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FRACTAL DIMENSIONS OF SETS DEFINED BY DIGIT RESTRICTIONS IN $\mathbb{R}^2$

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Abstract

We introduce a class of sets defined by digit restrictions in $\mathbb{R}^2$ and study its fractal dimensions. Let $E_{S,D}$ be a set defined by digit restrictions in $\mathbb{R}^2$. We obtain the Hausdorff and lower box dimensions of $E_{S,D}$. Under some condition, we gain the packing and upper box dimensions of $E_{S,D}$. We get the Assouad dimension of $E_{S,D}$ and show that it is 2 if and only if $E_{S,D}$ contains arbitrarily large arithmetic patches. Under some conditions, we study the upper spectrum, quasi-Assouad dimension and Assouad spectrum of $E_{S,D}$. Finally, we give an intermediate value property of fractal dimensions of the class of sets.

Keywords: Sets Defined by Digit Restrictions in $\mathbb{R}^2$; Assouad Dimension; Arithmetic Patch; Upper Spectrum; Quasi-Assouad Dimension; Assouad Spectrum.

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1. INTRODUCTION

Calculating fractal dimensions is one of the most important research topics in fractal geometry. Fractal dimensions describe complexity of fractal sets. Roughly speaking, the Hausdorff, packing, lower and upper box dimensions reflex the global complexity of fractal sets and the Assouad dimension and its variations give local complexity of fractal sets.

The Assouad dimension was popularized by Assouad in the 1970s.[1,2] The concept of Assouad dimension can be traced back to the work of many scholars, such as Bouligand,[3] Larman[4] and Furstenberg.[5,6] The Assouad dimension plays an important role in the study of embedding theory and quasi-conformal mappings.[7–11] In particular, the Assouad dimension has obtained a lot of attention in fractal geometry.[12–21] In the past few years, variations of the Assouad dimension have also become a hot topic in fractal geometry. Lü and Xi[22] introduced the quasi-Assouad dimension which is invariant under quasi-Lipschitz mappings. Fraser and Yu[23] introduced the Assouad spectrum to interpolate between the upper box and Assouad dimensions. Fraser *et al.*[24] proved that the upper spectrum can be expressed in terms of the Assouad spectrum and the Assouad spectrum approaches the quasi-Assouad dimension at the right-hand side of its domain. For more research on variations of the Assouad dimension, see Refs. 25–27.

The set defined by digit restrictions is an important fractal class and has gained a lot of interest in fractal geometry. Next, we introduce research on sets defined by digit restrictions in $\mathbb{R}$.

Let $q \geq 2$ be an integer and $\mathbb{N}$ be the set of positive integers. Let $S \subset \mathbb{N}$ and D be a nonempty proper subset of $\{0, 1, \ldots, q-1\}$.

Define

$$H_{S,D} := \left\{ \sum_{k=1}^{+\infty} \frac{a_k}{q^k} : a_k \in \{0, 1, \ldots, q-1\} \text{ if } k \in S \text{ and } a_k \in D \text{ if } k \notin S \text{ for each } k \in \mathbb{N} \right\}.$$

The set $H_{S,D}$ is called a set defined by digit restrictions in $\mathbb{R}$.

Bishop and Peres[28] introduced the definition of sets defined by digit restrictions in $\mathbb{R}$ for $q = 2$ and $D = \{0\}$ and computed the Hausdorff and lower and upper box dimensions of these sets. Dai *et al.*[29] introduced the definition of sets defined by digit restrictions in $\mathbb{R}$ for $q \geq 2$ being an integer and studied the Hausdorff, packing, lower and upper box and Assouad dimensions of these sets. Li *et al.*[30] studied the connection between Assouad dimension and arithmetic progressions for $H_{S,D}$ for $b \geq 2$ being an integer and they proved the Assouad dimension of $H_{S,D}$ is one if and only if $H_{S,D}$ contains arbitrarily long arithmetic progressions. Recall that $A \subset \mathbb{R}$ is called an arithmetic progression of length $n \in \mathbb{N}$ if $A = \{x + j\delta : j = 0, 1, \ldots, n-1\}$ for some $x \in \mathbb{R}$ and $\delta > 0$. One says that $E \subset \mathbb{R}$ contains arbitrarily long arithmetic progressions if for any $n \in \mathbb{N}$, there exists an arithmetic progression $A \subset \mathbb{R}$ of length n such that $A \subset E$. For other research on the set defined by digit restrictions and its variations, refer to Refs. 31–37 and the references therein.

In this paper, we introduce a class of sets defined by digit restrictions in $\mathbb{R}^2$ and study its fractal dimensions.

Let $\mathbb{N}$ be the set of positive integers. Let $\{n_k\}_{k\geq 1}$ be a sequence of positive integers and $n_k \geq 2$ for all $k \in \mathbb{N}$ and $q := \min\{n_k : k \in \mathbb{N}\}$. Let $S \subset \mathbb{N}$ and D be a nonempty proper subset of $\{0, 1, \ldots, q-1\}^2$.

Define

$$\tilde{E}_{S,D} := \left\{ \sum_{k=1}^{+\infty} \frac{\tau_k}{n_1 n_2 \cdots n_k} : \tau_k \in \{0, 1, \ldots n_k - 1\}^2 \text{ if } k \in S \text{ and } \tau_k \in D \text{ if } k \notin S \text{ for each } k \in \mathbb{N} \right\}.$$

The set $\tilde{E}_{S,D}$ is called a set defined by digit restrictions in $\mathbb{R}^2$.

We study fractal dimensions of the sets defined by digit restrictions $\tilde{E}_{S,D}$ in $\mathbb{R}^2$. We obtain the Hausdorff and lower box dimensions of $\tilde{E}_{S,D}$. Under some condition, we gain the packing and upper box dimensions of $\tilde{E}_{S,D}$. We get the Assouad dimension of $\tilde{E}_{S,D}$ and show that it is 2 if and only if $\tilde{E}_{S,D}$ contains arbitrarily large arithmetic patches. Under some conditions, we study the upper spectrum, quasi-Assouad dimension and Assouad spectrum of $\tilde{E}_{S,D}$. Finally, we give an intermediate value property of fractal dimensions of the class of sets to show that the class of sets defined by digit restrictions in $\mathbb{R}^2$ is complicated enough.

It is easy to see that when all n_ks are taken as some fixed positive integer and a special D is taken, then $\tilde{E}_{S,D}$ can degenerate into $H_{S,D}$, so the set defined by digit restrictions in $\mathbb{R}^2$ can be regarded as a generalization of the set defined by digit restrictions in $\mathbb{R}$. When n_ks are taken differently, $\tilde{E}_{S,D}$ can have complicated dimension formulae. Sets defined by digit restrictions in $\mathbb{R}^2$ are Moran sets. For the Moran sets, see Refs. 38–40. We remark that the sets defined by digit restrictions in $\mathbb{R}^2$ can be some Moran sets with zero infimum contraction, that is the case $\inf_{k\in\mathbb{N}} \frac{1}{n_k} = 0$ can happen. Calculating fractal dimensions of Moran sets with zero infimum contraction is difficult. Thus, our results can be regarded as a supplement to the fractal dimension theory of Moran sets with zero infimum contraction.

2. DEFINITIONS AND MAIN RESULTS

2.1. Sets Defined by Digit Restrictions in $\mathbb{R}^2$

For the convenience of proof, we give a fractal definition of sets defined by digit restrictions in $\mathbb{R}^2$.

Let $\mathbb{N}$ be the set of positive integers. Let $\{n_k\}_{k\geq 1}$ be a sequence of positive integers and $n_k \geq 2$ for all $k \in \mathbb{N}$ and $q := \min\{n_k : k \in \mathbb{N}\}$. Let $S \subset \mathbb{N}$ and D be a nonempty proper subset of $\{0, 1, \ldots, q-1\}^2$. We introduce some notations as follows:

$$\Omega := \{\tau_1\tau_2\cdots : \tau_k \in \{0, 1, \ldots, n_k - 1\}^2 \text{ for each } k \in \mathbb{N}\},$$

$$\Lambda := \{\tau_1\tau_2\cdots : \tau_k \in \{0, 1, \ldots, n_k - 1\}^2 \text{ if } k \in S \text{ and } \tau_k \in D \text{ if } k \notin S \text{ for each } k \in \mathbb{N}\}$$

and for any $k \in \mathbb{N}$,

$$\Lambda_k := \{\tau_1\tau_2\cdots\tau_k : \tau_i \in \{0, 1, \ldots, n_i - 1\}^2 \text{ if } i \in S \text{ and } \tau_i \in D \text{ if } i \notin S \text{ for each } i \in [1, k] \cap \mathbb{N}\}.$$

For any $k \in \mathbb{N}$, $j \in \{0, 1, \ldots, n_k - 1\}^2$, define

$$f_{k,j} : \mathbb{R}^2 \to \mathbb{R}^2$$
$$x \mapsto \frac{x + j}{n_k}.$$

Let $I := [0, 1]^2$. For any $\tau \in \Omega$, $k \in \mathbb{N}$, let $f_{\tau|_k} := f_{1,\tau_1} \circ \cdots \circ f_{k,\tau_k}$ and define

$$\Pi : \Omega \to \mathbb{R}^2$$
$$\tau \mapsto \bigcap_{k=1}^{+\infty} f_{\tau|_k}(I).$$

Define

$$E_{S,D} := \Pi(\Lambda).$$

The set $E_{S,D}$ is called a set defined by digit restrictions in $\mathbb{R}^2$. For any $\tau \in \Lambda$ and $k \in \mathbb{N}$, $f_{\tau|_k}(I)$ is called a $level-k$ set of $E_{S,D}$.

The above set $E_{S,D}$ can also be expressed as follows:

$$E_{S,D} = \tilde{E}_{S,D} = \left\{\sum_{k=1}^{+\infty} \frac{\tau_k}{n_1 n_2 \cdots n_k} : \tau_k \in \{0, 1, \ldots, n_k - 1\}^2 \text{ if } k \in S \text{ and } \tau_k \in D \text{ if } k \notin S \text{ for each } k \in \mathbb{N}\right\}.$$

2.2. Assouad Dimension and Its Variations

Falconer's textbook[41] includes a detailed introduction to the Hausdorff, packing, lower and upper box dimensions. In this paper, we denote the Hausdorff, packing, lower and upper box dimensions by $\dim_H$, $\dim_P$, $\underline{\dim}_B$ and $\overline{\dim}_B$, respectively. The following definition of the Assouad dimension and a more detailed introduction to the Assouad dimension can be found in Fraser's book.[15]

Let $\|\cdot\|$ be the Euclidean norm in $\mathbb{R}^d$ and $B(x, R) := \{y \in \mathbb{R}^d : \|y - x\| \leq R\}$ denote the closed ball with centre $x \in \mathbb{R}^d$ and radius $R > 0$. For nonempty bounded $E \subset \mathbb{R}^d$, let $N_\delta(E)$ denote the smallest number of closed balls of radius $\delta > 0$ required to cover E.

Definition (Ref. 15). For nonempty $F \subset \mathbb{R}^d$, the Assouad dimension of F, denoted by $\dim_A F$, is defined as

$$\dim_A F := \inf\left\{s \geq 0 : \text{there exist constants } C > 0, \rho > 0 \text{ such that, for all } 0 < r < R < \rho \text{ and } x \in F,\ N_r(B(x, R) \cap F) \leq C\left(\frac{R}{r}\right)^s\right\}.$$

In the above definition of the Assouad dimension, if we delete ρ, we can get the same infimum for bounded $F \subset \mathbb{R}^d$, see discussion on p. 14 in Ref. 15.

Definition (Refs. 22 and 24). For any $\theta \in (0, 1)$ and nonempty $F \subset \mathbb{R}^d$, the upper spectrum of F, denoted by $\overline{\dim}_A^{\theta} F$, is defined as

$$\overline{\dim}_A^{\theta} F := \inf \Big\{ s \geq 0 : \text{there exists a constant } C > 0 \text{ such that, for all } 0 < r \leq R^{1/\theta} < R < 1 \text{ and } x \in F, \ N_r(B(x,R) \cap F) \leq C \left(\frac{R}{r}\right)^s \Big\}.$$

The quasi-Assouad dimension of F, denoted by $\dim_{qA} F$, is defined as

$$\dim_{qA} F := \lim_{\theta \to 1} \overline{\dim}_A^{\theta} F.$$

Definition (Ref. 23). For any $\theta \in (0,1)$ and nonempty $F \subset \mathbb{R}^d$, the Assouad spectrum of F, denoted by $\dim_A^{\theta} F$, is defined as

$$\dim_A^{\theta} F := \inf \Big\{ s \geq 0 : \text{there exist a constant } C > 0 \text{ such that, for all } 0 < r = R^{1/\theta} < R < 1 \text{ and } x \in F, \ N_r(B(x,R) \cap F) \leq C \left(\frac{R}{r}\right)^s \Big\}.$$

2.3. Main Results

For any $k \in \mathbb{N}$, let

$$l_k := \begin{cases} n_k^2 & \text{if } k \in S, \\ \sharp D & \text{if } k \notin S, \end{cases}$$

here and in the sequel, $\sharp E$ denotes the cardinality of $E \subset \mathbb{R}^2$. If $S = \emptyset$, we adopt the convenience $\sup\{n_k : k \in S\} := 0$, thus if $S = \mathbb{N}$, we have $\sup\{n_k : k \in \mathbb{N}\backslash S\} = 0$. For any $x \in \mathbb{R}$, we use $\lfloor x \rfloor$ to denote the largest integer not exceeding x.

