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Self-similar structure on the intersection of middle-$(1-2\beta)$ Cantor sets with $\beta \in (1/3, 1/2)$

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Abstract

Let Γ_β be the middle-$(1-2\beta)$ Cantor set with $\beta \in (1/3, 1/2)$. We give all real numbers t with unique $\{-1, 0, 1\}$-code such that the intersections $\Gamma_\beta \cap (\Gamma_\beta + t)$ are self-similar sets. For a given $\beta \in (1/3, 1/2)$, a criterion is obtained to check whether or not a $t \in [-1, 1]$ has the unique $\{-1, 0, 1\}$-code from both geometric and algebraic views.

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1. Introduction

Let $\beta \in (0, 1/2)$ and $F_k(x) = \beta x + k(1-\beta)$, $k = -1, 0, 1$. The middle-$(1-2\beta)$ Cantor set $\Gamma_\beta \subset [0, 1]$ is a straightforward generalization of the classical middle third Cantor set which is defined as the invariant nonempty compact set under maps F_0 and F_1:

$$\Gamma_\beta = F_0(\Gamma_\beta) \cap F_1(\Gamma_\beta). \qquad (1)$$

In this case Γ_β is called a self-similar set generated by the iterated function system $\{F_0(x), F_1(x)\}$.

In general, a nonempty compact set $P \subseteq \mathbb{R}$ is said to be a *self-similar set* if there exists a finite collection of linear mappings $f_j(x) = r_j x + b_j$ with $0 < |r_j| < 1, b_j \in \mathbb{R}$ for $j = 1, \ldots, N$ such that

$$P = \bigcup_{j=1}^{N} f_j(P).$$

The collection $\{f_j(x)\}_{j=1}^N$, also called an *iterated function system* (IFS), is said to satisfy the *strong separation condition* (SSC) if $f_j(P), 1 \leqslant j \leqslant N$ are pairwise disjoint, and to satisfy the *open set condition* (OSC) if there exists a nonempty bounded open set O such that

$\bigcup_{j=1}^{N} f_j(O) \subseteq O$ with disjoint union on the left side. One can refer to [3, 4] for some more properties of self-similar sets. Clearly, $\{F_0(x), F_1(x)\}$ satisfy the SSC and OSC. $\Gamma_{1/3}$ is the classical middle third Cantor set.

In the past two decades, intersection of Cantor sets has been the subject of several studies [1, 5, 6–11, 12, 14–16]. An algebraic approach was used in [11] to determine the Hausdorff dimension of $\Gamma_\beta \cap (\Gamma_\beta + t)$ with $\beta \in (0, 1/3]$ and $t \in \Gamma_\beta - \Gamma_\beta$. When $\beta \in (1/3, 1/2)$ the intersections $\Gamma_\beta \cap (\Gamma_\beta + t)$ present a very complicated geometric structure and an algorithm was given in [16] for calculating their Hausdorff dimensions for some $\beta \in (1/3, 1/2)$.

Recently, Deng *et al* [2] obtained a delicate description on the structure of $\Gamma_{1/3} \cap (\Gamma_{1/3} + t)$, for $t \in \Gamma_{1/3} - \Gamma_{1/3} = [-1, 1]$. More exactly, they gave a necessary and sufficient characterization for those t such that the corresponding $\Gamma_{1/3} \cap (\Gamma_{1/3} + t)$ are self-similar sets. Motivated by [2], Li *et al* [13] extended the results in [2] to homogeneous symmetric Cantor sets E which are generated by a finite collection of contraction similitudes with same ratios. Approaches in [2, 13] greatly depend on the property that $\Gamma_{1/3} - \Gamma_{1/3}$ (or $E - E$) satisfies the OSC.

