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Limit Cycles for the Competitive Three Dimensional Lotka–Volterra System

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In the first part of this paper, it is proved that the number of limit cycles of the competitive three-dimensional Lotka–Volterra system in $\mathbf{R}_+^3$ is finite if this system has not any heteroclinic polycycles in $\mathbf{R}_+^3$. In the second part of this paper, a 3D competitive Lotka–Volterra system with two small parameters is discussed. This system always has a heteroclinic polycycle with three saddles. It is proved that there exists one parameter range in which the system is persistence and has at least two limit cycles, and there exists other parameter ranges in which the system is not persistence and has at least one limit cycle.

1. INTRODUCTION

Competitive Lotka–Volterra system modelling three mutually competing species, each of which, in isolation, would exhibit logistic growth, is expressed by

$$\dot{x} = X(b - Ax), \tag{1}$$

where $x = \mathrm{col}(x_1, x_2, x_3)$ is a three-dimensional state vector, $X = \mathrm{diag}(x_1, x_2, x_3)$ is a 3×3 diagonal matrix, $b = \mathrm{col}(b_1, b_2, b_3)$ is a positive real vector and $A = (a_{ij})_{3\times 3}$ is a positive matrix. According to a theorem of M. W. Hirsch in [11], there exists an invariant manifold Σ (i.e., carrying simplex) which is homeomorphic to the two-dimensional simplex and attracts all orbits except the origin. Therefore in 3D competitive systems the Poincaré–Bendixson theorem holds. Based on this, M. L. Zeeman [13] has given a classification of all possible stable phase portraits of 3D competitive Lotka–Volterra equations, thus extended a related classification of the game dynamical equation [14]. As one knows, it is very important to discuss existence of limit cycles of competitve Lotka–Volterra

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system from the point of view of both biology and mathematics. As the list of papers on the subject is very large, we content ourselves by referencing the works [7–10, 13]. In these references the limit cycles were all generated by Hopf bifurcation. In [8] the authors conjectured the number of limit cycles is at most two for Eq. (1). It is a very interesting conjecture. Now we are interested in (i) whether or not the number of limit cycles of Eq. (1) is finite; (ii) at most how many limit cycles there exist for Eq. (1) if they are finite; (iii) whether or not the limit cycles of Eq. (1) can be generated by other bifurcation.

In this paper we answer question (i) partly, i.e., we prove that the number of limit cycles of Eq. (1) in $\mathbf{R}^3_+$ is finite if Eq. (1) has not any heteroclinic polycycles in $\mathbf{R}^3_+$. Therefore the number of limit cycles of 33 stable equivalence classes, which were listed by Zeeman in [13], is finite except classes 26 and 27. For question (iii) we analyse a 3D competitive Lotka–Volterra system with two small parameters. The system has a limit cycle which isn't generated by Hopf bifurcation. Moreover we divide the parameter range of the system into four parts by Hopf bifurcation curve and Heteroclinic bifurcation curve. In bifurcation diagram (see Fig. 3) there exists a parameter range *II* such that in this range the system is persistence and has at least two limit cycles, and other parameter range *III* such that in that range the system is not persistence and has at least one limit cycle. Question (ii) is still open.

2. FINITENESS OF LIMIT CYCLE

It is a classical result that 2D Lotka–Volterra equations cannot have limit cycles: if there is a periodic orbit, then the interior singular point is a center (i.e., surrounded by a continuum of periodic orbits). Hence a center is a codimension one phenomenon for 2D Lotka–Volterra equations like that for linear equations. On the other hand, 3D Lotka–Volterra equations allow already complicated dynamics: the period doubling route to chaos and many other phenomena known from the interaction of the quadratic map have been observed by numerical simulation (Ref. [2]). For 3D competitive systems, the dynamical possibilities are more restricted, the compact limit sets of these systems are either singular points or periodic orbits by a theorem of M. W. Hirsch. The proof of following lemma can be found in [13].

LEMMA 2.1. *Every trajectory of Eq.* (1) *in* $\mathbf{R}^3_+ \setminus \{0\}$ *is asymptotic to one in* Σ, *and* Σ *is a Lipschitz submanifold homeomorphic to the unit simplex in* $\mathbf{R}^3_+$ *by radial projection.*

Hence all equilibria and periodic orbits of Eq. (1) lie on the Σ except the equilibrium O(0,0,0). If there is no equilibrium in interior of invariant manifold Σ, then the dynamics of Eq. (1) is trivial: every orbit converges to the boundary of Σ. Therefore we are interested only in the case that Eq. (1) has an equilibrium in the interior of Σ and the equilibrium is non-degenerative by the results in [13]. Without loss of generality, we can assume that $E=(1,1,1)$ is an equilibrium of Eq. (1), and the equilibrium $E(1,1,1)$ has no zero eigenvalues.

THEOREM 2.2. *There exists a small neighborhood of equilibrium* $E(1,1,1)$ *such that the number of limit cycles of Eq.* (1) *is finite in this neighborhood, i.e. infinitely many limit cycles of Eq.* (1) *can not accumulate on* $E(1,1,1)$.

Proof. We devide two cases for the equilibrium $E(1,1,1)$ to prove the theorem.