Theorem 2.1. *Let $E_{S,D}$ be a set defined by digit restrictions in $\mathbb{R}^2$. Then*

$$\dim_H E_{S,D} = \underline{\dim}_B E_{S,D} = \liminf_{k \to +\infty} \frac{\log(l_1 \cdots l_k)}{\log(n_1 \cdots n_{k+1}) - \log \sqrt{l_{k+1}}}.$$

Theorem 2.2. *Let $E_{S,D}$ be a set defined by digit restrictions in $\mathbb{R}^2$. If*

$$\lim_{k \to +\infty} \frac{\log(n_{k+1}^2 / l_{k+1})}{\log(n_1 \cdots n_k n_{k+1})} = 0,$$

then

$$\dim_P E_{S,D} = \overline{\dim}_B E_{S,D} = \limsup_{k \to +\infty} \frac{\log(l_1 \cdots l_k)}{\log(n_1 \cdots n_k)}.$$

Corollary 2.2.1. *Let $E_{S,D}$ be a set defined by digit restrictions in $\mathbb{R}^2$. If $\sup\{n_k : k \in \mathbb{N}\backslash S\} < +\infty$, then*

$$\dim_P E_{S,D} = \overline{\dim}_B E_{S,D} = \limsup_{k \to +\infty} \frac{\log(l_1 \cdots l_k)}{\log(n_1 \cdots n_k)}.$$

Remark. Some of the methods we used in the proof of Theorems 2.1 and 2.2 come from the study of Moran sets with zero infimum contraction.[38,42]

Theorem 2.3. *Let $E_{S,D}$ be a set defined by digit restrictions in $\mathbb{R}^2$. Then*

$$\dim_A E_{S,D} = \begin{cases} 2 & \text{if } \sup\{n_k : k \in S\} = +\infty, \\ \displaystyle\lim_{j \to +\infty} \sup_{k \geq j,\ m \geq j} \frac{\log(l_{k+1} \cdots l_{k+m})}{\log(n_{k+1} \cdots n_{k+m})} & \text{if } \sup\{n_k : k \in S\} < +\infty. \end{cases}$$

Remark. Li *et al.* obtained Assouad dimensions of Moran sets with positive infimum contraction in Ref. 43. Then Peng *et al.* established Assouad dimensions for uniform Cantor sets in Ref. 44. Inspired by Assouad dimension formulae for uniform Cantor sets in Ref. 44 the form of which also appears in Assouad dimension formulae of homogeneous perfect set in Ref. 45, we guess the above Assouad dimension formulae of sets defined by digit restrictions in $\mathbb{R}^2$. Recently, Zhu gained Assouad dimensions of two Moran classes with zero infimum contraction in Ref. 46. We remark that sets defined by digit restrictions in $\mathbb{R}^2$ can allow the two cases $\limsup_{m \to +\infty} \sup_{k \in \mathbb{N}} \frac{\log n_{k+m}}{\log(n_{k+1} n_{k+2} \cdots n_{k+m})} > 0$ and $\sup_{k \in \mathbb{N}} n_k = +\infty$ to happen, so Theorem 2.3 cannot be contained in Theorems 2.4 and 2.5 in Ref. 46.

Definition 2.1 (Ref. 47). One calls $A \subset \mathbb{R}^d$ an arithmetic patch of size $n \in \mathbb{N}$ in $\mathbb{R}^d$ if $A = \{x + j\delta : j \in \{0, 1, \ldots, n-1\}^d\}$ for some $x \in \mathbb{R}^d$ and $\delta > 0$. One says that $E \subset \mathbb{R}^d$ contains arbitrarily large arithmetic patches in $\mathbb{R}^d$ if for any $n \in \mathbb{N}$, there exists an arithmetic patch A of size n in $\mathbb{R}^d$ such that $A \subset E$.

Theorem 2.4. *Let $E_{S,D}$ be a set defined by digit restrictions in $\mathbb{R}^2$. Then $\dim_A E_{S,D} = 2$ if and only if $E_{S,D}$ contains arbitrarily large arithmetic patches in $\mathbb{R}^2$.*

Remark. Definition 2.1 is introduced by Fraser and Yu in Ref. 47. The main method we used in the proof of Theorem 2.4 is inspired by the proof method of Theorem 2.1 in Ref. 30. Li *et al.* in Theorem 2.1 in Ref. 30, proved that the Assouad dimension of $H_{S,D}$ is one if and only if $H_{S,D}$ contains arbitrarily long arithmetic progressions, where $H_{S,D}$ is the set defined by digit restrictions in $\mathbb{R}$.

Theorem 2.5. *Let $E_{S,D}$ be a set defined by digit restrictions in $\mathbb{R}^2$. For any $\theta \in (0,1)$,*

(1) *if*

$$\sup\{n_k : k \in S\} = +\infty \quad \textit{and}$$

$$\left(\frac{1}{n_1 \cdots n_{k-1}}\right)^{1/\theta} \geq \frac{1}{n_1 \cdots n_k}$$

for each $k \in S$ and $k \geq 2$, then

$$\overline{\dim}_A^{\theta} E_{S,D} = 2.$$

(2) *Let*

$$h(k,\theta) := \max\left\{1 \leq h \leq k : \frac{\log(n_h \cdots n_k)}{\log(n_1 \cdots n_k)} > 1-\theta\right\}$$

for each $k \in \mathbb{N}$. If

$$\lim_{k\to+\infty} \frac{\log n_k}{\log(n_1 \cdots n_k)} = 0,$$

then

$$\overline{\dim}_A^{\theta} E_{S,D} = \limsup_{k\to+\infty} \max_{1\leq h\leq h(k,\theta)} \frac{\log(l_h \cdots l_k)}{\log(n_h \cdots n_k)}$$

and

$$\dim_{qA} E_{S,D} = \lim_{\theta\to 1} \limsup_{k\to+\infty} \max_{1\leq h\leq h(k,\theta)} \frac{\log(l_h \cdots l_k)}{\log(n_h \cdots n_k)}.$$

(3) *If $\sup_{k\geq 1} n_k < +\infty$, then*

$$\dim_{qA} E_{S,D} = \lim_{\alpha\to 1} \limsup_{k\to+\infty} \max_{1\leq h\leq \alpha k} \frac{\log(l_h \cdots l_k)}{\log(n_h \cdots n_k)},$$

where $\alpha \in (0,1)$.

Remark. Theorems 2.5(2) and 2.5(3) are inspired by Theorem 1.14 in Ref. 22. Lü and Xi studied the quasi-Assouad dimension of homogeneous Moran sets in $\mathbb{R}$ with some condition in Theorem 1.14 in Ref. 22.

Theorem 2.6. *Let $E_{S,D}$ be a set defined by digit restrictions in $\mathbb{R}^2$. For any $\theta \in (0,1)$,*

(1) *if*

$$\sup\{n_k : k \in S\} = +\infty \quad \textit{and}$$

$$\left(\frac{1}{n_1 \cdots n_{k-1}}\right)^{1/\theta} \geq \frac{1}{n_1 \cdots n_k}$$

for each $k \in S$ and $k \geq 2$, then

$$\dim_A^{\theta} E_{S,D} = 2.$$

(2) *Let*

$$h(k,\theta) := \max\left\{1 \leq h \leq k : \frac{\log(n_h \cdots n_k)}{\log(n_1 \cdots n_k)} > 1-\theta\right\}$$

for each $k \in \mathbb{N}$. If

$$\lim_{k\to+\infty} \frac{\log n_k}{\log(n_1 \cdots n_k)} = 0$$

and there exists $L > 0$ such that, for all $k \geq 2$,

$$h(k,\theta) - h(k-1,\theta) < L,$$

then

$$\dim_A^{\theta} E_{S,D} = \limsup_{k\to+\infty} \frac{\log(l_{h(k,\theta)} \cdots l_k)}{(1-\theta)\log(n_1 \cdots n_k)}.$$

Remark. If $\inf_{k\in\mathbb{N}} \frac{1}{n_k} > 0$, the $\dim_A^{\theta} E_{S,D}$ can also be obtained in Corollary 6.2 in Ref. 48.

Example 2.1. Let $n_k = 2^k$ for each $k \in \mathbb{N}$, $S = \{1, 2^1, 2^2, \ldots\}$ and $D = \{(0,0)\}$, then by Theorem 2.6(2), $\dim_A^{\theta} E_{S,D} \equiv 0$ for all $\theta \in (0,1)$. Furthermore, we have $\dim_{qA} E_{S,D} = 0$ and $\dim_A E_{S,D} = 2$ by Theorem 2.3. The proof is in Sec. 8.

Theorem 2.7. *For any $0 < a < b < c \leq d \leq 2$, there exists a set defined by digit restrictions $E_{S,D}$ in $\mathbb{R}^2$ such that*

$$\dim_H E_{S,D} = a, \quad \overline{\dim}_B E_{S,D} = b,$$

$$\dim_{qA} E_{S,D} = c, \quad \dim_A E_{S,D} = d.$$

Remark. Theorem 2.7 is inspired by Proposition 1.6 in Ref. 22 and Theorem 1.2 in Ref. 29. Some of the methods we used in the proof of Theorem 2.7 is to combine the proof method of Proposition 1.6 in Ref. 22 with the proof method of Theorem 1.2 in Ref. 29. Lü and Xi studied the intermediate value property of fractal dimensions of homogeneous Moran sets in $\mathbb{R}$ in Proposition 1.6 in Ref. 22. Dai *et al.* studied the intermediate value property of some fractal dimensions, without the quasi-Assouad dimension, of the union of two sets defined by digit restrictions in $\mathbb{R}$ in Theorem 1.2 in Ref. 29.

Organization. Theorem 2.1 is proved in Sec. 3. In Sec. 4, we will prove Theorem 2.2 and Corollary 2.2.1. Section 5 is devoted to the proof of Theorems 2.3 and 2.4. The proof of Theorem 2.5 is given in Sec. 6. In Sec. 7, Theorem 2.6 is proved. The last section is devoted to the proof of Example 2.1 and Theorem 2.7.

3. HAUSDORFF AND LOWER BOX DIMENSIONS

Proof of Theorem 2.1. Let

$$t_* := \liminf_{k\to+\infty} \frac{\log(l_1\cdots l_k)}{\log(n_1\cdots n_{k+1}) - \log\sqrt{l_{k+1}}}.$$

For any $t > t_*$, there is a sequence $\{k_i\}_{i\geq 1}$ with $k_i \to +\infty$ as $i\to+\infty$ such that

$$\frac{\log(l_1\cdots l_{k_i})}{\log(n_1\cdots n_{k_i+1}) - \log\sqrt{l_{k_i+1}}} \leq t,$$

then

$$l_1\cdots l_{k_i}\left(\frac{\sqrt{l_{k_i+1}}}{n_1\cdots n_{k_i+1}}\right)^t \leq 1.$$

For any $i\in\mathbb{N}$, if $l_{k_i+1} = n_{k_i+1}^2$, then

$$\frac{l_1\cdots l_{k_i}}{(n_1\cdots n_{k_i})^t} \leq 1 \leq \sharp D.$$

For any $i\in\mathbb{N}$, if $l_{k_i+1} = \sharp D$, then

$$\frac{l_1\cdots l_{k_i} l_{k_i+1}}{(n_1\cdots n_{k_i} n_{k_i+1})^t} \leq l_{k_i+1}^{1-t/2} = (\sharp D)^{1-t/2} \leq \sharp D.$$

Let

$$k_i' := \begin{cases} k_i & \text{if } l_{k_i+1} = n_{k_i+1}^2, \\ k_i+1 & \text{if } l_{k_i+1} = \sharp D \end{cases}$$

for each $i\in\mathbb{N}$. Then $k_i'\to+\infty$ as $i\to+\infty$.

For $\{k_i'\}_{i\geq 1}$, we have

$$\frac{l_1\cdots l_{k_i'}}{(n_1\cdots n_{k_i'})^t} \leq \sharp D.$$

Let $\delta_{k_i'} := \frac{1}{n_1\cdots n_{k_i'}}$ for each $i\in\mathbb{N}$. We have

$$\begin{aligned}\underline{\dim}_B E_{S,D} &\leq \liminf_{i\to+\infty} \frac{\log N_{\delta_{k_i'}}(E_{S,D})}{-\log \delta_{k_i'}} \\ &\leq \liminf_{i\to+\infty} \frac{\log(l_1\cdots l_{k_i'})}{\log(n_1\cdots n_{k_i'})} \\ &\leq \liminf_{i\to+\infty} \frac{\log(\sharp D)}{\log(n_1\cdots n_{k_i'})} + t = t.\end{aligned}$$

It follows from the arbitrariness of t that $\underline{\dim}_B E_{S,D} \leq t_*$.

Next, we use the mass distribution principle to prove $\dim_H E_{S,D} \geq t_*$. There is a probability measure μ_0 on Λ such that, for all $k\in\mathbb{N}$ and $\tau := \tau_1\cdots\tau_k \in \Lambda_k$, $\mu_0(\tau) = \frac{1}{l_1\cdots l_k}$.

Let μ be the push-forward of μ_0 under

$$\Pi|_\Lambda : \Lambda \to \mathbb{R}$$

$$\tau \mapsto \Pi(\tau).$$

If $t_* = 0$, then $\dim_H E_{S,D} \geq t_*$, so we assume that $t_* > 0$. For any $0 < t < t_*$, there is $K\in\mathbb{N}$ such that $k \geq K$ implies

$$\frac{\log(l_1\cdots l_k)}{\log(n_1\cdots n_{k+1}) - \log\sqrt{l_{k+1}}} \geq t,$$

then

$$l_1\cdots l_k\left(\frac{\sqrt{l_{k+1}}}{n_1\cdots n_{k+1}}\right)^t \geq 1.$$

Let U be a nonempty subset of $\mathbb{R}^2$ and $|U| < \frac{1}{n_1\cdots n_K}$, where $|U|$ denotes the diameter of U. Then there exist $k\in\mathbb{N}$ and $k\geq K$ such that

$$\frac{1}{n_1\cdots n_{k+1}} \leq |U| < \frac{1}{n_1\cdots n_k}.$$

For one thing, U meets no more than 4 $level-k$ sets of $E_{S,D}$, then meets no more than $4l_{k+1}$ $level-(k+1)$ sets of $E_{S,D}$. For another, the number of $level-(k+1)$ sets of $E_{S,D}$ meeting U is not more than $(\frac{3|U|}{(n_1\cdots n_{k+1})^{-1}})^2$.

So

$$\begin{aligned}\mu(U) &\leq \frac{1}{l_1\cdots l_{k+1}} \\ &\quad\times \min\left\{\left(\frac{3|U|}{(n_1\cdots n_{k+1})^{-1}}\right)^2, 4l_{k+1}\right\} \\ &\leq \frac{1}{l_1\cdots l_{k+1}}\left(\frac{3|U|}{(n_1\cdots n_{k+1})^{-1}}\right)^t (4l_{k+1})^{1-t/2}\end{aligned}$$

$$\leq \frac{36}{l_1 \cdots l_{k+1}} \left(\frac{|U|\sqrt{l_{k+1}}}{\sqrt{l_{k+1}}/(n_1 \cdots n_{k+1})} \right)^t l_{k+1}^{1-t/2}$$
$$\leq \frac{36|U|^t}{l_1 \cdots l_k(\sqrt{l_{k+1}}/(n_1 \cdots n_{k+1}))^t}$$
$$\leq 36|U|^t.$$

By the mass distribution principle and the arbitrariness of t, we have

$$\dim_H E_{S,D} \geq t_*.$$

Finally, it follows from the well-known fact that $\dim_H F \leq \underline{\dim}_B F$ for bounded $F \subset \mathbb{R}^2$ that $\dim_H E_{S,D} = \underline{\dim}_B E_{S,D} = t_*$. □

4. PACKING AND UPPER BOX DIMENSIONS

Lemma 4.1 (See Proposition 2.4 on p. 33 in Ref. 41). *Let $E \subset \mathbb{R}^2$ be a bounded set. Then*

$$\overline{\dim}_B E = 2 + \limsup_{\delta \to 0} \frac{\log \mathcal{L}^2(E_\delta)}{-\log \delta},$$

where for any $\delta > 0$, $E_\delta := \{x \in \mathbb{R}^2 :$ there exists some $y \in E$ such that $\|x - y\| \leq \delta\}$ and $\mathcal{L}^2$ denotes the 2-dimensional Lebesgue measure in $\mathbb{R}^2$.