It is well known that points in Γ_β and $\Gamma_\beta - \Gamma_\beta$ (easy to check that the compact set $\Gamma_\beta - \Gamma_\beta$ is invariant under the collection $\{F_{-1}, F_0, F_1\}$) can be encoded by digits from $\{0, 1\}$ and from $\{-1, 0, 1\}$, respectively. This is done by the so-called coding mapping $\Pi : \{0, 1\}^{\mathbb{N}} \to \Gamma_\beta$ (or $\{-1, 0, 1\}^{\mathbb{N}} \to \Gamma_\beta - \Gamma_\beta$):

$$\Pi(J) = \sum_{i=1}^{\infty} j_k \beta^{k-1}(1 - \beta),$$

for $J = (j_k)_{k=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ (or $J = (j_k)_{k=1}^{\infty} \in \{-1, 0, 1\}^{\mathbb{N}}$). For each $x \in \Gamma_\beta$ (or $x \in \Gamma_\beta - \Gamma_\beta$), the $J \in \{0, 1\}^{\mathbb{N}}$ (or $J \in \{-1, 0, 1\}^{\mathbb{N}}$) satisfying $\Pi(J) = x$ is called a $\{0, 1\}$-code (or a $\{-1, 0, 1\}$-code) of x. Obviously, each $x \in \Gamma_\beta$ has a unique $\{0, 1\}$-code. However, the $\{-1, 0, 1\}$-code of $x \in \Gamma_\beta - \Gamma_\beta$ may not be unique. In fact, about codes of points in $\Gamma_\beta - \Gamma_\beta$ we have that

(I) each point of $\Gamma_\beta - \Gamma_\beta$ has a unique $\{-1, 0, 1\}$-code when $\beta \in (0, 1/3)$;

(II) all but a countable number of points of $\Gamma_{1/3} - \Gamma_{1/3} = [-1, 1]$ have a unique $\{-1, 0, 1\}$-code, and each exceptional point has two $\{-1, 0, 1\}$-codes;

(III) when $\beta \in (1/3, 1/2)$, $\Gamma_\beta - \Gamma_\beta = [-1, 1]$ and $\{F_{-1}, F_0, F_1\}$ does not satisfy the OSC and so a point of $\Gamma_\beta - \Gamma_\beta$ may have an infinite number of $\{-1, 0, 1\}$−codes.

For $\beta \in (0, 1/3)$, $\Gamma_\beta \cap (\Gamma_\beta + t)$ can be algebraically characterized as (cf [2, 11])

$$\begin{aligned} \Gamma_\beta \cap (\Gamma_\beta + t) &= \left\{ \sum_{i=1}^{\infty} x_i \beta^{i-1}(1 - \beta) : x_i \in \{0, 1\} \cap (\{0, 1\} + t_i) \right\} \\ &= \Pi\left(\prod_{i=1}^{\infty} \{0, 1\} \cap (\{0, 1\} + t_i) \right). \end{aligned} \tag{2}$$

where $(t_i)_{i=1}^{\infty} \in \{-1, 0, 1\}^{\mathbb{N}}$ is the unique $\{-1, 0, 1\}$-code of $t \in \Gamma_\beta - \Gamma_\beta$.

For $\beta = 1/3$, $\Gamma_\beta \cap (\Gamma_\beta + t)$ is either of form (2) when t has a unique $\{-1, 0, 1\}$-code or a finite set when t has two $\{-1, 0, 1\}$-codes.

When $\beta \in (1/3, 1/2)$ one can check that for each $t \in \Gamma_\beta - \Gamma_\beta = [-1, 1]$

$$\Gamma_\beta \cap (\Gamma_\beta + t) = \bigcup_{\tilde{t}} \Pi\left(\prod_{k=1}^{\infty} D_{k,\tilde{t}} \right), \tag{3}$$

where the union is taken over all $\{-1, 0, 1\}$-codes of t, and $D_{k,\tilde{t}} = \{0, 1\} \cap (\{0, 1\} + t_k)$ for an appointed $\{-1, 0, 1\}$-code $\tilde{t} = (t_k)_{k=1}^{\infty}$ of t. Thus $\Gamma_\beta \cap (\Gamma_\beta + t)$ is of the form (2) if and only if t has a unique $\{-1, 0, 1\}$-code.