Case I. If the equilibrium $E(1,1,1)$ is hyperbolic, then there exists a small neighborhood of $E(1,1,1)$ such that Eq. (1) has no periodic orbits in this neighborhoood. Thus the statement of the theorem is true.

Case II. If the equilibrium $E(1,1,1)$ is not hyperbolic, then we set $y_i=x_i-1$. Hence, Eq. (1) reads

$$\dot{y}_i=-(1+y_i)\left(\sum_{j=1}^{3} a_{ij}y_j\right), \qquad i=1,2,3. \tag{2}$$

To be more explicit, we use coordinate transformation such that Eq. (2) can be written into

$$\begin{aligned}\dot{x}&=-y+X(x,y,z)\\ \dot{y}&=x+Y(x,y,z)\\ \dot{z}&=\lambda z+Z(x,y,z),\end{aligned} \tag{3}$$

where $x,y,z\in\mathbf{R}$, X, Y and Z are real analytic functions which together with their first derivatives vanish at $(x,y,z)=(0,0,0)$. And λ is a negative real number. According to Bernd Aulbach [3], we know that the equilibrium $O(0,0,0)$ of Eqs. (3) is either a weak focus or a center.

If the equilibrium $O(0,0,0)$ is a center, then there exists a small neighborhood of $O(0,0,0)$ such that Eqs. (3) has no isolated periodic orbits in this neighborhood. Therefore the theorem holds.

If the equilibrium $O(0,0,0)$ is a weak focus, then the study of Eqs. (3) near $O(0,0,0)$ can be reduced to the study of the corresponding equations on a center manifold of $O(0,0,0)$. By local center manifold theorem in [4], we know that there exists a c^∞ local center manifold such that Eqs. (3) is locally topologically equivalent to the following equations

$$\begin{aligned} \dot{x} &= -y + W(x, y) \\ \dot{y} &= x + V(x, y) \\ \dot{z} &= \lambda z, \end{aligned} \tag{4}$$

where $x, y, z \in \mathbf{R}$, In the small neighborhood of $O(0, 0)$, $W(x, y)$ and $V(x, y)$ are real c^∞ functions which together with their first derivatives vanish at $(x, y) = (0, 0)$. And λ is a negative real number.

We consider the subsystem of Eqs. (4)

$$\begin{aligned} \dot{x} &= -y + W(x, y) \\ \dot{y} &= x + V(x, y). \end{aligned} \tag{5}$$

If the equilibrium $O(0, 0)$ of system (5) is a weak focus of multiplicity m, then the generic Hopf theorem, proved in Chapter II of [15], shows that at most m limit cycles can be generated from $O(0, 0)$ under any a small c^∞ perturbation of system (5). According to Lemma 2.1, we easily get that there exists a small neighborhood of $O(0, 0, 0)$ in the carrying simplex Σ such that Eqs. (3) has at most m limit cycles in this neighborhood. Thus there exists a small neighborhood of equilibrium $E(1, 1, 1)$ of Eq. (1) such that Eq. (1) has finite limit cycles in this neighborhood. Q.E.D.

To study orbits of Eq. (1) near equilibrium $E(1, 1, 1)$, property of eigenvalues of Eq. (1) at equilibrium $E(1, 1, 1)$ is very important. By the knowledge of linear algebra, we easily prove the following proposition.

PROPOSITION 2.3. *The necessary and sufficient conditions that the singular point $O(0, 0, 0)$ of Eqs. (2) has a negative real eigenvalue and a pair of purely imaginary eigenvalues are*

$$\det(A) = (A_{11} + A_{22} + A_{33})\, \mathrm{tr}(A) > 0,$$

here $\mathrm{tr}(A) = a_{11} + a_{22} + a_{33}$, $A_{11} = a_{22}a_{33} - a_{23}a_{32}$, $A_{22} = a_{11}a_{33} - a_{13}a_{31}$ *and* $A_{33} = a_{22}a_{11} - a_{12}a_{21}$. *Moreover the eigenvalues are* $-\mathrm{tr}(A)$ *and* $\pm\sqrt{(A_{11} + A_{22} + A_{33})}\, i$.

The proposition is very useful to discuss Hopf bifurcation in Section 3.

Next we consider dynamics near a periodic orbit Γ of Eq. (1). First we recall the definition of a real analytic function with two-variates.

Let $(x, y) \in D \subset \mathbf{R}^2$ and $(0, 0) \in D$. $f(x, y)$ is called a *real analytic function* in the neighborhood of $(0, 0)$ if there exists a convergence Taylor series expansion of $f(x, y)$ at $(0, 0)$ with all coefficients of the Taylor series expansion being real. It is obvious that the function $f(z_1, z_2)$ is a complex holomorphic (or analytic) function at a neighborhood of $(0, 0)$ in $\mathbf{C}^2$ if we substitute complex variables $(z_1, z_2) \in \mathbf{C}^2$ for real variables (x, y) of $f(x, y)$.

Suppose Eq. (1) has a periodic orbit Γ. Thus the Γ lies on the simplex Σ. Choosing a small tubular neighborhood W of Γ and a point O at Γ, we make a transversal plane Π passing through O of Γ. In Π we construct a rectangular coordinates system with the point O as the origin of coordinates. Let $D_0 = \Pi \cap W$ and define the Poincaré map $P\colon D_0 \to D_0$ by

$$P(x, y) = (f_1(x, y) - x, f_2(x, y) - y),$$

here $(f_1(x, y), f_2(x, y))$ is the coordinate of the first return point of the orbit passing through point (x, y) of Eq. (1) in D_0. According to [1], we know that the functions $f_1(x, y)$ and $f_2(x, y)$ are real analytic functions in D_0.