Lemma 4.2 (See Corollary 3.10 on p. 57 in Ref. 41). *Let $E \subset \mathbb{R}^2$ be compact and such that*

$$\overline{\dim}_B(E \cap V) = \overline{\dim}_B E$$

for any open set $V \subset \mathbb{R}^2$ with $V \cap E \neq \emptyset$. Then $\dim_P E = \overline{\dim}_B E$.

Proof of Theorem 2.2. If $\mathbb{N}\backslash S$ is empty or finite, then $E_{S,D}$ contains interior points, thus

$$\overline{\dim}_B E_{S,D} = 2 = \limsup_{k \to +\infty} \frac{\log(l_1 \cdots l_k)}{\log(n_1 \cdots n_k)}.$$

If $\mathbb{N}\backslash S$ is infinite, there exists δ_0, $K > 0$ such that, for any $0 < \delta < \delta_0$,

$$\log \mathcal{L}^2((E_{S,D})_\delta) < 0$$

and for any $k \geq K$,

$$\frac{9l_1 \cdots l_k}{(n_1 \cdots n_k)^2} < 1.$$

For any $0 < \delta < \min\{\frac{1}{n_1 \cdots n_K}, \delta_0\}$, there exists $k \geq K$ such that

$$\frac{1}{n_1 \cdots n_{k+1}} \leq \delta < \frac{1}{n_1 \cdots n_k},$$

then

$$\mathcal{L}^2((E_{S,D})_\delta) \leq l_1 \cdots l_k \left(\frac{3n_{k+1}}{n_1 \cdots n_k n_{k+1}} \right)^2 < 1.$$

By Lemma 4.1, we have

$$\begin{aligned}
&\overline{\dim}_B E_{S,D} \\
&\quad = 2 + \limsup_{\delta \to 0} \frac{\log \mathcal{L}^2((E_{S,D})_\delta)}{-\log \delta} \\
&\quad \leq 2 + \limsup_{k \to +\infty} \\
&\qquad \times \frac{\log(9l_1 \cdots l_k n_{k+1}^2) - \log(n_1 \cdots n_k n_{k+1})^2}{\log(n_1 \cdots n_k n_{k+1})} \\
&\quad = \limsup_{k \to +\infty} \frac{\log(l_1 \cdots l_k n_{k+1}^2)}{\log(n_1 \cdots n_k n_{k+1})} \\
&\quad = \limsup_{k \to +\infty} \frac{\log(l_1 \cdots l_k l_{k+1}) + \log(n_{k+1}^2/l_{k+1})}{\log(n_1 \cdots n_k n_{k+1})} \\
&\quad = \limsup_{k \to +\infty} \frac{\log(l_1 \cdots l_k)}{\log(n_1 \cdots n_k)}.
\end{aligned}$$

If

$$s^* := \limsup_{k \to +\infty} \frac{\log(l_1 \cdots l_k)}{\log(n_1 \cdots n_k)} = 0,$$

then it follows from

$$0 \leq \overline{\dim}_B E_{S,D} \leq s^*$$

that

$$\overline{\dim}_B E_{S,D} = 0 = s^*.$$

For any $0 < s < s^*$, there is a sequence $\{k_i\}_{i \geq 1}$ with $k_i \to +\infty$ as $i \to +\infty$ such that

$$\frac{\log(l_1 \cdots l_{k_i})}{\log(n_1 \cdots n_{k_i})} \geq s.$$

Let $\delta_{k_i} = \frac{1}{n_1 \cdots n_{k_i}}$ for each $i \in \mathbb{N}$. We have

$$\begin{aligned}
\overline{\dim}_B E_{S,D} &= \limsup_{\delta \to 0} \frac{\log N_\delta(E_{S,D})}{-\log \delta} \\
&\geq \limsup_{i \to +\infty} \frac{\log((l_1 \cdots l_{k_i})/16)}{\log(n_1 \cdots n_{k_i})} \\
&= \limsup_{i \to +\infty} \frac{\log(l_1 \cdots l_{k_i})}{\log(n_1 \cdots n_{k_i})} \geq s.
\end{aligned}$$

It follows from the arbitrariness of s that $\overline{\dim}_B E_{S,D} \geq s^*$.

For any open set $V \subset \mathbb{R}^2$ with $V \cap E_{S,D} \neq \emptyset$, there exist $\eta \in \Lambda$ and large enough $k \in \mathbb{N}$ such that $f_{\eta|_k}(I) \subset V$. For any $\omega \in \Lambda_k$,

$$\begin{aligned} f_\omega(I) \cap E_{S,D} &= \left\{ \sum_{i=k+1}^{+\infty} \frac{a_i}{n_1 n_2 \cdots n_k n_{k+1} \cdots n_i} : \right. \\ &\qquad a_i \in \{0, 1, \dots n_i - 1\}^2 \text{ if } i \in S \text{ and} \\ &\qquad \left. a_i \in D \text{ if } i \notin S \text{ for each } i \geq k+1 \right\} \\ &\qquad + f_\omega(0,0), \end{aligned}$$

where $A + b := \{x + b : x \in A\}$ for each $A \subset \mathbb{R}^2$ and $b \in \mathbb{R}^2$.

Then for any $\sigma,\ \tau \in \Lambda_k$,

$$\overline{\dim}_B(f_\sigma(I) \cap E_{S,D}) = \overline{\dim}_B(f_\tau(I) \cap E_{S,D}).$$

Since

$$E_{S,D} = \bigcup_{\omega \in \Lambda_k} (f_\omega(I) \cap E_{S,D}),$$

we have

$$\begin{aligned} \overline{\dim}_B E_{S,D} &= \max_{\omega \in \Lambda_k} \overline{\dim}_B(f_\omega(I) \cap E_{S,D}) \\ &= \overline{\dim}_B(f_{\eta|_k}(I) \cap E_{S,D}) \\ &\leq \overline{\dim}_B(E_{S,D} \cap V) \leq \overline{\dim}_B E_{S,D}. \end{aligned}$$

It follows from Lemma 4.2 that

$$\dim_P E_{S,D} = \overline{\dim}_B E_{S,D}. \qquad \square$$

Proof of Corollary 2.2.1. Let $M := \sup\{n_k : k \in \mathbb{N} \backslash S\}$. Then $0 \leq M < +\infty$.

Since

$$\frac{\log(n_{k+1}^2 / l_{k+1})}{\log(n_1 \cdots n_k n_{k+1})} \leq \frac{\log \max\{M^2/\sharp D,\ 1\}}{\log(n_1 \cdots n_k n_{k+1})}$$

for each $k \in \mathbb{N}$ and

$$\lim_{k \to +\infty} \frac{\log \max\{M^2/\sharp D,\ 1\}}{\log(n_1 \cdots n_k n_{k+1})} = 0,$$

we have

$$\lim_{k \to +\infty} \frac{\log(n_{k+1}^2/l_{k+1})}{\log(n_1 \cdots n_k n_{k+1})} = 0.$$

It follows from Theorem 2.2 that the corollary holds. $\square$

5. ASSOUAD DIMENSION AND ARITHMETIC PATCHES

Lemma 5.1 (See Theorem 2.4 in Ref. 47). *If $E \subset \mathbb{R}^d$ is bounded and contains arbitrarily large arithmetic patches in $\mathbb{R}^d$, then $\dim_A E = d$.*

Proof of Theorem 2.3. If $\sup\{n_k : k \in S\} = +\infty$, for any $n \in \mathbb{N}$, there is $k_n \in S$ such that $n_{k_n} \geq n$. Take any $\tau \in \Lambda_{k_n - 1}$ and $p \in D$. Let $x_i = \Pi(\tau i p^\infty)$ for each $i \in \{0, 1, \dots, n_{k_n} - 1\}^2$. Then $\{x_i : i \in \{0, 1, \dots, n_{k_n} - 1\}^2\}$ is an arithmetic patch of size n_{k_n} in $\mathbb{R}^2$. Thus, $E_{S,D}$ contains arbitrarily large arithmetic patches in $\mathbb{R}^2$, then it follows from Lemma 5.1 that $\dim_A E_{S,D} = 2$.

Next, consider the case $\sup\{n_k : k \in S\} < +\infty$. Let

$$\bar{s} := \lim_{j \to +\infty} \sup_{k \geq j,\ m \geq j} \frac{\log(l_{k+1} \cdots l_{k+m})}{\log(n_{k+1} \cdots n_{k+m})}.$$

For any $s > \bar{s}$, there is an integer $J > 0$ such that, for any $j \geq J$, $k \geq j$ and $m \geq j$, we have

$$\frac{\log(l_{k+1} \cdots l_{k+m})}{\log(n_{k+1} \cdots n_{k+m})} \leq s,$$

then

$$l_{k+1} \cdots l_{k+m} \leq (n_{k+1} \cdots n_{k+m})^s.$$

Fix any $0 < \rho < \frac{1}{n_1 \cdots n_J}$. For any $0 < r < R < \rho$, there exist $k \geq J,\ m \geq 0$ such that

$$\frac{1}{n_1 \cdots n_{k+1}} \leq R < \frac{1}{n_1 \cdots n_k} \quad \text{and}$$

$$\frac{1}{n_1 \cdots n_{k+1} \cdots n_{k+m+1}} \leq r < \frac{1}{n_1 \cdots n_k \cdots n_{k+m}}.$$

If $0 < r < \frac{1}{n_1 \cdots n_{k+J+1}} < R < \rho$, then $m \geq J + 1$. Let $A := \max\{\sharp D,\ \sup\{n_k^2 : k \in S\}\}$. For any $x \in E_{S,D}$, we have

$$\begin{aligned} N_r(B(x,R) \cap E_{S,D}) &\leq 9 l_{k+1} \cdots l_{k+m+1} \leq 9A^2 l_{k+2} \cdots l_{k+m} \\ &\leq 9A^2 (n_{k+2} \cdots n_{k+m})^s \leq 9A^2 \left(\frac{R}{r}\right)^s. \end{aligned}$$

If $\frac{1}{n_1 \cdots n_{k+J+1}} \leq r < R < \rho$, for any $x \in E_{S,D}$, we have

$$\begin{aligned} N_r(B(x,R) \cap E_{S,D}) &\leq 9 l_{k+1} \cdots l_{k+J+1} \\ &\leq 9A^{J+1} \leq 9A^{J+1} \left(\frac{R}{r}\right)^s. \end{aligned}$$

Let $C := 9A^{J+1}$. Then for any $s > \bar{s}$, there are $C > 0,\ \rho > 0$ such that, for all $0 < r < R < \rho$ and

$x \in E_{S,D}$, we have

$$N_r(B(x,R) \cap E_{S,D}) \le C\left(\frac{R}{r}\right)^s.$$

It follows from the arbitrariness of s that $\dim_A E_{S,D} \le \bar{s}$.

If $\bar{s} = 0$, it follows that $0 \le \dim_A E_{S,D} \le \bar{s} = 0$, then

$$\dim_A E_{S,D} = 0 = \bar{s}.$$

If $\bar{s} > 0$, for any $0 < s < \bar{s}$, there exists $\{(k_i,\ m_i)\}_{i=1}^{+\infty}$ with $k_i \to +\infty, m_i \to +\infty$ as $i \to +\infty$ such that

$$\frac{\log(l_{k_i+1}\cdots l_{k_i+m_i})}{\log(n_{k_i+1}\cdots n_{k_i+m_i})} \ge s,$$

then

$$l_{k_i+1}\cdots l_{k_i+m_i} \ge (n_{k_i+1}\cdots n_{k_i+m_i})^s.$$

For any $0 < \varepsilon < s$, $C > 0$ and $\rho > 0$, take i large enough such that $\frac{\sqrt{2}}{n_1\cdots n_{k_i}} < \rho$ and $2^{m_i\varepsilon} > 32C$. Then take $R_i = \frac{\sqrt{2}}{n_1\cdots n_{k_i}}$, $r_i = \frac{1}{n_1\cdots n_{k_i+m_i}}$ and any $x_i \in E_{S,D}$. Then

$$0 < r_i < R_i < \rho \quad \text{and}$$

$$\left(\frac{R_i}{r_i}\right)^\varepsilon = (\sqrt{2}n_{k_i+1}\cdots n_{k_i+m_i})^\varepsilon \ge 2^{m_i\varepsilon} > 32C,$$

then we have

$$\begin{aligned} &N_{r_i}(B(x_i,R_i) \cap E_{S,D}) \\ &\quad\ge \frac{1}{16} l_{k_i+1}\cdots l_{k_i+m_i} \ge \frac{1}{16\sqrt{2}^s}\left(\frac{R_i}{r_i}\right)^s \\ &\quad\ge \frac{1}{32}\left(\frac{R_i}{r_i}\right)^\varepsilon\left(\frac{R_i}{r_i}\right)^{s-\varepsilon} > C\left(\frac{R_i}{r_i}\right)^{s-\varepsilon}. \end{aligned}$$

Then $\dim_A E_{S,D} \ge s - \varepsilon$. By the arbitrariness of ε and s, we have $\dim_A E_{S,D} \ge \bar{s}$. □

Proof of Theorem 2.4. It follows from Lemma 5.1 that the sufficiency is true.

Next, we prove the necessity. Let $M := \sup\{n_k : k \in S\}$ and $N := \sup\{i \in \mathbb{N} : l_k = n_k^2, l_{k+1} = n_{k+1}^2, \ldots, l_{k+i-1} = n_{k+i-1}^2, l_{k+i} = \sharp D$ for some $k \in \mathbb{N}\}$. If $S = \emptyset$, we adopt the convenience $M := 0$ and $N := 0$.

If $M = +\infty$, it follows from the proof of Theorem 2.3 that $E_{S,D}$ contains arbitrarily large arithmetic patches in $\mathbb{R}^2$.