To state results in this paper, we need some notations. Let $\{0, 1\}^* = \bigcup_{n\geqslant 0}\{0, 1\}^n$ be the set of all finite words, where $\{0, 1\}^0$ contains only empty word $\emptyset$. For $I \in \{0, 1\}^m$ and $J \in \{0, 1\}^n$, let $IJ \in \{0, 1\}^{m+n}$ be the concatenation of I and J. For $I \in \{0, 1\}^*$, let $|I|$ denote its length and $\bar{I} := III \cdots \in \{0, 1\}^{\mathbb{N}}$, the infinite repeating of I. $K \in \{0, 1\}^{\mathbb{N}}$ is called *strong p-periodic* (or simply, strong periodic) if there exist two words $I, J \in \{0, 1\}^p$ such that $K = I\bar{J}$ and $I \preccurlyeq J$, where $I \preccurlyeq J$ means $i_n \leqslant j_n$, $1 \leqslant n \leqslant p$ for $I = i_1 \cdots i_p$, $J = j_1 \cdots j_p$. For a $J = (j_i)_{i=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ and $k \in \mathbb{N}$, let $J|k = (j_i)_{i=1}^{k} \in \{0, 1\}^k$. For $I, J \in \{0, 1\}^{\mathbb{N}}$, we say $I \preccurlyeq J$ if $I|n \preccurlyeq J|n$ for all $n \in \mathbb{N}$.

For $J = (j_i)_{i=1}^{k} \in \{0, 1\}^k$, denote

$$F_J = F_{j_1} \circ F_{j_2} \circ \cdots \circ F_{j_k}$$

with $F_\emptyset = I$, the identity map on $\mathbb{R}$. Thus (recall Γ_β is defined as in (1))

$$\Gamma_\beta = \bigcap_{k=1}^{\infty} \bigcup_{J\in\{0,1\}^k} F_J([0, 1]).$$

Let $S_0(x) = \beta x + t(1 - \beta)$, $S_1(x) = \beta x + (t + 1)(1 - \beta)$ and $S_J = S_{j_1} \circ S_{j_2} \circ \cdots \circ S_{j_k}$ for $J = (j_i)_{i=1}^{k} \in \{0, 1\}^k$. Similarly $S_\emptyset$ denotes the identity map. Then

$$S_J(t) = F_J(0) + t \qquad \text{for } J \in \{0, 1\}^*$$

and

$$\Gamma_\beta + t = S_0(\Gamma_\beta + t) \cup S_1(\Gamma_\beta + t) = \bigcap_{k=1}^{\infty} \bigcup_{J\in\{0,1\}^k} S_J([t, 1 + t]).$$

We call the sets $F_J([0, 1])$, $S_J([t, 1 + t])$ for $J \in \{0, 1\}^k$ the *k-level components* of Γ_β and $\Gamma_\beta + t$, respectively. Clearly, $[0, 1]$, $[t, 1 + t]$ are the 0-level components. For $J \in \{0, 1\}^k$ with $k \geqslant 0$, the *neighbourhood* of $F_J([0, 1])$ with respect to the k-level components of $\Gamma_\beta + t$ is defined as

$$N_{\Gamma_\beta,k}(F_J([0, 1])) = \{S_I([t, 1 + t]) : I \in \{0, 1\}^k,\ F_J([0, 1]) \cap S_I([t, 1 + t]) \neq \emptyset\}.$$

Note that for some $F_J([0, 1])$ with $J \in \{0, 1\}^k$, $N_{\Gamma_\beta,k}(F_J([0, 1]))$ may be empty. So these components do not meet any k-level components of $\Gamma_\beta + t$ and then they have no contribution to $\Gamma_\beta \cap (\Gamma_\beta + t)$. Let

$$\Lambda_k = \{J \in \{0, 1\}^k : N_{\Gamma_\beta,k}(F_J([0, 1])) \neq \emptyset\}.$$

One can readily check that

$$\Gamma_\beta \cap (\Gamma_\beta + t) = \bigcap_{k=1}^{\infty} \bigcup_{J\in\Lambda_k} F_J([0, 1]).$$

Thus, Λ_k with $k \in \mathbb{N} \cup \{0\}$ geometrically characterizes $\Gamma_\beta \cap (\Gamma_\beta + t)$. Let

$$M_{\Gamma_\beta,k} = \{F_J([0, 1]) : J \in \Lambda_k\} \qquad \text{and} \qquad M_{\Gamma_\beta} = \left\{F_J([0, 1]) : J \in \bigcup_{k\geqslant 0} \Lambda_k\right\}.$$