Obviously, $P(0, 0) = (0, 0)$ which corresponds to the periodic orbit Γ. By Lemma 2.1, we know that the orbit passing through point $(x_0, y_0) \in D_0$ of Eq. (1) is a periodic orbit if and only if $P(x_0, y_0) = (0, 0)$. Hence, in order to know if the periodic orbit Γ is isolated in the small tubular neighborhood W, we only need to discuss the property of $P(x, y)$ in D_0.

To study the property of $P(x, y)$, some related results on function of several complex variables are necessary. By $C\{z\}$, $C\{z_1, z_2\}$ we denote the rings consisting of analytic functions in a neighborhood of $0 \in \mathbf{C}$ and $(0, 0) \in \mathbf{C}^2$ respectively, i.e.,

$$C\{z\} = \left\{ f(z) : f(z) = \sum_{k=0}^{\infty} a_k z^k \text{ is a convergent power series in a neighborhood of the origin in } \mathbf{C} \right\},$$

$$C\{z_1, z_2\} = \left\{ f(z_1, z_2) : f(z_1, z_2) = \sum_{m, n=0}^{\infty} a_{mn} z_1^m z_2^n \text{ is a convergent power series in a neighborhood of the origin in } \mathbf{C}^2 \right\}.$$

A function $f(z_1, z_2) \in C\{z_1, z_2\}$ is called a *Weierstrass polynomial* in z_2 if

$$f(z_1, z_2) = z_2{}^n + \sum_{i=1}^{n} a_i(z_1)\, z_2{}^{n-i}, \qquad n \geqslant 1,$$

where $a_i(z_1) \in C\{z_1\}$ and $a_i(0) = 0$ for $i = 1, \cdots, n$.

The next theorem comes from some results of [4–6].

THEOREM 2.4. (1) *$C\{z_1, z_2\}$ is a unique decomposition integral domain;*

(2) (*Weierstrass Preparation Theorem*) *If $f(z_1, z_2) \in C\{z_1, z_2\}$ and $f(0, z_2)$ is not identically vanishing, then f can be uniquely represented as*

$$f(z_1, z_2) = u(z_1, z_2) \cdot w(z_1, z_2),$$

in a suitable neighbourhood of $(0, 0)$ where $w(z_1, z_2)$ is a Weierstrass polynomial and $u(z_1, z_2)$ is a unit in $C\{z_1, z_2\}$ (i.e., an invertible element in $C\{z_1, z_2\}$, which means $1/u(z_1, z_2) \in C\{z_1, z_2\}$, or equivalently $u(0, 0) \neq 0$).

(3) *If $f(z_1, z_2) = z_2^n + a_1(z_1) z_2^{n-1} + \cdots + a_n(z_1)$ is an irreducible Weierstrass polynomial, then there exists a disc $D = \{z_1 \in \mathbf{C}, |z_1| < \rho\}$ such that $f(z_1, z_2)$ has only simple roots for every $z_1 \neq 0$ and $z_1 \in D$. $f(z_1, z_2)$ can be represented as*

$$f(z_1, z_2) = \prod_{i=1}^{n} (z_2 - Z_i(z_1)),$$

where every $Z_i(z_1)$ is a single-valued holomorphic function on the disc D cutted along a half line (e.g., positive real axis) and each $Z_i(z_1)$ can arrives any other $Z_j(z_1)$, $i, j = 1, 2, ..., n$, by analytic extension around $z_1 = 0$. Moreover the map F: $\Delta = \{t \in \mathbf{C} : |t| < \rho^{1/n}\} \to \mathbf{C}$ defined by $F(t) = Z_i(t^n)$ is a single-valued holomorphic function. Q.E.D.

Now we give a property of real analytic functions.

PROPOSITION 2.5. *Suppose $f(x, y)$ and $g(x, y)$ are real analytic functions in the neighborhood of $(0, 0)$, and $f(0, 0) = 0$ and $g(0, 0) = 0$. Then one of the following results holds.*

(i) *$(0, 0)$ is an isolated zero of $f(x, y) = g(x, y) = 0$;*

(ii) *there exists a continous curve γ starting at $(0, 0)$ in the neighborhood of $(0, 0)$ such that $f(x, y) = g(x, y) = 0$ when $(x, y) \in \gamma$.*

Proof. Assume that $(0, 0)$ is not an isolated zero of $f(x, y) = g(x, y) = 0$. Then there exists a sequence of points (x_n, y_n) with $(x_n, y_n) \neq (0, 0)$ and $(x_n, y_n) \to (0, 0)$ as $n \to \infty$ such that $f(x_n, y_n) = g(x_n, y_n) = 0$. Without loss of generality we assume that there exist infinity many $x_n > 0$ (if necessary it suffices to rotate the rectangular coordinates system when defining the Poincaré map $P(x, y)$). In the following we only consider the subsequence (x_n, y_n) with $x_n > 0$, denoted still by (x_n, y_n) for sake of convenience.