If $M < +\infty$ and $N = +\infty$, for any $n \in \mathbb{N}$, there exists $k,\ i \in \mathbb{N}$ such that $l_k = n_k^2, l_{k+1} = n_{k+1}^2, \ldots, l_{k+i-1} = n_{k+i-1}^2$ and $n_k n_{k+1}\cdots n_{k+i-1} \ge n$. Take any $\tau \in \Lambda_{k-1}$ and $p \in D$. Then $\{\Pi(\tau\sigma_0\cdots\sigma_{i-1}p^\infty) : \sigma_0 \in \{0,1,\ldots,n_k - 1\}^2, \ldots, \sigma_{i-1} \in \{0,1,\ldots,n_{k+i-1} - 1\}^2\}$ is an arithmetic patch of size $n_k n_{k+1}\cdots n_{k+i-1}$ in $\mathbb{R}^2$, thus $E_{S,D}$ contains arbitrarily large arithmetic patches in $\mathbb{R}^2$.

If $M < +\infty$ and $N < +\infty$, we will prove this case does not happen. We discuss it in the following two cases.

(i) $S = \emptyset$, i.e. $M = N = 0$.

$$\begin{aligned} &\dim_A E_{S,D} \\ &\quad= \lim_{j\to+\infty} \sup_{k\ge j,\ m\ge j} \frac{2\log(l_{k+1}\cdots l_{k+m})}{\log(n_{k+1}\cdots n_{k+m})^2} \\ &\quad\le \frac{2\log(\sharp D)}{\log(\sharp D + 1)} < 2. \end{aligned}$$

(ii) $S \ne \emptyset$, i.e. $0 < M < +\infty$ and $0 < N < +\infty$.

For any $k > N+1$ and $m > N+1$, let$X_{k,m} := \{i : k+1 \le i \le k+m$ and $i \in S\}$ and $Y_{k,m} := \{k+1,\ldots,k+m\}\backslash S$. We have

$$\begin{aligned} &\frac{2\log(l_{k+1}\cdots l_{k+m})}{\log(n_{k+1}\cdots n_{k+m})^2} \\ &\quad\le \frac{2(\log\prod_{i\in X_{k,m}} l_i + \sharp Y_{k,m}\log\sharp D)}{\log\prod_{i\in X_{k,m}} l_i + \sharp Y_{k,m}\log(\sharp D+1)} \\ &\quad\le 2\frac{\log\prod_{i\in X_{k,m}} l_i + \sharp Y_{k,m}\log(\sharp D+1)}{\log\prod_{i\in X_{k,m}} l_i + \sharp Y_{k,m}\log(\sharp D+1)} \\ &\qquad - 2\frac{\sharp Y_{k,m}\log(\sharp D+1) - \sharp Y_{k,m}\log(\sharp D)}{\log\prod_{i\in X_{k,m}} l_i + \sharp Y_{k,m}\log(\sharp D+1)} \\ &\quad= 2 - 2\frac{1 - \frac{\log(\sharp D)}{\log(\sharp D+1)}}{\frac{\log\prod_{i\in X_{k,m}} l_i}{\sharp Y_{k,m}\log(\sharp D+1)} + 1} \end{aligned}$$

and

$$\begin{aligned} \frac{\log\prod_{i\in X_{k,m}} l_i}{\sharp Y_{k,m}\log(\sharp D+1)} &\le \frac{(m - \lfloor\frac{m}{N+1}\rfloor)\log M^2}{\lfloor\frac{m}{N+1}\rfloor\log(\sharp D+1)} \\ &\le \frac{m - \frac{m}{N+1} + 1}{\frac{m}{N+1} - 1}\cdot\frac{\log M^2}{\log(\sharp D+1)} \\ &= \frac{1 - \frac{1}{N+1} + \frac{1}{m}}{\frac{1}{N+1} - \frac{1}{m}}\cdot\frac{\log M^2}{\log(\sharp D+1)}. \end{aligned}$$

Since

$$\lim_{m\to+\infty} \frac{1 - \frac{1}{N+1} + \frac{1}{m}}{\frac{1}{N+1} - \frac{1}{m}}\cdot\frac{\log M^2}{\log(\sharp D+1)} = \frac{N\log M^2}{\log(\sharp D+1)},$$

there exists $J > N+1$ such that, for any $k \geq J$ and $m \geq J$, we have

$$\frac{\log(l_{k+1}\cdots l_{k+m})}{\log(n_{k+1}\cdots n_{k+m})} \leq 2 - 2\frac{1-\frac{\log(\sharp D)}{\log(\sharp D+1)}}{\frac{4N\log M}{\log(\sharp D+1)}+1} < 2.$$

It follows that

$$\dim_A E_{S,D} = \lim_{j\to+\infty}\sup_{k\geq j,\ m\geq j}\frac{\log(l_{k+1}\cdots l_{k+m})}{\log(n_{k+1}\cdots n_{k+m})} \leq 2 - 2\frac{1-\frac{\log(\sharp D)}{\log(\sharp D+1)}}{\frac{4N\log M}{\log(\sharp D+1)}+1} < 2. \qquad \square$$

6. UPPER SPECTRUM AND QUASI-ASSOUAD DIMENSION

Lemma 6.1 (See p. 82 in Ref. 8). *In $\mathbb{R}^d$, there exists $C_1 > 0$, only depending on d, such that, for all $x \in \mathbb{R}^d$ and $R > 0$, we have*

$$N_{\frac{R}{2}}(B(x,R)) \leq C_1.$$

Lemma 6.2. *For any $\theta \in (0,1)$ and nonempty $F \subset \mathbb{R}^d$,*

$$\overline{\dim}_{\tilde{A}}^{\theta} F := \inf\Big\{s \geq 0 : \text{there exist constants } C > 0 \text{ and } 0<\rho<1 \text{ such that, for all } 0 < r \leq R^{1/\theta} < R < \rho < 1 \text{ and } x \in F,\ N_r(B(x,R)\cap F) \leq C\left(\frac{R}{r}\right)^s\Big\},$$

then

$$\overline{\dim}_{\tilde{A}}^{\theta} F = \overline{\dim}_{A}^{\theta} F.$$

Proof of Lemma 6.2. It is obvious that

$$\overline{\dim}_{\tilde{A}}^{\theta} F \leq \overline{\dim}_{A}^{\theta} F.$$

For any $s > \overline{\dim}_{\tilde{A}}^{\theta} F$, there exist constants $C > 0$ and $0 < \rho < 1$ such that, for any $0 < r \leq R^{1/\theta} < R < \rho < 1$ and $x \in F$,

$$N_r(B(x,R)\cap F) \leq C\left(\frac{R}{r}\right)^s.$$

For $0 < \rho < 1$ above, then there exists $k \in \mathbb{N}$ such that

$$\frac{1}{2^k} < \rho \leq \frac{1}{2^{k-1}}.$$

When $R < \rho < 1$, for any $0 < r \leq R^{1/\theta} < R < \rho < 1$ and $x \in F$, we have

$$N_r(B(x,R)\cap F) \leq C\left(\frac{R}{r}\right)^s.$$

When $\rho \leq R < 1$, for any $0 < r \leq R^{1/\theta} < R < 1$ and $x \in F$, we consider the following two cases.

(i) If $r \leq \frac{1}{2^{k/\theta}}$, by Lemma 6.1, there exist at most C_1^{k+1} closed balls with radius $\frac{1}{2^{k+1}}$ covering $B(x,1)$ for any $x \in F$. Let $B(x_1, \frac{1}{2^{k+1}}), \ldots, B(x_t, \frac{1}{2^{k+1}})$, where $1 \leq t \leq C_1^{k+1}$, be all balls from the above cover meeting F. Then for any $1 \leq i \leq t$, there exists $y_i \in F \cap B(x_i, \frac{1}{2^{k+1}})$ such that $B(x_i, \frac{1}{2^{k+1}}) \subset B(y_i, \frac{1}{2^k})$. Then there exists $1 \leq i_0 \leq t$ such that

$$N_r\left(B\left(y_{i_0}, \frac{1}{2^k}\right)\cap F\right) = \max_{1\leq i\leq t} N_r\left(B\left(y_i, \frac{1}{2^k}\right)\cap F\right).$$

Then

$$\begin{aligned} N_r(B(x,R)\cap F) &\leq N_r(B(x,1)\cap F) \\ &\leq tN_r\left(B\left(y_{i_0}, \frac{1}{2^k}\right)\cap F\right)\Big) \\ &\leq C_1^{k+1}C\left(\frac{1/2^k}{r}\right)^s \leq C_1^{k+1}C\left(\frac{R}{r}\right)^s. \end{aligned}$$

(ii) If $r > \frac{1}{2^{k/\theta}} \geq \frac{1}{2^{k_1}}$, where $k_1 = \lfloor \frac{k}{\theta} \rfloor + 1$, by Lemma 6.1, we have

$$\begin{aligned} N_r(B(x,R)\cap F) &\leq N_{\frac{1}{2^{k/\theta}}}(B(x,1)\cap F) \\ &\leq N_{1/2^{k_1}}(B(x,1)\cap F) \\ &\leq N_{1/2^{k_1}}(B(x,1)) \leq C_1^{k_1} \\ &\leq C_1^{k_1}\left(\frac{R}{r}\right)^s. \end{aligned}$$

It follows from the arbitrariness of s that

$$\overline{\dim}_{A}^{\theta} F \leq \overline{\dim}_{\tilde{A}}^{\theta} F. \qquad \square$$

Proof of Theorem 2.5. (1) For any $0 < \varepsilon \leq 2$ and $C > 0$, there exists $k \in S$ with $k \geq 2$ such that

$$\frac{(\sqrt{2}n_k)^\varepsilon}{32} > C.$$

Let $x_k \in E_{S,D}$, $r_k = \frac{1}{n_1\cdots n_k}$ and $R_k = \frac{\sqrt{2}}{n_1\cdots n_{k-1}}$. Then

$$0 < r_k \le R_k^{1/\theta} < 1.$$

We have

$$\begin{aligned} &N_{r_k}(B(x_k, R_k)\cap E_{S,D}) \\ &\quad\ge \frac{1}{16}n_k^2 = \frac{1}{32}\left(\frac{R_k}{r_k}\right)^2 \\ &\quad= \frac{(\sqrt{2}n_k)^\varepsilon}{32}\left(\frac{R_k}{r_k}\right)^{2-\varepsilon} > C\left(\frac{R_k}{r_k}\right)^{2-\varepsilon}. \end{aligned}$$

It follows from the arbitrariness of ε that

$$\overline{\dim}_A^\theta E_{S,D} = 2.$$

(2) For any

$$s > \limsup_{k\to+\infty} \max_{1\le h\le h(k,\theta)} \frac{\log(l_h\cdots l_k)}{\log(n_h\cdots n_k)},$$

there exists $K_1 > 0$ such that, for any $k \ge K_1$, we have

$$\max_{1\le h\le h(k,\theta)} \frac{\log(l_h\cdots l_k)}{\log(n_h\cdots n_k)} < s,$$

furthermore, since

$$\lim_{k\to+\infty}\frac{\log n_k}{\log(n_1\cdots n_k)} = 0,$$

there exists $K_2 > 0$ such that for any $h \ge K_2$, we have

$$\frac{2\log n_h}{(1/\theta - 1)\log(n_1\cdots n_{h-1})} < s \quad \text{and}$$

$$\frac{2\log(n_h n_{h+1})}{(1-\theta)\log(n_1\cdots n_h)} < s.$$

Since

$$\lim_{k\to+\infty}\frac{\log n_k}{\log(n_1\cdots n_k)} = 0,$$

then for any $\varepsilon > 0$, there exists $K_3 > 0$ such that for any $k \ge K_3$ and $1 \le h < k$, we have

$$\frac{\log n_h + \log n_k}{\log(n_1\cdots n_k)} < (1-\theta)\varepsilon.$$

Let $K := \max\{K_1, K_2, K_3\}$. For any $0 < r < r^\theta \le R < \frac{1}{n_1\cdots n_K} < 1$ and $x \in E_{S,D}$, there exist $h, k > K$ such that

$$\frac{1}{n_1\cdots n_h} \le R < \frac{1}{n_1\cdots n_{h-1}} \quad \text{and}$$

$$\frac{1}{n_1\cdots n_k} \le r < \frac{1}{n_1\cdots n_{k-1}},$$

then

$$\frac{1}{(n_1\cdots n_k)^\theta} \le r^\theta \le R < \frac{1}{n_1\cdots n_{h-1}},$$

so

$$\frac{\log(n_h\cdots n_k)}{\log(n_1\cdots n_k)} > 1-\theta \quad \text{and} \quad 1 \le h \le h(k,\theta),$$

it follows that

$$\frac{\log(l_h\cdots l_k)}{\log(n_h\cdots n_k)} < s.$$

If $k > h+1$, then

$$\begin{aligned} \frac{\log(n_h n_k)}{\log(n_h\cdots n_k)} &= \frac{\log n_h + \log n_k}{\log(n_1\cdots n_k)}\cdot\frac{\log(n_1\cdots n_k)}{\log(n_h\cdots n_k)} \\ &< (1-\theta)\varepsilon\cdot\frac{1}{1-\theta} = \varepsilon \end{aligned}$$

and

$$\begin{aligned} &\frac{\log(l_h\cdots l_k)}{\log(n_h\cdots n_k) - \log(n_h n_k)} \\ &\quad\le \frac{\log(l_h\cdots l_k)}{\log(n_h\cdots n_k)(1 - \log(n_h n_k)/\log(n_h\cdots n_k))} \\ &\quad< \frac{1}{1-\varepsilon}\cdot\frac{\log(l_h\cdots l_k)}{\log(n_h\cdots n_k)} < \frac{s}{1-\varepsilon}. \end{aligned}$$

Thus,

$$\begin{aligned} N_r(B(x,R)\cap E_{S,D}) &\le 9 l_h\cdots l_k = 9\left(\frac{R}{r}\right)^{\frac{\log(l_h\cdots l_k)}{\log(R/r)}} \\ &\le 9\left(\frac{R}{r}\right)^{\frac{\log(l_h\cdots l_k)}{\log(n_h\cdots n_k)-\log(n_h n_k)}} \\ &\le 9\left(\frac{R}{r}\right)^{s/(1-\varepsilon)}. \end{aligned}$$