A graph with vertex set M_{Γ_β} is constructed as follows. For $I \in \Lambda_k$ and $J \in \Lambda_{k+1}$, if there exists an $\ell \in \{0, 1\}$ such that $J = I\ell$, then we connect a directed edge ℓ from $F_I([0, 1])$ to $F_J([0, 1])$. In this case, $F_I([0, 1])$ is called the parent of $F_J([0, 1])$ and $F_J([0, 1])$ an offspring of $F_I([0,1])$. Clearly each k-level component in $M_{\Gamma_\beta,k}(k \geqslant 1)$ of Γ_β has only one parent, but may have no offspring. The *reduced graph* is then constructed by removing those vertexes having no offspring and the edges going to them. The reduced graph determines a subset Λ of $\{0,1\}^{\mathbb{N}}$,

$$\Lambda = \{J = (j_i)_{i=1}^{\infty} \in \{0,1\}^{\mathbb{N}} : J|k \in \Lambda_k \text{ for any } k \in \mathbb{N}\}. \tag{4}$$

Therefore $\Gamma_\beta \cap (\Gamma_\beta + t) = \Pi(\Lambda)$.

Let $\#A$ denote the cardinality of set A. The following proposition comes from [16] which describes the geometric criterion for $\Gamma_\beta \cap (\Gamma_\beta + t)$ to be of the form (2), or equivalently, for t to have a unique $\{-1,0,1\}$-code.

Proposition 1.1 ([16], theorem 3.5). *Let Λ be defined as in (4). Let $\beta \in (1/3, 1/2)$. If $\#N_{\Gamma_\beta,k}(F_J([0,1])) \leqslant 1$ for all $J \in \{0,1\}^k$, $k \in \mathbb{N} \cup \{0\}$, then*

$$\Lambda = \prod_{i=1}^{\infty} D_i,$$

where $\emptyset \neq D_i \subseteq \{0,1\}$ consists of edges connecting a parent in $M_{\Gamma_\beta,i-1}$ to its offspring in $M_{\Gamma_\beta,i}$ in the reduced graph.

Assume that $\Gamma_\beta \cap (\Gamma_\beta + t) = \Pi(\prod_{i=1}^{\infty} D_i)$. From (3) it follows that t has the unique $\{-1,0,1\}$-code $(t_i)_{i=1}^{\infty}$ and so $D_i = \{0,1\} \cap (\{0,1\} + t_i)$. We will study its translation. Let

$$\Gamma = \Gamma_\beta \cap (\Gamma_\beta + t) - \gamma = \Pi\left(\prod_{i=1}^{\infty}(D_i - \gamma_i)\right), \tag{5}$$

where $\gamma = \min\{x | x \in \Gamma_\beta \cap (\Gamma_\beta + t)\} = \sum_{k=1}^{\infty} \gamma_k \beta^{k-1}(1-\beta)$ with $\gamma_k = \min\{z : z \in D_k\}$. Let

$$\gamma^* = (\gamma_k^*)_{k=1}^{\infty} \qquad \text{where } \gamma_k^* = \max\{z : z \in D_k - \gamma_k\} = \max_{z \in D_k} z - \min_{z \in D_k} z.$$

Then $\gamma_k^* = 1 - |t_k|$ and

$$\prod_{i=1}^{\infty}(D_i - \gamma_i) = \{(x_i)_{i=1}^{\infty} \in \{0,1\}^{\mathbb{N}} : x_k \leqslant \gamma_k^* \text{ for all } k \in \mathbb{N}\}. \tag{6}$$

The set Γ in (5) can be algebraically dealt with more advantageously than $\Gamma_\beta \cap (\Gamma_\beta + t)$ since Γ has the following properties.

(P1) $0 \in \Gamma$. If $\sum_{k=1}^{\infty} x_k \beta^{k-1}(1-\beta) \in \Gamma$, then $\sum_{k=1}^{\infty} y_k \beta^{k-1}(1-\beta) \in \Gamma$ for all $(y_k)_{k=1}^{\infty} \preccurlyeq (x_k)_{k=1}^{\infty}$.