If $f(0, y) \equiv 0$ and $g(0, y) \equiv 0$, then we can take γ as $x = 0$ and complete the proof. If $f(0, y) \equiv 0$ and $g(0, y) \not\equiv 0$, then there exists some $k \geqslant 1$ such that $f(x, y)=x^k f^*(x, y)$, where $f^*(x, y)$ is real analytic with $f^*(0, y)\not\equiv 0$ and $f^*(x_n, y_n)=0$. The same argument can be applied to the case when $f(0, y)\not\equiv 0$ and $g(0, y)\equiv 0$. Thus in the following discussion we assume that $f(0, y)\not\equiv 0$ and $g(0, y)\not\equiv 0$. Then

$$f(0, y)=b_0 y^{d_1}+b_1 y^{d_1+1}+\cdots, \qquad b_0\neq 0 \quad \text{and} \quad d_1\geqslant 1;$$
$$g(0, y)=c_0 y^{d_2}+c_1 y^{d_2+1}+\cdots, \qquad c_0\neq 0 \quad \text{and} \quad d_2\geqslant 1.$$

By Theorem 2.4(1),(2) there exists a neighborhood V of $(0, 0)\in \mathbf{C}^2$ such that

$$f(z_1, z_2)=q_1(z_1, z_2)\, f_1(z_1, z_2)\cdots f_{k_1}(z_1, z_2),$$
$$g(z_1, z_2)=q_2(z_1, z_2)\, g_1(z_1, z_2)\cdots g_{k_2}(z_1, z_2),$$

where $q_1(0, 0)\neq 0$, $q_2(0, 0)\neq 0$, $f_i(z_1, z_2)$ and $g_j(z_1, z_2)$ $(i=1, \cdots k_1, j=1, \cdots k_2)$ are irreducible Weierstress ploynomials. Thus there exist a subsequence (x_{n_i}, y_{n_i}) of (x_n, y_n) and some i_0 and j_0 such that $f_{i_0}(x_{n_i}, y_{n_i})= g_{j_0}(x_{n_i}, y_{n_i})=0$. Let

$$f_{i_0}(z_1, z_2)=z_2^n+a_1(z_1)\, z_2^{n-1}+\cdots+a_n(z_1),$$
$$g_{j_0}(z_1, z_2)=z_2^m+b_1(z_1)\, z_2^{m-1}+\cdots+b_m(z_1).$$

By $R(f, g)$ we denote the resultant of polynomials $f_{i_0}(z_1, z_2)$ and $g_{j_0}(z_1, z_2)$, and let

$$D(z_1):=R(f, g)=\det\begin{pmatrix} 1 & a_1(z_1) & \cdots & a_n(z_1) & 0 & \cdots & 0 \\ 0 & 1 & \cdots & a_{n-1}(z_1) & a_n(z_1) & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & a_1(z_1) & \cdots & a_n(z_1) \\ 1 & b_1(z_1) & \cdots & \cdots & b_m(z_1) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & b_1(z_1) & \cdots & b_m(z_1) \end{pmatrix}.$$

Then $D(z_1)$ is an analytic function in a neighborhood of the origin in $\mathbf{C}$. Since for each x_{n_i} the polynomials $f_{i_0}(x_{n_i}, z_2)$ and $g_{j_0}(x_{n_i}, z_2)$ have a common zero point $z_2=y_{n_i}$, According to knowledge of linear algebra we have $D(x_{n_i})=0$. Therefore $D(z_1)\equiv 0$ in some neighborhood of the origin in $\mathbf{C}$. Hence there exists a non-constants common factor between $f_{i_0}(z_1, z_2)$ and $g_{j_0}(z_1, z_2)$ for all z_1 lying in some neighborhood of the origin in $\mathbf{C}$.

On the other hand, according to Theorem 2.4(3) there exists a disc $D=\{z_1 \in \mathbf{C}, |z_1|<\rho\}$ such that

$$f_{i0}(z_1, z_2)=\Pi_{i=1}^{n}(z_2-\alpha_i(z_1)), \qquad g_{j0}(z_1, z_2)=\Pi_{j=1}^{m}(z_2-\beta_j(z_1)).$$

Here $\alpha_i(z_1)$ and $\beta_j(z_1)$ are single-valued holomorphic functions on the disc D cutted along positive real axis. By analytic extension around $z_1=0$ we can see that each $\alpha_i(z_1)$ can arrive any other $\alpha_j(z_1)$, $i, j=1, 2, ..., n$, and each $\beta_i(z_1)$ can arrive any other $\beta_j(z_1)$, $i, j=1, 2, ..., m$. Since there exists a non-constants common factor between $f_{i0}(z_1, z_2)$ and $g_{j0}(z_1, z_2)$ for all z_1 lying in some neighborhood of $z_1=0$, there exist i_1 and j_1 such that $\alpha_{i_1}(z_1)\equiv\beta_{j_1}(z_1)$ for all z_1 in D as we choose a suitable ρ. Therefore $m=n$ and $f_{i0}(z_1, z_2)\equiv g_{j0}(z_1, z_2)$ for all $(z_1, z_2)\in D$. Because $f(x_{n_i}, y_{n_i})=0$ there exists an infinite subsequence (x_{t_i}, y_{t_i}) of (x_{n_i}, y_{n_i}) and some $1\leqslant k\leqslant n$ such that $\alpha_k(x_{t_i})=y_{t_i}$.