If $k = h$, then

$$\frac{1}{n_1\cdots n_h} \le r < r^\theta \le R < \frac{1}{n_1\cdots n_{h-1}},$$

so

$$\frac{R}{r} \ge \frac{R}{R^{1/\theta}} = R^{1-1/\theta} \ge (n_1\cdots n_{h-1})^{1/\theta - 1}$$

and

$$\frac{\log l_h}{\log(R/r)} \le \frac{2\log n_h}{(1/\theta - 1)\log(n_1\cdots n_{h-1})} < s.$$

Thus,

$$N_r(B(x,R)\cap E_{S,D}) \le 9l_h = 9\left(\frac{R}{r}\right)^{\frac{\log l_h}{\log(R/r)}} < 9\left(\frac{R}{r}\right)^s.$$

If $k=h+1$, then

$$\frac{1}{n_1\cdots n_{h+1}} \le r < \frac{1}{n_1\cdots n_h} \le R < \frac{1}{n_1\cdots n_{h-1}},$$

so

$$\frac{R}{r} \ge r^{\theta-1} \ge (n_1\cdots n_h)^{1-\theta} \quad \text{and}$$

$$\frac{\log(l_h l_{h+1})}{\log(R/r)} \le \frac{2\log(n_h n_{h+1})}{(1-\theta)\log(n_1\cdots n_h)} < s.$$

Thus,

$$N_r(B(x,R)\cap E_{S,D}) \le 9l_h l_{h+1} = 9\left(\frac{R}{r}\right)^{\frac{\log(l_h l_{h+1})}{\log(R/r)}} < 9\left(\frac{R}{r}\right)^s.$$

It follows from Lemma 6.2 and the arbitrariness of ε and s that

$$\overline{\dim}_A^{\theta} E_{S,D} = \overline{\dim}_{\tilde{A}}^{\theta} E_{S,D} \le \limsup_{k\to+\infty} \max_{1\le h\le h(k,\theta)} \frac{\log(l_h\cdots l_k)}{\log(n_h\cdots n_k)}.$$

For any $s > \overline{\dim}_A^{\theta} E_{S,D}$, there exists $C>0$ such that, for any $0<r<r^\theta\le R<1$ and $x\in E_{S,D}$, we have

$$N_r(B(x,R)\cap E_{S,D}) \le C\left(\frac{R}{r}\right)^s.$$

For $k\ge 2$, let $r=\frac{1}{n_1\cdots n_k}$ and $x\in E_{S,D}$, $R=\frac{\sqrt{2}}{n_1\cdots n_{h-1}}$ with $1\le h\le h(k,\theta)$. Since every closed ball of radius r meets no more than 16 *level* $-\, k$ sets of $E_{S,D}$, we have

$$\frac{1}{16} l_h\cdots l_k \le N_r(B(x,R)\cap E_{S,D}) \le C\left(\frac{R}{r}\right)^s = C\sqrt{2}^s (n_h\cdots n_k)^s,$$

so

$$\frac{\log(l_h\cdots l_k)}{\log(n_h\cdots n_k)} \le \frac{\log(16\sqrt{2}^s C)}{\log(n_h\cdots n_k)} + s \le \frac{\log(16\sqrt{2}^s C)}{\log(n_{h(k,\theta)}\cdots n_k)} + s.$$

There exists an integer $K_0>0$ such that, for any $k\ge K_0$, we have

$$\frac{1}{(n_1\cdots n_k)^\theta} < \frac{1}{n_1}.$$

It follows from the definition of $h(k,\theta)$ that, for any $k\ge K_0$, $h(k,\theta)$ is the unique positive integer such that

$$\frac{1}{n_1\cdots n_{h(k,\theta)}} \le \frac{1}{(n_1\cdots n_k)^\theta} < \frac{1}{n_1\cdots n_{h(k,\theta)-1}}.$$

Then we have

$$\lim_{k\to+\infty} h(k,\theta) = +\infty \quad \text{and}$$

$$(n_1\cdots n_{h(k,\theta)-1})^{1-\theta} < (n_{h(k,\theta)}\cdots n_k)^\theta$$

for each $k\ge K_0$, so

$$\lim_{k\to+\infty} n_{h(k,\theta)}\cdots n_k = +\infty.$$

It follows from the arbitrariness of s that

$$\overline{\dim}_A^{\theta} E_{S,D} \ge \limsup_{k\to+\infty} \max_{1\le h\le h(k,\theta)} \frac{\log(l_h\cdots l_k)}{\log(n_h\cdots n_k)}.$$

(3) Let $A := \sup_{k\ge1} n_k$. For any $1\le h\le k$, we have

$$\frac{(k-h+1)\log 2}{k\log A} \le \frac{\log(n_h\cdots n_k)}{\log(n_1\cdots n_k)} \le \frac{(k-h+1)\log A}{k\log 2}.$$

For any $0<\theta<1$ and $1\le h\le k$, if

$$\frac{(k-h+1)\log A}{k\log 2} \le 1-\theta,$$

we have

$$k+1-\frac{(1-\theta)k\log 2}{\log A} \le h,$$

then

$$h(k,\theta) < k+1-\frac{(1-\theta)k\log 2}{\log A}.$$

For any $0<\theta<1$ and $1\le h\le k$, if

$$\frac{(k-h+1)\log 2}{k\log A} > 1-\theta,$$

we have

$$k+1-\frac{(1-\theta)k\log A}{\log 2} > h,$$

then

$$h(k,\theta) \ge k-\frac{(1-\theta)k\log A}{\log 2}.$$

Thus, for any $0 < \theta < 1$ and $1 \le h \le k$, we have

$$k\left(1 - \frac{(1-\theta)\log A}{\log 2}\right) \le h(k,\theta) < k\left(1 - \frac{(1-\theta)\log 2}{\log A} + \frac{1}{k}\right).$$

For any $0 < \theta < 1$, there exists $K > 0$ such that, for any $k \ge K$, we have

$$0 < 1 - \frac{(1-\theta)\log 2}{\log A} + \frac{1}{k} < 1,$$

then there exists $0 < \alpha_0 < 1$ such that, for any $\alpha_0 < \alpha < 1$ and $k \ge K$, we have

$$h(k,\theta) < k\left(1 - \frac{(1-\theta)\log 2}{\log A} + \frac{1}{k}\right) < \alpha_0 k < \alpha k,$$

then

$$\limsup_{k\to+\infty} \max_{1\le h\le h(k,\theta)} \frac{\log(l_h\cdots l_k)}{\log(n_h\cdots n_k)} \le \lim_{\alpha\to 1}\limsup_{k\to+\infty} \max_{1\le h\le \alpha k} \frac{\log(l_h\cdots l_k)}{\log(n_h\cdots n_k)},$$

so

$$\lim_{\theta\to 1}\limsup_{k\to+\infty} \max_{1\le h\le h(k,\theta)} \frac{\log(l_h\cdots l_k)}{\log(n_h\cdots n_k)} \le \lim_{\alpha\to 1}\limsup_{k\to+\infty} \max_{1\le h\le \alpha k} \frac{\log(l_h\cdots l_k)}{\log(n_h\cdots n_k)}.$$

For any $0 < \alpha < 1$, there exists $K = \frac{1}{\alpha} > 0$ such that, for any $k \ge K$, we have $\alpha k \ge 1$, then there exists $0 < \theta_0 < 1$ such that, for any $\theta_0 < \theta < 1$ and $k \ge K$, we have

$$\alpha k < k\left(1 - \frac{(1-\theta)\log A}{\log 2}\right) \le h(k,\theta),$$

then

$$\limsup_{k\to+\infty} \max_{1\le h\le \alpha k} \frac{\log(l_h\cdots l_k)}{\log(n_h\cdots n_k)} \le \lim_{\theta\to 1}\limsup_{k\to+\infty} \max_{1\le h\le h(k,\theta)} \frac{\log(l_h\cdots l_k)}{\log(n_h\cdots n_k)},$$

so

$$\lim_{\alpha\to 1}\limsup_{k\to+\infty} \max_{1\le h\le \alpha k} \frac{\log(l_h\cdots l_k)}{\log(n_h\cdots n_k)} \le \lim_{\theta\to 1}\limsup_{k\to+\infty} \max_{1\le h\le h(k,\theta)} \frac{\log(l_h\cdots l_k)}{\log(n_h\cdots n_k)}. \qquad \square$$

7. ASSOUAD SPECTRUM

Lemma 7.1 (See Proposition 3.1 and Corollary 3.2 in Ref. 23). *For any nonempty bounded* $F \subset \mathbb{R}^d$,

$$\overline{\dim}_B F \le \dim_A^\theta F \le \min\left\{\frac{\overline{\dim}_B F}{1-\theta}, \dim_A F\right\}$$

and

$$\lim_{\theta\to 0}\dim_A^\theta F = \overline{\dim}_B F.$$

Lemma 7.2 (See Theorem 2.1 in Ref. 24). *For any* $\theta \in (0,1)$ *and nonempty* $F \subset \mathbb{R}^d$,

$$\overline{\dim}_A^\theta F = \sup_{0<\theta'\le\theta} \dim_A^{\theta'} F.$$

Lemma 7.3 (See Corollary 3.5 in Ref. 23). *For any nonempty* $F \subset \mathbb{R}^d$, *the function* $\theta \mapsto \dim_A^\theta F$ *is continuous in* $\theta \in (0,1)$.

Lemma 7.4 (See Corollary 3.6 in Ref. 23). *For any nonempty* $F \subset \mathbb{R}^d$, *if for some* $\theta \in (0,1)$, *we have* $\dim_A^\theta F = \dim_A F$, *then*

$$\dim_A^{\theta'} F = \dim_A F$$

for all $\theta' \in [\theta, 1)$.

Lemma 7.5 (Ref. 23). *For any* $\theta \in (0,1)$ *and nonempty* $F \subset \mathbb{R}^d$,

$$\dim_{\tilde{A}}^\theta F := \inf\Big\{s \ge 0 : \text{there exist constants } C > 0 \text{ and } 0 < \rho < 1 \text{ such that, for all } 0 < r = R^{1/\theta} < R < \rho < 1 \text{ and } x \in F,\ N_r(B(x,R)\cap F) \le C\left(\frac{R}{r}\right)^s\Big\},$$

then $\dim_{\tilde{A}}^\theta F = \dim_A^\theta F$.

Remark. The $\dim_{\tilde{A}}^\theta$ is the initial definition of Assouad spectrum introduced by Fraser and Yu in Ref. 23. The current definition $\dim_A^\theta$ of Assouad spectrum is introduced in Ref. 24. The fact that $\dim_{\tilde{A}}^\theta = \dim_A^\theta$ is known. We give a proof just to check it.

Proof of Lemma 7.5. It is obvious that

$$\dim_{\tilde{A}}^\theta F \le \dim_A^\theta F.$$

For any $s > \dim_{\tilde{A}}^\theta F$, there exist constants $C > 0$ and $0 < \rho < 1$ such that, for any $0 < r = R^{1/\theta} <$

$R < \rho < 1$ and $x \in F$,

$$N_r(B(x,R) \cap F) \le C \left(\frac{R}{r}\right)^s.$$

For $0 < \rho < 1$ above, there exists $k \in \mathbb{N}$ such that

$$\frac{1}{2^k} < \rho^{1/\theta} \le \frac{1}{2^{k-1}}.$$

When $R < \rho < 1$, for any $0 < r = R^{1/\theta} < R < \rho < 1$ and $x \in F$, we have

$$N_r(B(x,R) \cap F) \le C \left(\frac{R}{r}\right)^s.$$

When $\rho \le R < 1$, for any $0 < r = R^{1/\theta} < R < 1$ and $x \in F$, we have

$$r = R^{1/\theta} \ge \rho^{1/\theta},$$

then by Lemma 6.1 we have

$$\begin{aligned} N_r(B(x,R) \cap F) &\le N_{\rho^{1/\theta}}(B(x,1) \cap F) \\ &\le N_{1/2^k}(B(x,1) \cap F) \\ &\le N_{1/2^k}(B(x,1)) \le C_1^k \\ &\le C_1^k \left(\frac{R}{r}\right)^s. \end{aligned}$$

It follows from the arbitrariness of s that

$$\dim_A^\theta F \le \dim_{\tilde{A}}^\theta F. \qquad \square$$

Proof of Theorem 2.6. (1) It follows from Theorem 2.5 that

$$\overline{\dim}_A^\theta E_{S,D} = 2 = \dim_A E_{S,D}.$$

If

$$\overline{\dim}_A^{\theta'} E_{S,D} \equiv 2$$

for all $\theta' \in (0,1)$, then it easily follows from Lemmas 7.1 and 7.2 that

$$\overline{\dim}_B E_{S,D} = \lim_{\theta' \to 0} \overline{\dim}_A^{\theta'} E_{S,D} = 2.$$

It follows from Lemma 7.1 that

$$\overline{\dim}_B E_{S,D} \le \dim_A^{\theta'} E_{S,D} \le 2$$

for all $\theta' \in (0,1)$, then

$$\dim_A^{\theta'} E_{S,D} \equiv 2$$

for all $\theta' \in (0,1)$, then

$$\dim_A^\theta E_{S,D} = 2.$$

Next, we need the obvious facts that the function $t \mapsto \overline{\dim}_A^t$ is non-decreasing in $t \in (0,1)$ and

$$\dim_A^t \le \overline{\dim}_A^t$$

for each $t \in (0,1)$.

