^{j_1} (c_{i_1}\overline{c_{i_2}})^{l_1} (c_{i_2}\overline{c_{i_2}})^{j_2} \cdots (c_{i_{n-1}}\overline{c_{i_n}})^{l_{n-1}} (c_{i_n}\overline{c_{i_n}})^{j_n}$$

or their reflections, where $0 \le i_1 < i_2 < \cdots < i_n < \ell$ *are integers,* $0 \le l_i \le 1$, $0 \le j, j_i \le \infty$ *for all* $i \ge 1$, *and*

$$\omega \in \{1, \cdots, m-1\} \cup \bigcup_{N=1}^{\infty} \{0^N b : 0 < b \le k+1\} \cup \bigcup_{N=1}^{\infty} \{m^N b : k-1 \le b < m\}.$$

Corollary 5.6. *Let* M, c_ℓ *be given in* (5.1) *and* (5.2). *If* (q_0, q_0^*) *is a connected component of* $(M, m+1] \setminus \overline{\mathcal{U}}$ *related to* $c_0^- = a_1 \cdots a_n$ *with* $q_0 > M$, *and* $q \in (q_\ell, q_{\ell+1}]$ *for some* $\ell \ge 1$. *Then,* $(b_i) \in \mathcal{U}'_q \setminus \mathcal{U}'_{q_0}$ *if and only if* (b_i) *is formed by sequences of the form*

$$\omega (c_0^-)^j (c_{i_1}\overline{c_{i_1}})^{j_1} (c_{i_1}\overline{c_{i_2}})^{l_1} (c_{i_2}\overline{c_{i_2}})^{j_2} \cdots (c_{i_{n-1}}\overline{c_{i_n}})^{l_{n-1}} (c_{i_n}\overline{c_{i_n}})^{j_n}$$

or their reflections, where

$$\omega \in \bigcup_{p=1}^{\infty} \{u_1 \cdots u_p : u_1 \cdots u_p \text{ is matched to } a_1 \cdots a_n\},$$

$0 \le i_1 < i_2 < \cdots < i_n < \ell$ *are integers,* $0 \le l_i \le 1$, $j \in \{0, \infty\}$ *and* $0 \le j_i \le \infty$ *for all* $i \ge 1$.

Remark. We give an example to show inequality $q_{j+1} \le \min \mathcal{B}_2^{(2j)}(m)$ no longer holds for some $j \ge 1$ if $m > 1$. Let $q \approx 3.627$ be the real root of $x^3 - 3x^2 - 2x - 1 = 0$ and $m = 4$, we have $(04432^\infty)_q = (112^\infty)_q$. In this case, $q_2 = q_f(4) \approx 3.562$ and $q_{KL} \approx 3.667$, then $q_f(4) < q < q_{KL}$, and hence, $q \in (q_n, q_{n+1})$ for some $n > 1$. Moreover, $\alpha(q_f(4)) = (31)^\infty$, thus $4432^\infty, 12^\infty \in \mathcal{U}'_q$ by Lemmas 2.1 and 2.2. Since the proof of [15, Lemma 6.1] is independent of the alphabet, it follows from this lemma and Theorem 3.2 that $q \in \mathcal{B}_2^{(2n)}(4) \cap (q_n, q_{n+1})$.

6. Appendix

- Page 11, calculation of Table 1:

Let

$$\begin{aligned}
h(x) &= -x^6 + 4x^5 - 2x^4 - 4x^3 + 3x^2 - 6x + 5 \\
h'(x) &= 2(-3x^5 + 10x^4 - 4x^3 - 6x^2 + 3x - 3) \\
h''(x) &= 2(-15x^4 + 40x^3 - 12x^2 - 12x + 3) \\
h'''(x) &= 24(-5x^3 + 10x^2 - 2x - 1) \\
h^{(4)}(x) &= 24(-15x^2 + 20x - 2).
\end{aligned}$$

It follows from $h^{(4)}(x) < 0$ for $x \in [q_f(1), 2]$ that $h'''(x)$ is strictly decreasing in $[q_f(1), 2]$. Since

$$h'''(q_f(1)) = 24(-5(q_f(1))^3 + 10(q_f(1))^2 - 2q_f(1) - 1) = 24(3q_f(1) - 6) < 0,$$

we have $h'''(x) < 0$ for $x \in [q_f(1), 2]$. So $h''(x)$ is strictly decreasing in $[q_f(1), 2]$. Now, $h''(2) = 22$, and so $h''(x) > 0$ for $x \in [q_f(1), 2]$. Note that

$$\begin{aligned}
h'(x) &= 2(-3x^5 + 10x^4 - 4x^3 - 6x^2 + 3x - 3) \\
&= 2((x^3 - 2x^2 + x - 1)(-3x^2 + 4x + 7) + x^2 + 4).
\end{aligned}$$