(P2) Γ is centrally symmetric, i.e. $\Gamma = \Pi(\gamma^*) - \Gamma$. Thus, when Γ is generated by an IFS, say $f_i(x) = r_i x + b_i$, $i = 1, 2, \ldots, N$, one can require all $r_i > 0$. Otherwise, one can replace $f_i(x)$ by $f_i^*(x) = -r_i x + r_i \Pi(\gamma^*) + b_i$ if $r_i < 0$. This can be seen by (cf [2, 13])

$$f_i^*(\Gamma) = -r_i \Gamma + r_i \Pi(\gamma^*) + b_i = r_i(\Pi(\gamma^*) - \Gamma) + b_i = r_i \Gamma + b_i = f_i(\Gamma).$$

Furthermore, one can assume $0 = b_1 \leqslant b_2 \leqslant \cdots \leqslant b_N$ since $0 \in \Gamma$.

The first theorem in this paper is to answer when the intersections $\Gamma_\beta \cap (\Gamma_\beta + t)$ have self-similar structure for the case of t having a unique $\{-1,0,1\}$-code. Its proof is done along with the idea in [2]. The properties (P1) and (P2) play a very important role. As to the case of t having multiple $\{-1,0,1\}$-codes, $\Gamma_\beta \cap (\Gamma_\beta + t)$ is of the form (3) and seems not to be a self-similar set. But we cannot prove it.

Theorem 1.2. *Let $\beta \in (1/3, 1/2)$. Suppose that $t \in (\Gamma_\beta - \Gamma_\beta) = [-1,1]$ has a unique $\{-1,0,1\}$-code $(t_k)_{k=1}^{\infty}$. Then $\Gamma_\beta \cap (\Gamma_\beta + t)$ is a self-similar set if and only if $(1 - |t_k|)_{k=1}^{\infty}$ is strong periodic.*

For a $\beta \in (1/3, 1/2)$ we denote by $\mathcal{B}_\beta$ the set of $t \in [-1, 1]$ which have a unique $\{-1, 0, 1\}$-code. Clearly, for all $\beta \in (1/3, 1/2)$, $\mathcal{B}_\beta$ contains $-1, 0$ and 1. The following theorem gives a way to check whether or not a t belongs to $\mathcal{B}_\beta$ from both geometric and algebraic views.

Theorem 1.3. *Let $\beta \in (1/3, 1/2)$ and $t \in \Gamma_\beta - \Gamma_\beta = [-1, 1]$. Then the following three statements are equivalent.*

(i) $\#N_{\Gamma_\beta,k}(F_J([0, 1])) \leqslant 1$ for all $k \in \mathbb{N} \cup \{0\}$ and all $J \in \{0, 1\}^k$;
(ii) t has a unique $\{-1, 0, 1\}$-code;
(iii) there exists a $\{-1, 0, 1\}$-code $(t_i)_{i=1}^\infty$ of t such that for each $k \in \mathbb{N}$

$$\begin{cases} \left| y_k\beta^{k-1}(1-\beta) - \sum_{n=k}^{\infty} t_n\beta^{n-1}(1-\beta) \right| > \beta^k, & t_k = 0, \\ y_k\beta^{k-1}(1-\beta) - \sum_{n=k}^{\infty} t_n\beta^{n-1}(1-\beta) < -\beta^k, & t_k = 1, \\ y_k\beta^{k-1}(1-\beta) - \sum_{n=k}^{\infty} t_n\beta^{n-1}(1-\beta) > \beta^k, & t_k = -1, \end{cases} \tag{7}$$

where $y_k = \pm 1$ if $t_k = 0$ and $y_k = 0$ if $t_k = \pm 1$.

We now apply theorems 1.2 and 1.3 for a simple example. Some more complicated examples can be constructed in a similar way.