If there exists $\theta_1 \in (0,\theta)$ such that

$$\overline{\dim}_A^{\theta_1} E_{S,D} < 2,$$

then it follows from Lemma 7.2 and the above facts that

$$\begin{aligned} \overline{\dim}_A^\theta E_{S,D} &= \sup_{\theta' \in (0,\theta]} \dim_A^{\theta'} E_{S,D} \\ &= \sup_{\theta' \in [\theta_1,\theta]} \dim_A^{\theta'} E_{S,D} = 2. \end{aligned}$$

By Lemma 7.3, there exists $\theta_2 \in [\theta_1, \theta]$ such that

$$\dim_A^{\theta_2} E_{S,D} = 2 = \dim_A E_{S,D}.$$

It follows from Lemma 7.4 that

$$\dim_A^\theta E_{S,D} = 2.$$

(2) For any

$$s > \limsup_{k \to +\infty} \frac{\log(l_{h(k,\theta)} \cdots l_k)}{(1-\theta)\log(n_1 \cdots n_k)},$$

there exists $K_1 > 0$ such that, for any $k \ge K_1$, we have

$$\frac{\log(l_{h(k,\theta)} \cdots l_k)}{(1-\theta)\log(n_1 \cdots n_k)} < s.$$

Since

$$\lim_{k \to +\infty} \frac{\log n_k}{\log(n_1 \cdots n_k)} = 0,$$

we have

$$\lim_{k \to +\infty} \frac{\log(n_1 \cdots n_k)}{\log(n_1 \cdots n_{k-1})} = 1,$$

so, for any $\delta > 0$, there exists $K_2 > 0$ such that, for each $k \ge K_2$, we have

$$\frac{\log(n_1 \cdots n_k)}{\log(n_1 \cdots n_{k-1})} < 1 + \delta.$$

There exists an integer $K_0 \ge 2$ such that, for any $k \ge K_0$,

$$\frac{1}{(n_1 \cdots n_k)^\theta} < \frac{1}{n_1}.$$

It follows from the definition of $h(k,\theta)$ that, for any $k \ge K_0$, $h(k,\theta)$ is the unique positive integer such that

$$\frac{1}{n_1\cdots n_{h(k,\theta)}} \le \frac{1}{(n_1\cdots n_k)^\theta} < \frac{1}{n_1\cdots n_{h(k,\theta)-1}}.$$

Then, for any $k \ge K_0$, we have

$$h(k,\theta) \ge h(k-1,\theta)$$

and

$$\lim_{k\to+\infty} h(k,\theta) = +\infty.$$

Since

$$\lim_{k\to+\infty} \frac{\log n_k}{\log(n_1\cdots n_k)} = 0$$

and there exists $L > 0$ such that, for all $k \ge 2$,

$$h(k,\theta) - h(k-1,\theta) < L,$$

we have

$$\lim_{k\to+\infty} \frac{2\log(n_{h(k-1,\theta)}\cdots n_{h(k,\theta)-1}n_{h(k,\theta)}/n_{h(k,\theta)})}{\log(n_1\cdots n_k)} = 0,$$

so, for any $\varepsilon > 0$, there exists $K_3 > 0$ such that, for each $k \ge K_3$, we have

$$\frac{2\log(n_{h(k-1,\theta)}\cdots n_{h(k,\theta)-1}n_{h(k,\theta)}/n_{h(k,\theta)})}{\log(n_1\cdots n_k)} < \varepsilon.$$

Let $K := \max\{K_0, K_1, K_2, K_3\}$. For any $0 < r < r^\theta = R < \frac{1}{n_1\cdots n_K} < 1$ and $x \in E_{S,D}$, there exist $h, k > K$ such that

$$\frac{1}{n_1\cdots n_h} \le R < \frac{1}{n_1\cdots n_{h-1}} \text{ and } \frac{1}{n_1\cdots n_k} \le r < \frac{1}{n_1\cdots n_{k-1}},$$

then

$$\frac{1}{n_1\cdots n_h} \le R = r^\theta < \frac{1}{(n_1\cdots n_{k-1})^\theta}$$

and

$$\begin{aligned} N_r(B(x,R)\cap E_{S,D}) &\le 9 l_h\cdots l_k \\ &= 9\left(\frac{R}{r}\right)^{\frac{\log(l_h\cdots l_k)}{\log(R/r)}} \\ &\le 9\left(\frac{R}{r}\right)^{\frac{\log(l_h\cdots l_k)}{(1-\theta)\log(n_1\cdots n_{k-1})}}. \end{aligned}$$

For h, k above, since

$$\frac{1}{n_1\cdots n_h} \le R = r^\theta < \frac{1}{(n_1\cdots n_{k-1})^\theta},$$

we have

$$h \ge h(k-1,\theta),$$

then

$$\begin{aligned} &\frac{\log(l_h\cdots l_k)}{(1-\theta)\log(n_1\cdots n_{k-1})} \\ &\le \frac{\log(l_{h(k-1,\theta)}\cdots l_k)}{(1-\theta)\log(n_1\cdots n_k)}\cdot\frac{\log(n_1\cdots n_k)}{\log(n_1\cdots n_{k-1})} \\ &\le (1+\delta)\left(\frac{\log(l_{h(k-1,\theta)}\cdots l_{h(k,\theta)-1}\times l_{h(k,\theta)}/l_{h(k,\theta)})}{(1-\theta)\log(n_1\cdots n_k)} + \frac{\log(l_{h(k,\theta)}\cdots l_k)}{(1-\theta)\log(n_1\cdots n_k)}\right) \\ &\le (1+\delta)\left(\frac{2\log(n_{h(k-1,\theta)}\cdots n_{h(k,\theta)-1}\times n_{h(k,\theta)}/n_{h(k,\theta)})}{(1-\theta)\log(n_1\cdots n_k)} + \frac{\log(l_{h(k,\theta)}\cdots l_k)}{(1-\theta)\log(n_1\cdots n_k)}\right) \\ &\le (1+\delta)\left(\frac{\varepsilon}{1-\theta} + \frac{\log(l_{h(k,\theta)}\cdots l_k)}{(1-\theta)\log(n_1\cdots n_k)}\right) \\ &\le (1+\delta)\left(\frac{\varepsilon}{1-\theta} + s\right) \\ &= s + s\delta + \frac{\varepsilon(1+\delta)}{1-\theta}. \end{aligned}$$

It follows from Lemma 7.5 and the arbitrariness of δ, ε and s that

$$\dim_A^\theta E_{S,D} = \dim_{\tilde{A}}^\theta E_{S,D} \le \limsup_{k\to+\infty} \frac{\log(l_{h(k,\theta)}\cdots l_k)}{(1-\theta)\log(n_1\cdots n_k)}.$$

For any $s > \dim_A^\theta E_{S,D}$, there exists $C > 0$ such that, for any $0 < r < r^\theta = R < 1$ and $x \in E_{S,D}$,

$$N_r(B(x,R)\cap E_{S,D}) \le C\left(\frac{R}{r}\right)^s.$$

There exists an integer $K_0 \ge 2$ such that, for any $k \ge K_0$,

$$\frac{1}{(n_1\cdots n_k)^\theta} < \frac{1}{n_1}.$$

Since

$$\lim_{k\to+\infty} \frac{\log n_k}{\log(n_1\cdots n_k)} = 0,$$

it follows from the definition of $h(k,\theta)$ that there exists $K \geq K_0$, for any $k \geq K$, we have

$$h(k,\theta) < k.$$

For any $k \geq K$, let $x \in E_{S,D}$ and $r = \frac{\sqrt{2}^{1/\theta}}{n_1 \cdots n_k}$.

It follows from the definition of $h(k,\theta)$ that, for any $k \geq K_0$, $h(k,\theta)$ is the unique positive integer such that

$$\frac{1}{n_1 \cdots n_{h(k,\theta)}} \leq \frac{1}{(n_1 \cdots n_k)^\theta} < \frac{1}{n_1 \cdots n_{h(k,\theta)-1}}.$$

Then

$$\frac{\sqrt{2}}{n_1 \cdots n_{h(k,\theta)}} \leq \frac{\sqrt{2}}{(n_1 \cdots n_k)^\theta} = r^\theta$$
$$= R < \frac{\sqrt{2}}{n_1 \cdots n_{h(k,\theta)-1}}.$$

Since every closed ball of radius r meets no more than $(2\sqrt{2}^{1/\theta}+2)^2$ $level-k$ sets of $E_{S,D}$, we have

$$\frac{1}{(2\sqrt{2}^{1/\theta}+2)^2} l_{h(k,\theta)+1} \cdots l_k$$
$$\leq N_r(B(x,R) \cap E_{S,D}) \leq C\left(\frac{R}{r}\right)^s$$
$$= Cr^{s(\theta-1)} = C\sqrt{2}^{\frac{s(\theta-1)}{\theta}} (n_1 \cdots n_k)^{s(1-\theta)}.$$

It follows that

$$\log(l_{h(k,\theta)+1} \cdots l_k)$$
$$\leq \log C_1 + s(1-\theta)\log(n_1 \cdots n_k),$$

where $C_1 = C(2\sqrt{2}^{1/\theta}+2)^2 \sqrt{2}^{\frac{s(\theta-1)}{\theta}}$, then

$$\frac{\log(l_{h(k,\theta)+1} \cdots l_k)}{(1-\theta)\log(n_1 \cdots n_k)}$$
$$\leq \frac{\log C_1}{(1-\theta)\log(n_1 \cdots n_k)} + s,$$

so

$$\frac{\log(l_{h(k,\theta)} \cdots l_k)}{(1-\theta)\log(n_1 \cdots n_k)} = \frac{\log l_{h(k,\theta)}}{(1-\theta)\log(n_1 \cdots n_k)}$$
$$+ \frac{\log(l_{h(k,\theta)+1} \cdots l_k)}{(1-\theta)\log(n_1 \cdots n_k)}$$
$$\leq \frac{2\log n_{h(k,\theta)}}{(1-\theta)\log(n_1 \cdots n_k)}$$
$$+ \frac{\log C_1}{(1-\theta)\log(n_1 \cdots n_k)}$$
$$+ s.$$

Since

$$\lim_{k\to+\infty} \frac{\log n_k}{\log(n_1 \cdots n_k)} = 0 \quad \text{and} \quad 1 \leq h(k,\theta) \leq k$$

for each $k \in \mathbb{N}$, we have

$$\lim_{k\to+\infty} \frac{2\log n_{h(k,\theta)}}{(1-\theta)\log(n_1 \cdots n_k)} = 0.$$

It follows from the arbitrariness of s that

$$\dim_A^\theta E_{S,D} \geq \limsup_{k\to+\infty} \frac{\log(l_{h(k,\theta)} \cdots l_k)}{(1-\theta)\log(n_1 \cdots n_k)}. \qquad \square$$

8. EXAMPLE 2.1 AND THEOREM 2.7

Proof of Example 2.1. Let $n_k = 2^k$ for each $k \in \mathbb{N}$, $S = \{1, 2^1, 2^2, \ldots\}$ and $D = \{(0,0)\}$.

Then

$$\lim_{k\to+\infty} \frac{\log n_k}{\log(n_1 \cdots n_k)} = \lim_{k\to+\infty} \frac{k}{k(k+1)/2} = 0.$$

For any $\theta \in (0,1)$, there exists $\tilde{K}$ such that, for any $k \geq \tilde{K}$,

$$\frac{1}{(n_1 \cdots n_k)^\theta} < \frac{1}{n_1} \quad \text{and} \quad \sqrt{\theta}k \geq 2.$$

It follows from the definition of $h(k,\theta)$ for any $\theta \in (0,1)$ and $k \geq \tilde{K}$ in Theorem 2.6 that $h(k,\theta)$ is the unique positive integer such that

$$\frac{1}{n_1 \cdots n_{h(k,\theta)}} \leq \frac{1}{(n_1 \cdots n_k)^\theta} < \frac{1}{n_1 \cdots n_{h(k,\theta)-1}}.$$

For any $\theta \in (0,1)$ and $k \geq \tilde{K}$,

$$\frac{1}{n_1 \cdots n_{\lfloor\sqrt{\theta}k\rfloor+1}} = \frac{1}{2^{(\lfloor\sqrt{\theta}k\rfloor+1)(\lfloor\sqrt{\theta}k\rfloor+2)/2}}$$
$$< \frac{1}{2^{\sqrt{\theta}k(\sqrt{\theta}k+\sqrt{\theta})/2}} = \frac{1}{2^{\theta k(k+1)/2}}$$
$$= \frac{1}{(n_1 \cdots n_k)^\theta}$$

and

$$\frac{1}{(n_1 \cdots n_{k-1})^\theta} = \frac{1}{2^{\theta k(k-1)/2}} = \frac{1}{2^{\sqrt{\theta}k(\sqrt{\theta}k-\sqrt{\theta})/2}}$$
$$< \frac{1}{2^{\lfloor\sqrt{\theta}k\rfloor(\lfloor\sqrt{\theta}k\rfloor-1)/2}}$$
$$= \frac{1}{n_1 \cdots n_{\lfloor\sqrt{\theta}k\rfloor-1}},$$

then

$$h(k,\theta) \leq \lfloor\sqrt{\theta}k\rfloor + 1 \quad \text{and} \quad h(k-1,\theta) \geq \lfloor\sqrt{\theta}k\rfloor,$$

then

$$h(k,\theta)-h(k-1,\theta)\le 1.$$

Then there exists $L>0$ such that, for all $k\ge 2$,

$$h(k,\theta)-h(k-1,\theta)< L.$$

It follows from Theorem 2.6 that

$$\begin{aligned}\dim_A^\theta E_{S,D} &= \limsup_{k\to+\infty}\frac{\log(l_h\cdots l_k)}{(1-\theta)\log(n_1\cdots n_k)}\\ &\le \limsup_{k\to+\infty}\frac{\log(l_1\cdots l_k)}{(1-\theta)\log(n_1\cdots n_k)}\\ &\le \limsup_{k\to+\infty}\frac{\log(l_1\cdots l_{2^k})}{(1-\theta)\log(n_1\cdots n_{2^k})}\\ &= \limsup_{k\to+\infty}\frac{2\log(2^1\cdot 2^{2^1}\cdot 2^{2^2}\cdots 2^{2^k})}{(1-\theta)\log(2^1\cdot 2^2\cdot 2^3\cdots 2^{2^k})}\\ &= \lim_{k\to+\infty}\frac{2(2^{k+1}-1)}{(1-\theta)2^{k-1}(1+2^k)}=0.\end{aligned}$$

then

$$\dim_A^\theta E_{S,D}\equiv 0$$

for all $\theta\in(0,1)$.

Moreover, it follows from Lemma 7.2 and Theorem 2.3 that

$$\dim_{qA} E_{S,D}=0 \quad\text{and}\quad \dim_A E_{S,D}=2. \qquad \square$$

Proof of Theorem 2.7. Let $n_k\equiv 2$ for each $k\in\mathbb{N}$ and $D=\{(0,0)\}$.

Let $\{K_i\}_{i\ge 1}$ be a sequence of positive integers satisfying

(i) $K_i\ge \frac{1+\lfloor 2/c\rfloor}{(c-a)/(c-b)-1}$ for each $i\in\mathbb{N}$,

(ii) $K_{i+1}\ge \lfloor\frac{c-a}{c-b}K_i\rfloor+i_0+i+\lfloor\frac{2}{a}\rfloor$ for each $i\in\mathbb{N}$, where $i_0=\lfloor\frac{2}{d}\rfloor$,

(iii) $\frac{K_1+K_2+\cdots+K_i}{K_{i+1}}\le\frac{1}{2}$ for each $i\in\mathbb{N}$,

(iv) $\lim_{i\to+\infty}\frac{K_1+K_2+\cdots+K_i}{K_{i+1}}=0$. Then $\lim_{i\to+\infty}\frac{i}{K_i}=0$.