Thus, $h'(q_f(1)) = 2(q_f(1))^2 + 8 > 0$. Finally, we have $h(2) > 0$ and

$$\begin{aligned}
h(x) &= -x^6 + 4x^5 - 2x^4 - 4x^3 + 3x^2 - 6x + 5 \\
&= (x^3 - 2x^2 + x - 1)(-x^3 + 2x^2 + 3x - 1) - 2x + 4.
\end{aligned}$$

Thus, $h(q_f(1)) = -2q_f(1) + 4 > 0$. Therefore, $h(x) > 0$ for $x \in [q_f(1), 2]$ by Proposition 3.8.

- Page 13, calculation of Table 2:

Let

$$\begin{aligned}
h(x) &= -x^5 + 8x^4 - 16x^3 + 8x^2 - 10x + 8 \\
h'(x) &= -5x^4 + 32x^3 - 48x^2 + 16x - 10 \\
h''(x) &= 4(-5x^3 + 24x^2 - 24x + 4) \\
h'''(x) &= 12(-5x^2 + 16x - 8).
\end{aligned}$$

We have $h'''(x) < 0$ for $x \in [q_f(3), 4]$. Recall that $q_f(3)$ satisfies

$$(q_f(3))^3 - 3(q_f(3))^2 + q_f(3) - 2 = 0. \tag{6.1}$$

In fact,

$$-5x^4 + 32x^3 - 48x^2 + 16x - 10 = (x^3 - 3x^2 + x - 2)(-5x + 17) + 8x^2 - 11x + 24.$$

Thus, by (6.1), we have

$$\begin{aligned} h'(q_f(3)) &= 8(q_f(3))^2 - 11q_f(3) + 24 \\ &= 8\left((q_f(3))^2 - 3q_f(3) + 1\right) + 13q_f(3) + 16 = \frac{16}{q_f(3)} + 13q_f(3) + 16 > 0. \end{aligned}$$

Similarly, $h''(q_f(3)) > 0$ and $h(q_f(3)) > 0$ by means of (6.1). Finally, $h(4) = 96 > 0$.

- Page 14, calculation of Table 3:

Let

$$\begin{aligned} h(x) &= -x^4 + 6x^3 - 3x^2 - 20x + 15 \\ h'(x) &= -4x^3 + 18x^2 - 6x - 20 \\ h''(x) &= -12x^2 + 36x - 6. \end{aligned}$$

It follows from $h''(x) < 0$ for $x \in [q_f(3), 4]$ that $h'(x)$ is strictly decreasing in $[q_f(3), 4]$. But $h'(x) > 0$ in $[q_f(3), \alpha)$ and $h'(x) < 0$ in $(\alpha, 4]$ for some $q_f(3) < \alpha < 4$. Moreover, $h(q_f(3))$ and $h(4)$ are positive.

- Page 14, calculation of Table 4:

Let

$$g(x) = x^3 - (k+3)x^2 + (k+4)x - 2k - 3.$$

$g'''(x)$ satisfies condition (I) of Proposition 3.8 in $[q_f(2k+1), 2k+2]$, and $g'(q_f(2k+1)) > 0$, $g''(q_f(2k+1)) > 0$ and $g(2k+2) > 0$. Note that

$$\begin{aligned} &x^3 - (k+3)x^2 + (k+4)x - 2k - 3 \\ &\quad = (x^3 - (k+2)x^2 + x - k - 1) + (-x^2 + (k+3)x - k - 2). \end{aligned}$$

Recall that $q_f(2k+1)$ is the root of $x^3 - (k+2)x^2 + x - k - 1 = 0$. Since $k + 1 < q_f(2k+1) < k + 2$, we have

$$\begin{aligned} g(q_f(2k+1)) &= -(q_f(2k+1))^2 + (k+3)q_f(2k+1) - k - 2 \\ &= -(q_f(2k+1) - 1)(q_f(2k+1) - k - 2) > 0. \end{aligned}$$

• Page 15, calculation of Tables 5 and 6:

Let

$$\begin{aligned} h(x) &= -x^3 + (2k+8)x^2 - (10k+16)x + 6k + 8 \\ h'(x) &= -3x^2 + 4(k+4)x - 10k - 16. \end{aligned}$$

The roots $x_1 < x_2$ of $h'(x) = 0$ satisfy, for $k \geq 3$,

$$\begin{aligned} x_1 &= \frac{2k+8-\sqrt{4k^2+2k+16}}{3} < k+1 < q_f(2k+1) \\ k+1 < x_2 &= \frac{2k+8+\sqrt{4k^2+2k+16}}{3} < m+1 = 2k+2, \end{aligned}$$

and for $k = 2$,

$$x_1 = 2 < k+1 < q_f(5) < k+2 < x_2 = 6 = 2k+2.$$

Hence, $h(x)$ is strictly increasing in $[k+1, x_2]$, and strictly decreasing in $[x_2, m+1]$ for $k \geq 3$, and $h(k+1) = k^3 - k^2 - 5k - 1 > 0, h(m+1) = 4k^2 + 2k > 0$; $h(x)$ is strictly increasing in $[q_f(5), 6]$ for $k = 2$, and $h(q_f(5)) > 0$.

• Page 16, calculation of Table 7:

Let

$$\begin{aligned} h(x) &= -x^4 + 2(k+3)x^3 - (4k+12)x^2 - (6k-6)x + 6k \\ h'(x) &= 2(-2x^3 + 3(k+3)x^2 - (4k+12)x - 3k + 3) \\ h''(x) &= -4(3x^2 - 3(k+3)x + 2k + 6) \\ h'''(x) &= -4(6x - 3(k+3)). \end{aligned}$$

We have $h'(k+1) = 2(k^3 + 5k^2 - 4k - 2) > 0$, $h''(k+1) = 16k > 0$ and $h'''(x) < 0$ in $[k+1, m+1]$. In addition,

$$\begin{aligned} h(k+1) &= k^4 + 4k^3 - 8k^2 - 6k - 1 > 0 \\ h(m+1) &= h(2k+2) = (8k^2 - 6k - 2)(2k+2) + 6k > 0. \end{aligned}$$

• Page 17, calculation of Table 8:

Let

$$\begin{aligned} h(x) &= -x^4 + (2k+4)x^3 - (k+2)x^2 - (12k+8)x + 3(3k+2) \\ h'(x) &= 2(-2x^3 + 3(k+2)x^2 - (k+2)x - 6k - 4) \end{aligned}$$

$$h''(x) = -2(6x^2 - 6(k+2)x + k + 2)$$
$$h'''(x) = -12(2x - (k+2)).$$

Then $h'''(x) < 0$ in $[k+1, m+1]$, $h''(k+1) = -2(-5k-4) > 0$ and $h'(k+1) = 2(k^3 + 5k^2 - 2) > 0$. Since

$$h(k+1) = (k^3 + 4k^2 - 8k - 7)(k+1) + 9k + 6 > 0$$
$$h(m+1) = (6k^2 - 2k - 4)(2k+2) + 9k + 6 > 0,$$

we have $h(x) > 0$ for $x \in [k+1, m+1] \supseteq [q_f(2k+1), 2k+2]$ by Proposition 3.8.

- Page 19, calculation of Table 9 for (2) case (i):

  We have

$$g(x) = x^4 - (k+2)x^3 + kx^2 - (k-2)x - k - 1$$
$$g'(x) = 4x^3 - 3(k+2)x^2 + 2kx - k + 2$$
$$g''(x) = 2(6x^2 - 3(k+2)x + k).$$

The roots of $g''(x)$ satisfy

$$x_1 < x_2 = \frac{3k + 6 + \sqrt{9k^2 + 12k + 36}}{12} < k + 1 < q_f(2k)$$

and $g'(k+1) = k^3 + 2k^2 - 2k > 0$. Hence, $g(x)$ is strictly increasing in $[q_f(2k), 2k+1]$. Next, we shall show that $g(q_f(2k)) > 0$ for $k \geq 2$. Recall $q_f(2k) = (k + 1 + \sqrt{k^2 + 6k + 1})/2$ is the largest real root of $x^2 - (k+1)x - k = 0$, and

$$\begin{aligned} g(x) &= x^4 - (k+2)x^3 + kx^2 - (k-2)x - k - 1 \\ &= (x^2 - x + k - 1)(x^2 - (k+1)x - k) + (k-1)^2(x+1) - 2. \end{aligned}$$

Thus, $g(q_f(2k)) > 0$ for $k \geq 2$.