Next we will show that $\alpha_k(z_1)$ takes real values as $z_1\in(0, \rho)$. To do this let $F(z_1)=\alpha_k(z_1{}^n)$. Theorem 2.4(3) tells us that $F(z_1)$ is a single-valued holomorphic function in $\Delta=\{t\in\mathbf{C} : |t|<\rho^{1/n}\}$. Let

$$F(x+iy)=U(x, y)+iV(x, y), \qquad \alpha_k(x+iy)=u(x, y)+iv(x, y).$$

Note that $V(x, 0)$ is real analytic when x is in the interval $(-\rho^{1/n}, \ \rho^{1/n})$, and $V(x_{t_i}^{1/n}, 0)=0$. Thus $V(x, 0)\equiv 0$ for every x in the interval $(-\rho^{1/n}, \rho^{1/n})$. Furthermore $V(x, 0)=v(x^n, 0)$ when x is in the interval $(0, \ \rho^{1/n})$. Thus $v(x, 0)\equiv 0$ as x in the interval $(0, \ \rho)$, i.e., $\alpha_k(z_1)$ is real when z_1 is in $(0, \rho)$. Taking γ: $y=\alpha_k(x)$, $x\in(0, \rho)$, we complete our proof. Q.E.D.

Remark 2.1. Note that when taking $f(x, y)=g(x, y)$ in the Proposition 2.5, an important property on the zero points of the two-variables real analytic function is given.

Now we state and prove the main result in this section.

THEOREM 2.6. *If Γ is a periodic orbit of Eq.* (1) *in $\mathbf{R}^3_+$, then there exists a small neighborhood of Γ in $\mathbf{R}^3_+$ such that there exist only finite limit cycles of Eq.* (1) *in this neighborhood, i.e. infinitely many limit cycles can not accumulate on Γ.*

Proof. As stated above, we can choose a small tubular neighborhood W of Γ and a transversal plane Π. Let $D_0=\Pi\cap W$ and define a Poincaré map P: $D_0\to D_0$ by

$$P(x, y)=(f_1(x, y)-x, f_2(x, y)-y), \qquad P(0, 0)=(0, 0),$$

here $f_1(x, y)$ and $f_2(x, y)$ are real analytic functions in D_0.

Let $f(x, y) = f_1(x, y) - x$ and $g(x, y) = f_2(x, y) - y$. Then $f(x, y)$ and $g(x, y)$ are real analytic functions with $f(0, 0) = 0$ and $g(0, 0) = 0$. By Proposition 2.5 there exists a small neighborhood D_1 of the origin in D_0 such that either the set $\{(x, y)\colon\ P(x, y) = (0, 0)\}$ is a singleton $\{(0, 0)\}$ in D_1 or there exists a continuous curve $\gamma\colon y = h(x)$ with $h(0) = 0$ such that $P(x, h(x)) = (0, 0)$ in D_1. Note that any orbit passing through the point in the set $\{(x, y) : P(x, y) = (0, 0)\}$ is a periodic orbit, and every non-trivial periodic orbit of Eq. (1) is in the carrying simplex Σ of Eq. (1) by Lemma 2.1. Hence the set $\{(x, y) : P(x, y) = (0, 0)$, and $(x, y) \in D_1\} \subset \Sigma$. Therefore there exists a neighborhood Λ of Γ on Σ such that either the Γ is the unique periodic orbit of Eq. (1) or every orbit of Eq. (1) is a periodic orbit in Λ.

Q.E.D.

Summarizing the above Theorem 2.2 and Theorem 2.6, it follows the next theorem.

THEOREM 2.7. *If Eq.* (1) *has not any heteroclinic polycycles, then the number of limit cycles of Eq.* (1) *is finite.*

Remark 2.2. The number of limit cycles of Eq. (1) is finite except classes 26 and 27 in [13].

Remark 2.3. It is a very interesting problem to prove whether or not the number of limit cycles of Eq. (1) is finite in the small neighborhood of heteroclinic polycycle.

Remark 2.4. It is still a open question that at most how many limit cycles of Eq. (1) there are if Eq. (1) has no heteroclinic polycycle.

3. BIFURCATIONS FOR A 3D COMPETITIVE LOTKA–VOLTERRA SYSTEM

In this section we investigate bifurcations of a 3D competitive Lotka–Volterra system with two small parameters. The system always has a heteroclinic polycycles with three saddles. We prove that limit cycles of this system can be generated by Hopf bifurcation and heteroclinic bifurcation.

Consider system

$$\begin{aligned}
\dot{x}_1 &= x_1[(1 - x_1) + (1 - x_2) + (1 - x_3)] \\
\dot{x}_2 &= x_2[(1 - x_1) + (1 - x_2) + 2(1 - x_3)] \\
\dot{x}_3 &= x_3[(\tfrac{13}{5} + \varepsilon_1)(1 - x_1) + (\tfrac{8}{5} + \varepsilon_2)(1 - x_2) + 3(1 - x_3)],
\end{aligned} \tag{6}$$

where ε_1 and ε_2 are small parameters. Write $\varepsilon = \mathrm{col}(\varepsilon_1, \varepsilon_2)$.