Now we construct the set $S\subset\mathbb{N}$. For any $i\in\mathbb{N}$, we divide $(K_i,K_{i+1}]$ into

$$\left(K_i,\left\lfloor\frac{c-a}{c-b}K_i\right\rfloor\right],\left(\left\lfloor\frac{c-a}{c-b}K_i\right\rfloor,\left\lfloor\frac{c-a}{c-b}K_i\right\rfloor+i_0+i\right]\quad\text{and}\quad\left(\left\lfloor\frac{c-a}{c-b}K_i\right\rfloor+i_0+i,K_{i+1}\right].$$

For any $i\in\mathbb{N}$, let

$$A_i:=\left\{K_i+\left\lfloor\frac{2}{c}\right\rfloor,K_i+\left\lfloor\frac{2\times 2}{c}\right\rfloor,\ldots,K_i+\left\lfloor\frac{2M_i}{c}\right\rfloor\right\},$$

where M_i is the largest positive integer such that

$$K_i+\left\lfloor\frac{2M_i}{c}\right\rfloor\le\left\lfloor\frac{c-a}{c-b}K_i\right\rfloor,$$

$$B_i:=\left\{\left\lfloor\frac{c-a}{c-b}K_i\right\rfloor+\left\lfloor\frac{2}{d}\right\rfloor,\left\lfloor\frac{c-a}{c-b}K_i\right\rfloor+\left\lfloor\frac{2\times 2}{d}\right\rfloor,\ldots,\left\lfloor\frac{c-a}{c-b}K_i\right\rfloor+\left\lfloor\frac{2M_i'}{d}\right\rfloor\right\},$$

where M_i' is the largest positive integer such that

$$\left\lfloor\frac{c-a}{c-b}K_i\right\rfloor+\left\lfloor\frac{2M_i'}{d}\right\rfloor\le\left\lfloor\frac{c-a}{c-b}K_i\right\rfloor+i_0+i,$$

$$C_i:=\left\{\left\lfloor\frac{c-a}{c-b}K_i\right\rfloor+i_0+i+\left\lfloor\frac{2}{a}\right\rfloor,\left\lfloor\frac{c-a}{c-b}K_i\right\rfloor+i_0+i+\left\lfloor\frac{2\times 2}{a}\right\rfloor,\ldots,\left\lfloor\frac{c-a}{c-b}K_i\right\rfloor+i_0+i+\left\lfloor\frac{2M_i''}{a}\right\rfloor\right\},$$

where M_i'' is the largest positive integer such that

$$\left\lfloor\frac{c-a}{c-b}K_i\right\rfloor+i_0+i+\left\lfloor\frac{2M_i''}{a}\right\rfloor\le K_{i+1}.$$

From the condition (i) of the sequence $\{K_i\}_{i\ge 1}$, we have, for any $i\in\mathbb{N}$,

$$\left\lfloor\frac{c-a}{c-b}K_i\right\rfloor-K_i>\frac{c-a}{c-b}K_i-1-K_i=\left(\frac{c-a}{c-b}-1\right)K_i-1\ge\left\lfloor\frac{2}{c}\right\rfloor,$$

then A_i is nonempty. From the condition (ii) of the sequence $\{K_i\}_{i\ge 1}$, we have, for any $i\in\mathbb{N}$, B_i and C_i are nonempty.

Let

$$S_i:=A_i\cup B_i\cup C_i$$

for each $i\in\mathbb{N}$ and

$$S:=\bigcup_{i=1}^{+\infty}S_i.$$

For any $l', m' \in \mathbb{N}$, $0 < e \leq 2$ and $h, k \in \mathbb{N}$ with $l' + \lfloor \frac{2}{e} \rfloor \leq h \leq k \leq l' + \lfloor \frac{2m'}{e} \rfloor$, there exist $\tilde{h}, \tilde{k} \in \{1, 2, \ldots, m'\}$ such that

$$l' + \left\lfloor \frac{2\tilde{h}}{e} \right\rfloor \leq h < l' + \left\lfloor \frac{2(\tilde{h}+1)}{e} \right\rfloor \quad \text{and}$$

$$l' + \left\lfloor \frac{2\tilde{k}}{e} \right\rfloor \leq k < l' + \left\lfloor \frac{2(\tilde{k}+1)}{e} \right\rfloor,$$

then

$$l' + \frac{2\tilde{h}}{e} - 1 < l' + \left\lfloor \frac{2\tilde{h}}{e} \right\rfloor \leq h < l' + \left\lfloor \frac{2(\tilde{h}+1)}{e} \right\rfloor \leq l' + \frac{2(\tilde{h}+1)}{e}$$

and

$$l' + \frac{2\tilde{k}}{e} - 1 < l' + \left\lfloor \frac{2\tilde{k}}{e} \right\rfloor \leq k < l' + \left\lfloor \frac{2(\tilde{k}+1)}{e} \right\rfloor \leq l' + \frac{2(\tilde{k}+1)}{e}.$$

It follows that

$$\begin{aligned} &\frac{e}{2}\sharp\{h, \ldots, k\} - 1 - e \\ &\quad = \frac{e}{2}(k - l') - 1 - \frac{e}{2}(h - l' + 1) < \tilde{k} - \tilde{h} \\ &\quad \leq \sharp\left(\left\{l' + \left\lfloor \frac{2}{e} \right\rfloor, l' + \left\lfloor \frac{2\times 2}{e} \right\rfloor, \ldots, l' + \left\lfloor \frac{2m'}{e} \right\rfloor\right\} \cap \{h, h+1, \ldots, k\}\right) \\ &\quad \leq \tilde{k} - \tilde{h} + 1 < \frac{e}{2}(k - l' + 1) - \left(\frac{e}{2}(h - l') - 1\right) + 1 \\ &\quad = \frac{e}{2}\sharp\{h, \ldots, k\} + 2. \qquad (*) \end{aligned}$$

For any $k \geq K_1$, there exists a unique positive integer $i(k)$ such that $K_{i(k)} \leq k < K_{i(k)+1}$.

(1) From Theorem 2.3, $(*)$, (iii) and (iv), we have

$$\begin{aligned} \dim_A E_{S,D} &= \lim_{j\to+\infty} \sup_{k\geq j,\ m\geq j} \frac{\log(l_{k+1}\cdots l_{k+m})}{\log(n_{k+1}\cdots n_{k+m})} \\ &= \lim_{j\to+\infty} \sup_{k\geq j,\ m\geq j} \frac{2\sharp(S \cap \{k+1, \ldots, k+m\})}{m} \\ &\leq \lim_{j\to+\infty} \sup_{k\geq j,\ m\geq j} \frac{2((dm)/2 + 2\times 3 \times (\sharp(\{k+1, \ldots, k+m\} \cap \{K_1, K_2, \ldots\}) + 1))}{m} \\ &\leq d + \lim_{j\to+\infty} \sup_{k\geq j,\ m\geq j} \frac{12\sharp(\{k+1, \ldots, k+m\} \cap \{K_1, K_2, \ldots\})}{m} \\ &\leq d + \lim_{j\to+\infty} \sup_{m\geq j} \left(\frac{12\sharp(\{K_1, K_1+1, \ldots, K_1+m-1\} \cap \{K_1, K_2, \ldots\})}{K_1+m-1} \cdot \frac{K_1+m-1}{m}\right) \\ &\leq d + \lim_{j\to+\infty} \sup_{m\geq j} \frac{12i(K_1+m-1)}{K_{i(K_1+m-1)}} \cdot \frac{K_1+m-1}{m} \\ &= d. \end{aligned}$$

Let $k_i := \lfloor \frac{c-a}{c-b} K_i \rfloor + \lfloor \frac{2}{d} \rfloor - 1$, $m_i := \lfloor \frac{2M_i'}{d} \rfloor - \lfloor \frac{2}{d} \rfloor + 1$ for each $i \in \mathbb{N}$.

From $(*)$, we have, for any $i \in \mathbb{N}$,

$$\frac{2((dm_i)/2 - 1 - d)}{m_i} \leq \frac{2\sharp(S \cap \{k_i+1, \ldots, k_i+m_i\})}{m_i} \leq \frac{2((dm_i)/2 + 2)}{m_i},$$

then

$$\dim_A E_{S,D} \geq \lim_{i\to+\infty} \frac{2\sharp(S \cap \{k_i+1, \ldots, k_i+m_i\})}{m_i} = d.$$

(2) From Theorem 2.5 and $(*)$, we have

$$\begin{aligned} \dim_{qA} E_{S,D} &= \lim_{\alpha\to 1} \limsup_{k\to+\infty} \max_{1\leq h\leq \alpha k} \frac{\log(l_h\cdots l_k)}{\log(n_h\cdots n_k)} \\ &= \lim_{\alpha\to 1} \limsup_{k\to+\infty} \max_{1\leq h\leq \alpha k} \frac{2\sharp(S \cap \{h, \ldots, k\})}{k-h+1} \end{aligned}$$

$$\le c + \lim_{\alpha\to 1} \limsup_{k\to+\infty} \max_{K_1\le h\le \alpha k} \frac{2\left(\sum_{i=i(h)}^{i(k)} \sharp B_i + 2\times 3(i(k)-i(h)+1)\right)}{(1-\alpha)k+1}$$

$$\le c + \lim_{\alpha\to 1} \limsup_{k\to+\infty} \frac{2(K_{i(k)-1} + (i_0+i(k)-1) + (i_0+i(k)) + 6(i(k)-i(h)+1))}{(1-\alpha)k+1}$$

$$\le c + \lim_{\alpha\to 1} \limsup_{k\to+\infty} \frac{2(K_{i(k)-1} + (i_0+i(k)-1) + (i_0+i(k)) + 6(i(k)-i(h)+1))}{(1-\alpha)K_{i(k)}+1}$$

$$= c.$$

Let $h_i := K_i + \lfloor \frac{2}{c} \rfloor$ and $k_i := K_i + \lfloor \frac{2M_i}{c} \rfloor$ for each $i \in \mathbb{N}$.

For any $i \in \mathbb{N}$, it follows from the definition of M_i that

$$K_i + \left\lfloor \frac{2M_i}{c} \right\rfloor \le \left\lfloor \frac{c-a}{c-b} K_i \right\rfloor < K_i + \left\lfloor \frac{2(M_i+1)}{c} \right\rfloor,$$

then

$$\frac{c-a}{c-b}K_i - 1 \le K_i + \frac{2M_i+2}{c},$$

then

$$\frac{c-a}{c-b}K_i - 1 - \frac{2}{c} \le K_i + \frac{2M_i}{c}. \qquad (2*)$$

Since $\frac{c-a}{c-b} > 1$, there exists $\alpha_0 > 0$ such that, for any $\alpha_0 \le \alpha < 1$, we have

$$\alpha\frac{c-a}{c-b} - 1 > 0.$$

For α_0 above, since

$$\lim_{i\to+\infty} K_i = +\infty,$$

there exists $i_0 > 0$ such that, for any $i \ge i_0$,

$$\left(\alpha_0\frac{c-a}{c-b} - 1\right)K_i - 2 - \frac{4}{c} \ge 0.$$

Then, for any $\alpha_0 \le \alpha < 1$ and $i \ge i_0$, we have

$$\left(\alpha\frac{c-a}{c-b} - 1\right)K_i - 2 - \frac{4}{c} \ge \left(\alpha_0\frac{c-a}{c-b} - 1\right)K_i - 2 - \frac{4}{c} \ge 0,$$

then it follows from (2∗) that

$$\alpha k_i - h_i = \alpha\left(K_i + \left\lfloor\frac{2M_i}{c}\right\rfloor\right) - K_i - \left\lfloor\frac{2}{c}\right\rfloor \ge \alpha\left(K_i + \frac{2M_i}{c} - 1\right) - K_i - \left\lfloor\frac{2}{c}\right\rfloor$$

$$\ge \alpha\left(\frac{c-a}{c-b}K_i - 2 - \frac{2}{c}\right) - K_i - \left\lfloor\frac{2}{c}\right\rfloor$$

$$= \left(\alpha\frac{c-a}{c-b} - 1\right)K_i - 2\alpha - \frac{2\alpha}{c} - \left\lfloor\frac{2}{c}\right\rfloor$$

$$\ge \left(\alpha\frac{c-a}{c-b} - 1\right)K_i - 2 - \frac{4}{c} \ge 0. \qquad (3*)$$

It follows from (∗) and (3∗) that

$$\dim_{qA} E_{S,D} \ge \limsup_{i\to+\infty} \frac{2\sharp(S\cap\{h_i,\ldots,k_i\})}{k_i-h_i+1}$$

$$\ge \limsup_{i\to+\infty} \frac{2((c/2)(k_i-h_i+1)-1-c)}{k_i-h_i+1}$$

$$\ge \limsup_{i\to+\infty} \left(c - \frac{2+2c}{k_i-h_i+1}\right)$$

$$\ge \limsup_{i\to+\infty} \left(c - \frac{2+2c}{(1-\alpha_0)k_i+1}\right)$$

$$= c.$$

(3) From Theorem 2.2, we have

$$\overline{\dim}_B E_{S,D} = \limsup_{k\to+\infty} \frac{\log(l_1\cdots l_k)}{\log(n_1\cdots n_k)} = \limsup_{k\to+\infty} \frac{2\sharp(S\cap\{1,\ldots,k\})}{k}.$$

For any positive integer $k \ge K_2$, there exists a unique positive integer $i(k)$ such that $K_{i(k)} \le k < K_{i(k)+1}$.