Example 1.4. *$\Gamma_\beta \cap (\Gamma_\beta + t)$ are self-similar sets for all $t \in \{\pm(1-\beta)\beta^\ell, \pm(1-\beta)(1+\beta^\ell) : \ell \in \mathbb{N}\}$ and $\beta \in (1/3, (3-\sqrt{5})/2)$.*

Proof. When $t = \pm(1-\beta)\beta^\ell$ with $\ell \in \mathbb{N}$, we first check that t satisfies (7) (so $t \in \mathcal{B}_\beta$).

For $1 \leqslant k \leqslant \ell$, (7) is equivalent to

$$(1-\beta)(1-\beta^{\ell-k+1}) > \beta,$$

which is true since $(1-\beta)(1-\beta^{\ell-k+1}) \geqslant (1-\beta)^2 > \beta$ for $\beta \in (1/3, (3-\sqrt{5})/2)$.

For $k \geqslant \ell + 1$, (7) is equivalent to $(1-\beta)\beta^{k-1} > \beta^k$ which is true. Thus, the desired results follow just from theorem 1.2.

When $t = \pm(1-\beta)(1+\beta^\ell)$ with $\ell \in \mathbb{N}$, one can also check that they satisfy (7) and so lead to the desired results by theorem 1.2. □

The rest of this paper is organized as follows. We prove theorem 1.3 in section 2. For the reader's convenience the proof of proposition 1.1 is also included in this section. The proof of theorem 1.2 is arranged in section 3.

2. Proofs of proposition 1.1 and theorem 1.3

To prove proposition 1.1, we now establish an equivalence relation for members in M_{Γ_β}. Since the k-level components of Γ_β and $\Gamma_\beta + t$ are intervals of length β^k, they are completely determined by their left-end points. For $I = (i_\ell)_{\ell=1}^k \in \{0, 1\}^k$, the endpoints of intervals $F_I([0, 1])$ and $S_I([t, t+1])$ are

$$F_I(0) = F_{i_1} \circ \cdots \circ F_{i_k}(0) = \sum_{j=1}^{k} i_j(1-\beta)\beta^{j-1}$$

and

$$S_I(t) = S_{i_1} \circ \cdots \circ S_{i_k}(t) = \sum_{j=1}^{k} i_j(1-\beta)\beta^{j-1} + t.$$

For $\tau \in M_{\Gamma_\beta,k}$, let

$$R_{\Gamma_\beta,k}(\tau) = \{\beta^{-k}(\hat{\delta} - \hat{\tau}) : \delta \in N_{\Gamma_\beta,k}(\tau)\},$$

where $\hat{\delta}$, $\hat{\tau}$ are the left-end points of δ, τ, respectively. τ_1 and τ_2 are said to be *equivalent* (or of *the same type*), denoted by $\tau_1 \sim \tau_2$, if $R_{\Gamma_\beta,k_1}(\tau_1) = R_{\Gamma_\beta,k_2}(\tau_2)$. Zou *et al* [16] obtained the following results (see [16, lemma 3.1]), which will be used in the proof of proposition 1.1.

Lemma 2.1. *Let $I \in \Lambda_{k_1}$ and $J \in \Lambda_{k_2}$. If $F_I([0,1]) \sim F_J([0,1])$, then for $\ell \in \{0,1\}$, $I\ell \in \Lambda_{k_1+1}$ if and only if $J\ell \in \Lambda_{k_2+1}$. Further, $F_{I\ell}([0,1]) \sim F_{J\ell}([0,1])$.*

Proof of proposition 1.1. First let D_k denote the union of edges connecting a parent belonging to $M_{\Gamma_\beta,k-1}$ to its offspring belonging to $M_{\Gamma_\beta,k}$. We begin with $k = 1$. Then there are two possible cases.

Case 1. $\#N_{\Gamma_\beta,1}(F_0([0,1])) = \#N_{\Gamma_\beta,1}(F_1([0,1])) = 1$. In this case, $N_{\Gamma_\beta,1}(F_0([0,1])) = S_0([t,t+1])$ and $N_{\Gamma_\beta,1}(F_1([0,1])) = S_1([t,t+1])$, implying $F_0([0,1]) \sim F_1([0,1])$.

Case 2. There exists a unique $j \in \{0,1\}$ such that $\#N_{\Gamma_\beta,1}(F_j([0,1])) = 1$ and then $F_j([0,1]) \sim F_j([0,1])$.