When $\varepsilon = 0$, Eqs. (6) has an interior nondegenarate singular point $E(1,1,1)$ which is a center, the three hyperbolic saddle points $R_1(3,0,0)$, $R_2(0,4,0)$, $R_3(0,0,\frac{12}{5})$ and an unstable node $O(0,0,0)$. And there exists a carrying simplex Σ_0 that is a plane, and the boundary of the plane Σ_0 is heteroclinic polycycle $R_1 R_2 R_3$ (see Fig. 1).

First we consider the singular point $E(1,1,1)$ of Eqs. (6) as $0 < \|\varepsilon\| \ll 1$. Proposition 2.3 tells that the necessary condition of occurrence of Hopf bifurcation at the singular point $E(1,1,1)$ for Eqs. (6) is

$$\det(A) = \operatorname{tr}(A)(A_{11} + A_{22} + A_{33}) > 0. \tag{7}$$

Here

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ \frac{13}{5} + \varepsilon_1 & \frac{8}{5} + \varepsilon_2 & 3 \end{pmatrix}$$

and $0 < \|\varepsilon\| \ll 1$. By calculating we obtain that Eq. (7) is equivalent to $3\varepsilon_2 + 2\varepsilon_1 = 0$. And when $3\varepsilon_2 + 2\varepsilon_1 = 0$ the eigenvalues of Eqs. (6) at $E(1,1,1)$ are $\lambda_1 = -5$, $\lambda_{2,3} = \pm i\sqrt{1/5 - \varepsilon_1}$. In addition it is easy to check

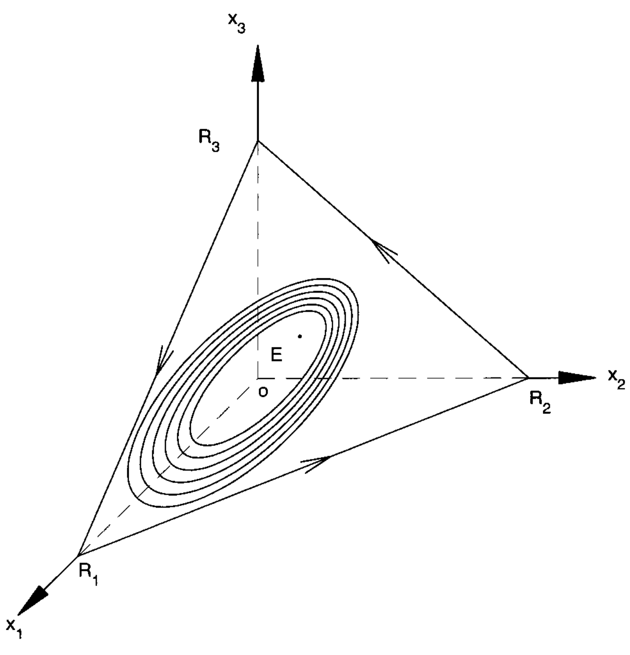

FIG. 1. The phase portrait of Eqs. (6) as $\varepsilon = 0$.

that the singular point $E(1, 1, 1)$ is an unstable focus when $3\varepsilon_2 + 2\varepsilon_1 > 0$ and a stable focus when $3\varepsilon_2 + 2\varepsilon_1 < 0$. Next we discuss stability of $E(1, 1, 1)$ when $3\varepsilon_2 + 2\varepsilon_1 = 0$, $0 < \|\varepsilon\| \ll 1$.

Let $y_i = 1 - x_i$, $i = 1, 2, 3$, and set $y = Tz$, $t = \tau/\sqrt{1/5 - \varepsilon_1}$. Then Eqs. (6) are transferred to

$$\begin{aligned} \dot{z}_1 &= -\frac{5}{\sqrt{1/5 - \varepsilon_1}} z_1 + O(z^2) \\ \dot{z}_2 &= z_3 + O(z^2) \\ \dot{z}_3 &= -z_2 + O(z^2), \end{aligned} \tag{8}$$

where $y = \mathrm{col}(y_1, y_2, y_3)$, $z = \mathrm{col}(z_1, z_2, z_3)$, and

$$T = \begin{pmatrix} 6 & -1 & \sqrt{\frac{1}{5} - \varepsilon_1} \\ 9 & 1 & 2\sqrt{\frac{1}{5} - \varepsilon_1} \\ 10 & \sqrt{\frac{1}{5} - \varepsilon_1} & -2\sqrt{\frac{1}{5} - \varepsilon_1} \end{pmatrix}.$$