If $k \in [K_{i(k)}, \lfloor \frac{c-a}{c-b} K_{i(k)} \rfloor)$, using $(*)$, we have

$$\begin{aligned}
&\frac{2\sharp(S \cap \{1, \dots, k\})}{k} \\
&\quad \le \frac{2\sharp(S \cap [1, \lfloor \frac{c-a}{c-b} K_{i(k)-1} \rfloor + i_0 + i(k) - 1]) + 2\sharp C_{i(k)-1}}{k} \\
&\qquad + \frac{2\sharp(A_{i(k)} \cap (K_{i(k)}, k])}{k} \\
&\quad \le \frac{2(\frac{c-a}{c-b} K_{i(k)-1} + i_0 + i(k)) + 2(\frac{a}{2} K_{i(k)} + 2) + 2(\frac{c}{2}(k - K_{i(k)}) + 2)}{k} \\
&\quad \le \frac{aK_{i(k)} + ck - cK_{i(k)} + 2(\frac{c-a}{c-b} K_{i(k)-1} + i_0 + i(k)) + 8}{k} \\
&\quad \le c - \frac{(c-a)K_{i(k)}}{\frac{c-a}{c-b} K_{i(k)}} \\
&\qquad + \frac{2(\frac{c-a}{c-b} K_{i(k)-1} + i_0 + i(k)) + 8}{K_{i(k)}} \\
&\quad = b + \frac{2(\frac{c-a}{c-b} K_{i(k)-1} + i_0 + i(k)) + 8}{K_{i(k)}}. \qquad (4*)
\end{aligned}$$

If $k \in [\lfloor \frac{c-a}{c-b} K_{i(k)} \rfloor, K_{i(k)+1})$, using $(*)$, we have

$$\begin{aligned}
&\frac{2\sharp(S \cap \{1, \dots, k\})}{k} \\
&\quad \le \frac{2\sharp(S \cap [1, \lfloor \frac{c-a}{c-b} K_{i(k)-1} \rfloor + i_0 + i(k) - 1]) + 2\sharp C_{i(k)-1}}{k} \\
&\qquad + \frac{2\sharp A_{i(k)} + 2\sharp(S \cap B_{i(k)}) + 2\sharp(S \cap C_{i(k)})}{k} \\
&\quad \le \frac{2(\frac{c-a}{c-b} K_{i(k)-1} + i_0 + i(k)) + 2(\frac{a}{2} K_{i(k)} + 2)}{k} \\
&\qquad + \frac{2(\frac{c}{2}(\lfloor \frac{c-a}{c-b} K_{i(k)} \rfloor - K_{i(k)}) + 2) + 2(i_0 + i(k))}{k} \\
&\qquad + \frac{2(\frac{b}{2}(k - \lfloor \frac{c-a}{c-b} K_{i(k)} \rfloor) + 2)}{k} \\
&\quad \le \frac{aK_{i(k)} + c\frac{c-a}{c-b} K_{i(k)} - cK_{i(k)} + bk - b\frac{c-a}{c-b} K_{i(k)} + b}{k} \\
&\qquad + \frac{2(\frac{c-a}{c-b} K_{i(k)} + i_0 + i(k)) + 2(i_0 + i(k)) + 12}{k} \\
&\quad \le b + \frac{b + 2(\frac{c-a}{c-b} K_{i(k)-1} + i_0 + i(k)) + 2(i_0 + i(k)) + 12}{K_{i(k)}}. \qquad (5*)
\end{aligned}$$

Since

$$\lim_{k \to +\infty} \frac{2(\frac{c-a}{c-b} K_{i(k)-1} + i_0 + i(k)) + 8}{K_{i(k)}} = 0$$

and

$$\lim_{k \to +\infty} \frac{b + 2(\frac{c-a}{c-b} K_{i(k)-1} + i_0 + i(k)) + 2(i_0 + i(k)) + 12}{K_{i(k)}} = 0,$$

it follows from $(4*)$ and $(5*)$ that

$$\overline{\dim}_B E_{S,D} \le b.$$

Let $k_i := K_i + \left\lfloor \frac{2M_i}{c} \right\rfloor$ for each $i \in \mathbb{N}$. It follows from definitions of M_i'' and M_i for each $i \in \mathbb{N}$ that, for any $i \ge 2$,

$$K_i < \left\lfloor \frac{c-a}{c-b} K_{i-1} \right\rfloor + i_0 + i - 1 + \left\lfloor \frac{2(M_{i-1}'' + 1)}{a} \right\rfloor$$

and

$$\left\lfloor \frac{c-a}{c-b} K_i \right\rfloor < K_i + \left\lfloor \frac{2(M_i + 1)}{c} \right\rfloor,$$

then

$$M_{i-1}'' \ge \frac{a}{2} \left(K_i - \frac{c-a}{c-b} K_{i-1} - i_0 - i \right) - 1$$

and

$$M_i \ge \frac{c}{2} \left(\frac{c-a}{c-b} K_i - K_i - 1 \right) - 1.$$

Furthermore, from $(*)$, we have

$$\begin{aligned}
&\overline{\dim}_B E_{S,D} \\
&\quad \ge \limsup_{i \to +\infty} \frac{2\sharp(S \cap \{1, \dots, k_i\})}{k_i} \\
&\quad \ge \limsup_{i \to +\infty} \frac{2(M_{i-1}'' + M_i)}{K_i + \lfloor \frac{2M_i}{c} \rfloor} \\
&\quad \ge \limsup_{i \to +\infty} \frac{2(M_{i-1}'' + M_i)}{\frac{c-a}{c-b} K_i}
\end{aligned}$$

$$\begin{aligned}
&\geq \limsup_{i\to+\infty} \frac{\begin{array}{c} aK_i - a(\frac{c-a}{c-b}K_{i-1} + i_0 + i) \\ -2 + c(\frac{c-a}{c-b}K_i - K_i) - c - 2 \end{array}}{\frac{c-a}{c-b}K_i} \\
&\geq \limsup_{i\to+\infty} \frac{\begin{array}{c} (a - c + c\frac{c-a}{c-b})K_i \\ -a(\frac{c-a}{c-b}K_{i-1} + i_0 + i) - c - 4 \end{array}}{\frac{c-a}{c-b}K_i} \\
&= \limsup_{i\to+\infty} \left(b - \frac{a(\frac{c-a}{c-b}K_{i-1} + i_0 + i) + c + 4}{\frac{c-a}{c-b}K_i} \right) \\
&= b.
\end{aligned}$$

(4) From Theorem 2.1 and $n_k \equiv 2$ for each $k \in \mathbb{N}$, we have

$$\begin{aligned}
\dim_H E_{S,D} &= \liminf_{k\to+\infty} \frac{\log(l_1 \cdots l_k)}{\log(n_1 \cdots n_{k+1}) - \log\sqrt{l_{k+1}}} \\
&= \liminf_{k\to+\infty} \frac{\log(l_1 \cdots l_k)}{\log(n_1 \cdots n_k)} \\
&= \liminf_{k\to+\infty} \frac{2\sharp(S \cap \{1, \ldots, k\})}{k}.
\end{aligned}$$

For any positive integer $k \geq K_2$, there exists a unique positive integer $i(k)$ such that $K_{i(k)} \leq k < K_{i(k)+1}$.

If $k \in [K_{i(k)}, \lfloor \frac{c-a}{c-b}K_{i(k)} \rfloor)$, it follows from definitions of M_i and M_i'' for each $i \in \mathbb{N}$ that

$$\begin{aligned}
&\left\lfloor \frac{c-a}{c-b}K_{i(k)} \right\rfloor - \left(K_{i(k)} + \left\lfloor \frac{2M_{i(k)}}{c} \right\rfloor \right) \\
&\quad \leq K_{i(k)} + \left\lfloor \frac{2(M_{i(k)}+1)}{c} \right\rfloor \\
&\qquad - \left(K_{i(k)} + \left\lfloor \frac{2M_{i(k)}}{c} \right\rfloor \right) \leq \frac{2}{c} + 1
\end{aligned}$$

and

$$\begin{aligned}
K_{i(k)} &< \left\lfloor \frac{c-a}{c-b}K_{i(k)-1} \right\rfloor + i_0 + i(k) - 1 \\
&\quad + \left\lfloor \frac{2(M''_{i(k)-1}+1)}{a} \right\rfloor,
\end{aligned}$$

then

$$\begin{aligned}
&a\left(K_{i(k)} - \left(\left\lfloor \frac{c-a}{c-b}K_{i(k)-1} \right\rfloor + i_0 + i(k) - 1 \right) \right) - 2 \\
&\quad \leq 2M''_{i(k)-1}.
\end{aligned}$$

Furthermore, using $(*)$, we have

$$\begin{aligned}
&\frac{2\sharp(S \cap \{1, \ldots, k\})}{k} \\
&\quad \geq \frac{2\sharp(A_{i(k)} \cap [1,k]) + 2\sharp C_{i(k)-1}}{k} \\
&\quad \geq \frac{\begin{array}{c} 2(\frac{c}{2}(k - K_{i(k)} - \lfloor \frac{2}{c} \rfloor + 1) - 1 - c) \\ -2(\frac{2}{c}+1) + 2M''_{i(k)-1} \end{array}}{k} \\
&\quad \geq \frac{c(k - K_{i(k)}) - c\lfloor \frac{2}{c} \rfloor - 2 - c - 2(\frac{2}{c}+1)}{k} \\
&\qquad + \frac{\begin{array}{c} a(K_{i(k)} - (\lfloor \frac{c-a}{c-b}K_{i(k)-1} \rfloor \\ + i_0 + i(k) - 1) - 2 \end{array}}{k} \\
&\quad \geq \frac{\begin{array}{c} a(k - K_{i(k)}) + aK_{i(k)} - c\lfloor \frac{2}{c} \rfloor \\ -2 - c - 2(\frac{2}{c}+1) \end{array}}{k} \\
&\qquad + \frac{-a(\lfloor \frac{c-a}{c-b}K_{i(k)-1} \rfloor + i_0 + i(k) - 1) - 2}{k} \\
&\quad = a + \frac{\begin{array}{c} -c\lfloor \frac{2}{c} \rfloor - 2 - c - 2(\frac{2}{c}+1) \\ -a(\lfloor \frac{c-a}{c-b}K_{i(k)-1} \rfloor \\ + i_0 + i(k) - 1) - 2 \end{array}}{k}. \qquad (6*)
\end{aligned}$$

If $k \in [\lfloor \frac{c-a}{c-b}K_{i(k)} \rfloor, K_{i(k)+1})$, it follows from the definition of M_i for each $i \in \mathbb{N}$ that

$$\begin{aligned}
\left\lfloor \frac{c-a}{c-b}K_{i(k)} \right\rfloor &< K_{i(k)} + \left\lfloor \frac{2(M_{i(k)}+1)}{c} \right\rfloor \\
&\leq K_{i(k)} + \frac{2(M_{i(k)}+1)}{c}.
\end{aligned}$$

Then

$$\begin{aligned}
&a\left(\left\lfloor \frac{c-a}{c-b}K_{i(k)} \right\rfloor - K_{i(k)} \right) - 2 \\
&\quad \leq c\left(\left\lfloor \frac{c-a}{c-b}K_{i(k)} \right\rfloor - K_{i(k)} \right) - 2 \leq 2M_{i(k)}.
\end{aligned}$$

Using the definition of M_i'' for each $i \in \mathbb{N}$, we have

$$\begin{aligned}
&K_{i(k)+1} - \left(\left\lfloor \frac{c-a}{c-b}K_{i(k)} \right\rfloor + i_0 + i(k) + \left\lfloor \frac{2M''_{i(k)}}{a} \right\rfloor \right) \\
&\quad \leq \left\lfloor \frac{2(M''_{i(k)}+1)}{a} \right\rfloor - \left\lfloor \frac{2M''_{i(k)}}{a} \right\rfloor \leq \frac{2}{a} + 1.
\end{aligned}$$

Furthermore, using $(*)$, we have

$$\begin{aligned}
&\frac{2\sharp(S\cap\{1,\ldots,k\})}{k}\\
&\quad\geq \frac{2\sharp C_{i(k)-1}+2\sharp A_{i(k)}+2\sharp(C_{i(k)}\cap[1,k])}{k}\\
&\quad\geq \frac{2M''_{i(k)-1}+2M_{i(k)}+2(\frac{a}{2}(k-\lfloor\frac{c-a}{c-b}K_{i(k)}\rfloor - i_0-i(k)-\lfloor\frac{2}{a}\rfloor+1)-1-a)}{k}\\
&\qquad+\frac{-2(\frac{2}{a}+1)}{k}\\
&\quad\geq \frac{a(K_{i(k)}-(\lfloor\frac{c-a}{c-b}K_{i(k)-1}\rfloor+i_0+i(k)-1)) -2+a(\lfloor\frac{c-a}{c-b}K_{i(k)}\rfloor-K_{i(k)})}{k}\\
&\qquad+\frac{-2+a(k-\lfloor\frac{c-a}{c-b}K_{i(k)}\rfloor-i_0-i(k) -\lfloor\frac{2}{a}\rfloor+1)-2(\frac{2}{a}+1)-2-2a}{k}\\
&\quad= a+\frac{-a(\lfloor\frac{c-a}{c-b}K_{i(k)-1}\rfloor+i_0+i(k)-1) +a(-i_0-i(k)-\lfloor\frac{2}{a}\rfloor+1)}{k}\\
&\qquad+\frac{-2(\frac{2}{a}+1)-6-2a}{k}. \qquad (7*)
\end{aligned}$$

Since

$$\lim_{k\to+\infty}\frac{-c\lfloor\frac{2}{c}\rfloor-2-c-2(\frac{2}{c}+1) -a(\lfloor\frac{c-a}{c-b}K_{i(k)-1}\rfloor+i_0+i(k)-1)-2}{k}=0$$

and

$$\lim_{k\to+\infty}\frac{-a(\lfloor\frac{c-a}{c-b}K_{i(k)-1}\rfloor+i_0+i(k)-1) +a(-i_0-i(k)-\lfloor\frac{2}{a}\rfloor+1) -2(\frac{2}{a}+1)-6-2a}{k}=0,$$

it follows from $(6*)$ and $(7*)$ that

$$\dim_H E_{S,D}\geq a.$$

Let $k_i:=\lfloor\frac{c-a}{c-b}K_i\rfloor+i_0+i+\lfloor\frac{2M''_i}{a}\rfloor$ for each $i\in\mathbb{N}$. It follows from the definition of M''_i for each $i\in\mathbb{N}$ that, for any $i\in\mathbb{N}$,

$$K_{i+1}-k_i\leq\left\lfloor\frac{2(M''_i+1)}{a}\right\rfloor-\left\lfloor\frac{2M''_i}{a}\right\rfloor\leq\frac{2}{a}+1.$$

Furthermore, from $(*)$, we have

$$\begin{aligned}
\dim_H E_{S,D}&=\liminf_{k\to+\infty}\frac{2\sharp(S\cap\{1,\ldots,k\})}{k}\\
&\leq\liminf_{i\to+\infty}\frac{2\sharp(S\cap\{1,\ldots,k_i\})}{k_i}\\
&\leq\liminf_{i\to+\infty}\frac{2(\lfloor\frac{c-a}{c-b}K_i\rfloor+i_0+i)+2\sharp C_i}{k_i}\\
&\leq\liminf_{i\to+\infty}\frac{2(\lfloor\frac{c-a}{c-b}K_i\rfloor+i_0+i)+a(K_{i+1} -(\lfloor\frac{c-a}{c-b}K_i\rfloor+i_0+i))+4}{K_{i+1}-\frac{2}{a}-1}\\
&=a. \qquad\square
\end{aligned}$$

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