- Page 19, Table 9 for (2) case (ii):

  We have

$$g(x) = x^4 - (k+3)x^3 + (3k+1)x^2 - (3k-2)x - k - 1$$
$$g'(x) = 4x^3 - 3(k+3)x^2 + 2(3k+1)x - 3k + 2$$
$$g''(x) = 2(6x^2 - 3(k+3)x + 3k + 1).$$

The roots of $g''(x)$ satisfy for $k \geq 2$,

$$x_1 < x_2 = \frac{3k + 9 + \sqrt{9k^2 - 18k + 57}}{12} < k + 1 < q_f(2k).$$

Thus, $g'(x) > 0$ in $[k + 1, 2k + 1]$ by $g'(k + 1) > 0$. So, $g(x)$ is strictly increasing in $[q_f(2k), 2k + 1]$. Finally,

$$\begin{aligned} g(x) &= x^4 - (k + 3)x^3 + (3k + 1)x^2 - (3k - 2)x - k - 1 \\ &= (x^2 - 2x + 2k - 1)(x^2 - (k + 1)x - k) + (2k^2 - 4k + 1)x + (2k^2 - 2k - 1). \end{aligned}$$

- Page 20, calculation of Tables 10 and 11:

  Let

$$\begin{aligned} h(x) &= -x^4 + (2k + 4)x^3 - (4k - 1)x^2 - (6k + 10)x + 6(k + 1) \\ h'(x) &= 2(-2x^3 + 3(k + 2)x^2 - (4k - 1)x - 3k - 5) \\ h''(x) &= 2(-6x^2 + 6(k + 2)x - 4k + 1). \end{aligned}$$

The roots x_1, x_2 of $h''(x)$ satisfy for $k \geq 2$,

$$x_1 < k + 1 < x_2 = \frac{3k + 6 + \sqrt{9k^2 + 12k + 42}}{6} < k + 2.$$

Hence, $h'(x)$ is strictly increasing in $[k+1, x_2]$, and strictly decreasing in $[x_2, m+1]$ for $k \geq 2$. Note that

$$h'(k + 1) = 2(k^3 + 2k^2 + 3k) > 0.$$

In addition, we have $h'(m + 1) = h'(2k + 1)$ is positive when $k = 2$, but negative for $k \geq 3$. This means that in $[k+1, m+1]$ the function $h(x)$ is strictly increasing if $k = 2$, and $h(x)$ is strictly increasing firstly, and then strictly decreasing if $k \geq 3$. However, we have for all $k \geq 2$,

$$\begin{aligned} h(k + 1) &= (k^3 + k^2 - 2k)(k + 1) > 0 \\ h(m + 1) &= (4k^2 + 4k - 6)(2k + 1) + 6k + 6 > 0. \end{aligned}$$

- Page 20, calculation of Table 12 for (3)(i) case 2:

  Let

$$h(x) = -x^5 + (2k + 4)x^4 - (4k + 2)x^3 + (6k - 4)x^2 - (18k - 3)x + 12k.$$

Then, $h'''(x) = 12(-5x^2 + 4(k+2)x - 2k - 1) < 0$ for $x \in [k+1, m+1]$, and

$$h''(k+1) > 0, h'(k+1) > 0, h(k+1) > 0 \qquad \text{and} \qquad h(m+1) > 0.$$

Therefore, $h(x) > 0$ for $x \in [k+1, m+1] \supseteq [q_f(2k), m+1]$ by Proposition 3.8.

- Page 20, calculation of Table 12 for (3)(ii) case 2:

  Let

$$\begin{aligned} h(x) &= -x^4 + (2k+6)x^3 - (10k+3)x^2 + (6k-8)x + 6 \\ h'(x) &= 2(-2x^3 + 3(k+3)x^2 - (10k+3)x + 3k - 4) \\ h''(x) &= 2(-6x^2 + 6(k+3)x - 10k - 3). \end{aligned}$$

We have $h'''(x) = -24x + 12(k+3) < 0$ in $[k+1, m+1]$. In addition, one can check

$$h''(k+1) > 0, h'(k+1) > 0, h(k+1) \geq 0 \qquad \text{and} \qquad h(m+1) > 0.$$

From Proposition 3.8, it follows that $h(x) > 0$ for $x \in [q_f(2k), 2k+1]$. Here, we would like to remark that $h(k+1) = 0$ for $k = 2$, however, $h(k+1) > 0$ for $k \geq 3$. This does not affect us to get the result in $[q_f(2k), 2k+1] \subsetneqq [k+1, 2k+1]$.