According to the method of Bernd Aulbach in [3], we subject the given Eqs. (8) to the tranformation $z_1 = rw$, $z_2 = r\cos\theta$, $z_3 = r\sin\theta$. Here r, θ and $w \in \mathbf{R}$, which yields a system of the form

$$\begin{aligned} \dot{r} &= rR(\theta, r, w) \\ \dot{\theta} &= 1 + \Theta(\theta, r, w) \\ \dot{w} &= -\frac{5}{\sqrt{\frac{1}{5} - \varepsilon_1}} w + \Phi(\theta)\, r + W(\theta, r, w), \end{aligned} \tag{9}$$

where R, Θ, W are analytic functions which can be represented as power series in r, w around $(r, w) = (0, 0) \in \mathbf{R} \times \mathbf{R}$. The coefficients of those series as well as the function $\Phi(\theta)$ are analytic and 2π-periodic on $\mathbf{R}$. For each $\theta \in [0, 2\pi]$ these power series converge for $|r| \leqslant \delta$, $|w| \leqslant \delta$, here δ is a positive constant independent of θ. The Taylor series of R and Θ (around $(r, w) = (0, 0)$) begin with the first order terms in r, w, whereas the Taylor series of W begins with the second order terms in r, w. Because of the particular form of the θ-equation it is possible to eliminate the independent variable τ from Eqs. (9). This leads to a 2-dimensional system of the form

$$\begin{aligned} \frac{dr}{d\theta} &= r\bar{R}(\theta, r, w) \\ \frac{dw}{d\theta} &= -\frac{5}{\sqrt{\frac{1}{5} - \varepsilon_1}} w + \Phi(\theta)\, r + \bar{W}(\theta, r, w), \end{aligned} \tag{10}$$

where the functions on the right-hand side can be written

$$r\bar{R}(\theta, r, w) = \sum_{i\geqslant 1, i+j\geqslant 2} R_{i,j}(\theta)\, r^i w^j$$
$$\bar{W}(\theta, r, w) = \sum_{i+j\geqslant 2} W_{i,j}(\theta)\, r^i w^j. \tag{11}$$

The coefficients $R_{i,j}(\theta)$ and $W_{i,j}(\theta)$ are analytic and 2π-periodic functions. The series in (11) are absolutely convergent in a polycylinder $|r| \leqslant k_1$, $|w| \leqslant k_1$, $\theta \in [0, 2\pi]$ for some positive k_1 and its coefficients satisfy the Cauchy estimates

$$|R_{i,j}| \leqslant \frac{M}{k_1^{i+j}}, \qquad |W_{ij}| \leqslant \frac{M}{k_1^{i+j}},$$

where M is the maximum of $|r\bar{R}(\theta, r, w)|$ and $|\bar{W}(\theta, r, w)|$ on the above polycyclinder.

Let $r = \sum_{i=1}^{\infty} \alpha_i(\theta)\, c^i$, $w = \sum_{i=1}^{\infty} \beta_i(\theta)\, c^i$ be a solution of (10) with initial value $(\theta, r, w) = (0, c, c)$, $0 \leqslant \|c\| \ll 1$. Substituting $\sum_{i=1}^{\infty} \alpha_i(\theta)\, c^i$ and $\sum_{i=1}^{\infty} \beta_i(\theta)\, c^i$ for r and w respectively in Eqs. (10) yields that $\alpha_i(\theta)$ and $\beta_i(\theta)$ satisfies the scalar differential equations with initial value $\alpha_1(0) = \beta_1(0) = 1$ and $\alpha_i(0) = \beta_i(0) = 0$ for $i \geqslant 2$. Solving the scalar differential equations, we find the singular point $E(1, 1, 1)$ is an unstable focus with positive first focal value at curve $3\varepsilon_2 + 2\varepsilon_1 = 0$ by Theorem 4.1 in [3]. On the other hand, the singular point $E(1, 1, 1)$ is unstable when the parameters $(\varepsilon_1, \varepsilon_2)$ lie in the range $3\varepsilon_2 + 2\varepsilon_1 > 0$ and stable when the parameters $(\varepsilon_1, \varepsilon_2)$ lie in the range $3\varepsilon_2 + 2\varepsilon_1 < 0$. By Hopf bifurcation theorem (subcritical case) in [4], we obtain that an unstable limit cycle of Eqs. (6) is born in a small neighborhood of $E(1, 1, 1)$, and the curve $3\varepsilon_2 + 2\varepsilon_1 = 0$ is a Hopf bifurcation curve (see Fig. 2).

Now we consider stability of heteroclinic polycycle. When parameter $\varepsilon \neq 0$, the heteroclinic polycycle $R_1 R_2 R_3$ is changed to heteroclinic polycycle $R_1^* R_2^* R_3^*$. The saddles are $R_1^*(3, 0, 0)$, $R_2^*(0, 4, 0)$ and $R_3^*(0, 0, \frac{12}{5} + \frac{1}{3}(\varepsilon_1 + \varepsilon_2))$. The external eigenvalues at equilibrium R_i^*, $i = 1, 2, 3$ in the direction j are given by $\lambda_{12} = 1$, $\lambda_{23} = \frac{4}{5} + \varepsilon_1 - 3\varepsilon_2$, $\lambda_{31} = 3/5 - (\varepsilon_1 + \varepsilon_2)/3$, $\lambda_{21} = -1$, $\lambda_{13} = -\frac{3}{5} - 2\varepsilon_1 + \varepsilon_2$, $\lambda_{32} = -\frac{4}{5} - 2(\varepsilon_1 + \varepsilon_2)$.