Thus, for both cases described above we have that all $F_I([0,1]), I \in \Lambda_1 := D_1$ are equivalent and $\emptyset \neq D_1 \subseteq \{0,1\}$.

Now suppose that all $F_I([0,1]), I \in \Lambda_k$ are equivalent and $\Lambda_k = \prod_{i=1}^{k} D_i$, where $D_i = \{0,1\} \subseteq \{0,1\}$. Then for each $F_I([0,1])$ with $I \in \Lambda_k$, either it has only one offspring in $M_{\Gamma_\beta,k+1}$ and so $F_{I0} \sim F_{I0}$ (or $F_{I1} \sim F_{I1}$) or it has two offspring in $M_{\Gamma_\beta,k+1}$ and these two offspring are of the same type, i.e. $F_{I0} \sim F_{I1}$, by the same argument as that in case 1. On the other hand, for $I, J \in \Lambda_k$ lemma 2.1 shows that they connect their offspring in the same way and $F_{Ii}([0,1]) \sim F_{Ji}([0,1])$ with $i \in \{0,1\}$. Therefore, we obtain that for each $k \in \mathbb{N}$, $F_I([0,1]), I \in \Lambda_k$ are of the same type and $\Lambda_k = \prod_{i=1}^{k} D_i$ with $D_i \subseteq \{0,1\}$. By the definition of Λ, we have $\Lambda = \prod_{i=1}^{\infty} D_i$. Finally, $\Gamma_\beta \cap (\Gamma_\beta + t) \neq \emptyset$ implies $\Lambda \neq \emptyset$ and so each D_k is not empty. □

Now we are ready to give the proof of theorem 1.3.

Proof of theorem 1.3.

(i) $\Longrightarrow$ (ii) and (i) $\Longrightarrow$ (iii). By proposition 1.1 we have

$$\Gamma_\beta \cap (\Gamma_\beta + t) = \Pi\left(\prod_{\ell=1}^{\infty} D_\ell\right) \qquad \text{with } \emptyset \neq D_\ell \subseteq \{0,1\}.$$

Thus, the $\{-1,0,1\}$-code of t is unique by (3). We denote by $(t_k)_{k=1}^{\infty}$ the unique $\{-1,0,1\}$-code of t. Then

$$t_k = \begin{cases} 0 & \text{if } D_k = \{0,1\}, \\ -1 & \text{if } D_k = \{0\}, \\ 1 & \text{if } D_k = \{1\}. \end{cases}$$

For each $x \in \Gamma_\beta \cap (\Gamma_\beta + t)$ there exists unique $x^* \in \Gamma_\beta$, called the *accompanying* point of x, such that $x = x^* + t$. Let $J = (j_k)_{k=1}^\infty \in \prod_{\ell=1}^\infty D_\ell$ and $J^* = (j_k^*)_{k=1}^\infty \in \{0,1\}^{\mathbb{N}}$ be the codes of x and x^*, respectively. The code J^* is called the *accompanying* code of J. Then

$$(j_k^*)_{k=1}^\infty = (j_k)_{k=1}^\infty - (t_k)_{k=1}^\infty = (j_k - t_k)_{k=1}^\infty,$$

since $(t_k)_{k=1}^\infty$ is the unique $\{-1,0,1\}$-code of t. In addition, $F_{J|k}([0,1]) \cap S_{J^*|k}([t,t+1]) \neq \emptyset$ for all $k \in \mathbb{N} \cup \{0\}$.

In the following we verify that $(t_k)_{k=1}^\infty$ satisfies (7). Let $J = (j_\ell)_{\ell=1}^\infty \in \prod_{\ell=1}^\infty D_\ell$ with $J^* = (j_\ell^*)_{\ell=1}^\infty \in \{0,1\}^{\mathbb{N}}$ its accompanying code. Let $L \in \{0,1\}^k$ be such that $L|(k-1) = J^*|(k-1)$ and $L \neq J^*|k$. Then

$$F_{J|k}([0,1]) \cap S_L([t,t+1]) = F_{J|k}([0,1]) \cap (F_L([0,1]+t) = \emptyset \qquad \text{for all } k \in \mathbb{N}, \tag{8}$$

where we have used the fact that $S_L([t,t+1]) = F_L([0,1]) + t$. Thus the distance between the left endpoint $F_{J|k}(0)$ of $F_{J|k}([0,1])$ and the left endpoint $F_L(0) + t$ of $S_L([t,t+1])$ is bigger than β^k.