Acknowledgements. The authors would like to thank the referees and Professor Vilmos Komornik for helpful suggestions as well as for his generous hospitality during their visit at the University of Strasbourg.

References

[1] J.-P. Allouche, Théorie des nombres et automates, Doctoral Thesis, *Université Sciences et Technologies, Bordeaux I*, 1983.

[2] C. Baiocchi and V. Komornik, Greedy and quasi-greedy expansions in non-integer bases, arXiv:0710.3001.

[3] S. Baker, Generalized golden ratios over integer alphabets, *Integers* **14** (2014), Paper No. A15, 28 pp.

[4] S. Baker and N. Sidorov, Expansions in non-integer bases: lower order revisited, *Integers* **14** (2014), Paper No. A57, 15 pp.

[5] Y. Cai and V. Komornik, Hausdorff distance of univoque sets, arXiv:2006.16130.

[6] Z. Daróczy and I. Kátai, Univoque sequences, *Publ. Math. Debrecen* **42** (1993), 397–407.

[7] Z. Daróczy and I. Kátai, On the structure of univoque numbers, *Publ. Math. Debrecen* **46** (1995), 385–408.

[8] M. de Vries and V. Komornik, Unique expansions of real numbers, *Adv. Math.* **221** (2009), 390–427.

[9] M. de Vries and V. Komornik, A two-dimensional univoque set, *Fund. Math.* **212** (2011), 175–189.

[10] M. de Vries, V. Komornik and P. Loreti, Topology of univoque bases, *Topology Appl.* **205** (2016), 117–137.

[11] P. Erdős, M. Horváth and I. Joó, On the uniqueness of the expansions $1 = \sum q^{-n_i}$, *Acta Math. Hungar.* **59** (1991), 333–342.

[12] P. Erdős and I. Joó, On the number of expansions $1 = \sum q^{-n_i}$, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **35** (1992), 129–132.

[13] P. Erdős, I. Joó and V. Komornik, Characterization of the unique expansions $1 = \sum q^{-n_i}$ and related problems, *Bull. Soc. Math. France* **118** (1990), 377–390.

[14] P. Glendinning and N. Sidorov, Unique representations of real numbers in non-integer bases, *Math. Res. Lett.* **8** (2001), 535–543.

[15] V. Komornik and D. Kong, Bases in which some numbers have exactly two expansions, *J. Number Theory* **195** (2019), 226–268.

[16] V. Komornik, D. Kong, and W. Li, Hausdorff dimension of univoque sets and Devil's staircase, *Adv. Math.* **305** (2017), 165–196.

[17] V. Komornik and P. Loreti, Unique developments in non-integer bases, *Amer. Math. Monthly* **105** (1998), 636–639.

[18] V. Komornik and P. Loreti, Subexpansions, superexpansions and uniqueness properties in non-integer bases, *Period. Math. Hungar.* **44** (2002), 197–218.

[19] V. Komornik and P. Loreti, On the topological structure of univoque sets, *J. Number Theory* **122** (2007), 157–183.

[20] D. Kong and W. Li, Hausdorff dimension of unique beta expansions, *Nonlinearity* **28** (2015), 187–209.

[21] D. Kong, W. Li, and Y. Zou, On small bases which admit points with two expansions, *J. Number Theory* **173** (2017), 100–128.

[22] W. Parry, On the β-expansions of real numbers, *Acta Math. Acad. Sci. Hungar.* **11** (1960), 401–416.

[23] N. Sidorov, Expansions in non-integer bases: lower, middle and top orders, *J. Number Theory* **129** (2009), 741–754.

[24] N. Sidorov and A. Vershik, Ergodic properties of the Erdős measure, the entropy of the golden shift, and related problems, *Monatsh. Math.* **126** (1998), 215–261.

YI CAI
COLLEGE OF SCIENCES
SHANGHAI INSTITUTE OF TECHNOLOGY
SHANGHAI 201418
P. R. CHINA

E-mail: 546823540@qq.com

WENXIA LI
SCHOOL OF MATHEMATICAL SCIENCES
SHANGHAI KEY LABORATORY OF PMMP
EAST CHINA NORMAL UNIVERSITY
SHANGHAI 200062
P. R. CHINA

E-mail: wxli@math.ecnu.edu.cn

(Received December 26, 2021; revised November 24, 2022)