In order to test the stability of heteroclinic polycycle $R_1^* R_2^* R_3^*$, we denote

$$p = \lambda_{12}\lambda_{23}\lambda_{31} + \lambda_{21}\lambda_{13}\lambda_{32} = (\varepsilon_1 + \varepsilon_2)\left(-\frac{37}{15} + 3\varepsilon_2 - \frac{13\varepsilon_1}{3}\right).$$

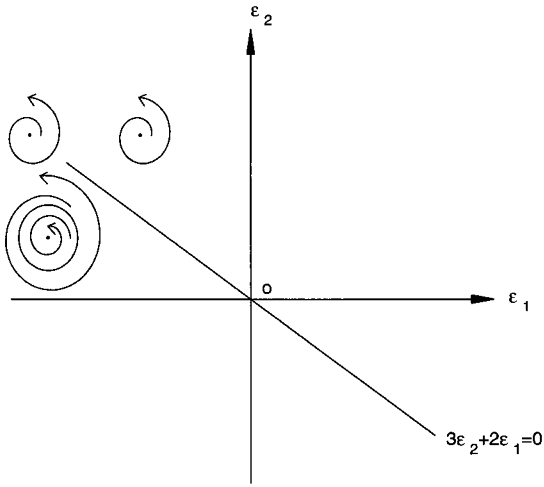

FIG. 2. The Hopf bifurcation diagram and the corresponding phase portraits near the equilibrium E of Eqs. (6).

The results in [7] tell that the heteroclinic polycycle is unstable when $p > 0$ and the heteroclinic polycycle is stable when $p < 0$. However, when $p = 0$ we cannot directly decide the stability of heteroclinic polycycle. The heteroclinic bifurcation curve is

$$\varepsilon_1 + \varepsilon_2 = 0, \qquad \text{as } 0 < \|\varepsilon\| \ll 1.$$

The heteroclinic polycycle is unstable when $\varepsilon_1 + \varepsilon_2 < 0$ and stable when $\varepsilon_1 + \varepsilon_2 > 0$.

We note that the Poincaré–Bendixson theorem holds in the carrying complex Σ_ε of Eqs. (6). Hence Eqs. (6) have at least two limit cycles in Σ_ε when $(\varepsilon_1, \varepsilon_2)$ belongs to the parameter range *II*, $II = \{\varepsilon : 0 < \|\varepsilon\| \ll 1, \varepsilon_1 + \varepsilon_2 < 0 \text{ and } 2\varepsilon_1 + 3\varepsilon_2 < 0\}$, and Eqs. (6) are persistence, however, Eqs. (6) have at least one limit cycle in Σ_ε when $(\varepsilon_1, \varepsilon_2)$ belongs to the parameter range *III*, $III = \{\varepsilon : 0 < \|\varepsilon\| \ll 1, \varepsilon_1 + \varepsilon_2 > 0 \text{ and } 2\varepsilon_1 + 3\varepsilon_2 < 0\}$, and the heteroclinic polycycle is stable, which means Eqs. (6) are not persistence. Moreover the Eqs. (6) have at least one limit cycle in Σ_ε when $(\varepsilon_1, \varepsilon_2)$ is in the range *I*, $I = \{\varepsilon : 0 < |\varepsilon| \ll 1, \varepsilon_1 + \varepsilon_2 < 0 \text{ and } 2\varepsilon_1 + 3\varepsilon_2 > 0\}$ (see Fig. 3).

Remark 3.1. We guess that Eqs. (6) have not any periodic orbits in Σ_ε when $(\varepsilon_1, \varepsilon_2)$ is in the range *IV*, $IV = \{\varepsilon : 0 < |\varepsilon| \ll 1, \varepsilon_1 + \varepsilon_2 > 0 \text{ and } 2\varepsilon_1 + 3\varepsilon_2 > 0\}$. If Eqs. (6) have hyperbolic periodic orbits in this parameter range, then Eqs. (6) have at least three periodic orbits in some parameter range.

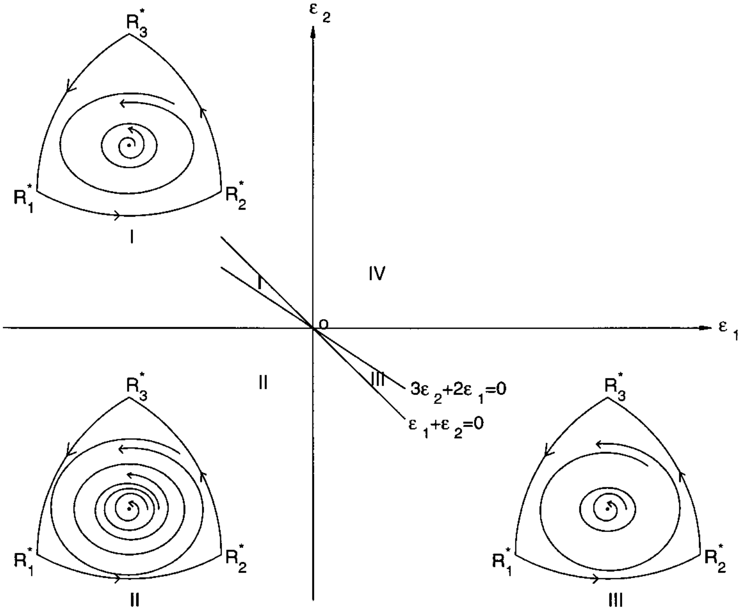

FIG. 3. The bifurcation diagram and the corresponding phase portraits with limit cycles of Eqs. (6).

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