Case 1. Suppose $t_k = 1$. Then $D_k = \{1\}$ and so $j_k = 1$ and $j_k^* = 0$. Thus, $L = (J^*|(k-1))1$. Note that $F_{J|k}([0,1]) \cap S_L([t,t+1]) = \emptyset$ (by (8)) and $F_{J|k}([0,1]) \cap S_{J^*|k}([t,t+1]) \neq \emptyset$. Thus $S_L([t,t+1])$ lies on the right side of $F_{J|k}([0,1])$. So (at this moment $y_k = 0$ for $t_k = 1$)

$$\begin{aligned}
y_k\beta^{k-1}(1-\beta) &- \sum_{n=k}^\infty t_n\beta^{n-1}(1-\beta) \\
&= \sum_{n=1}^{k-1} t_n\beta^{n-1}(1-\beta) + y_k\beta^{k-1}(1-\beta) - \sum_{n=1}^\infty t_n\beta^{n-1}(1-\beta) \\
&= \sum_{n=1}^{k-1}(j_n - j_n^*)\beta^{n-1}(1-\beta) + (1-1)\beta^{k-1}(1-\beta) - \sum_{n=1}^\infty t_n\beta^{n-1}(1-\beta) \\
&= F_{J|k}(0) - F_L(0) - t = F_{J|k}(0) - S_L(t) < -\beta^k,
\end{aligned}$$

which leads to (7).

Case 2. Suppose $t_k = -1$. Then $D_k = \{0\}$ and so $j_k = 0$ and $j_k^* = 1$. One can check (7) by the same argument as above.

Case 3. Suppose $t_k = 0$. Then $D_k = \{0,1\}$ and so $j_k = 0$ or 1 and, correspondingly, $j_k^* = 0$ or 1, respectively. Therefore, by the same argument as those in case 1

$$\begin{cases} F_{J|k}(0) - S_L(t) < -\beta^k & \text{if } j_k = 0, \\ F_{J|k}(0) - S_L(t) > \beta^k & \text{if } j_k = 1. \end{cases}$$

So the first inequality in (7) is verified where $y_k = 1$ corresponds to $j_k = 1$ and $y_k = -1$ corresponds to $j_k = 0$.

(iii) $\Longrightarrow$ (i). Clearly, $N_{\Gamma_\beta,0}(F_\emptyset([0,1])) = \{[t,t+1]\}$. Let

$$D_\ell = \{0,1\} \cap (\{0,1\} + t_\ell) \qquad \text{for } \ell \in \mathbb{N}.$$

Below we will prove by induction that for any $k \in \mathbb{N}$ and $J \in \{0,1\}^k$

$$N_{\Gamma_\beta,k}(F_J([0,1])) = \begin{cases} \{S_{J-(t_i)_{i=1}^k}([t,t+1])\} & \text{if } J \in \prod_{i=1}^k D_i, \\ \emptyset & \text{otherwise.} \end{cases} \tag{9}$$

When $k = 1$, formula (7) becomes

$$\begin{cases} t > \beta & \text{if } t_1 = 1, \\ t < -\beta & \text{if } t_1 = -1, \\ |1-\beta-t| > \beta \quad \text{and} \quad |\beta-1-t| > \beta & \text{if } t_1 = 0. \end{cases}$$

Thus, when $t_1 = 1$ we have

$$N_{\Gamma_\beta,1}(F_0([0,1])) = \emptyset \qquad \text{and} \qquad N_{\Gamma_\beta,1}(F_1([0,1])) = \{S_0([t,t+1])\}.$$

When $t_1 = -1$ we have

$$N_{\Gamma_\beta,1}(F_0([0,1])) = \{S_1([t,t+1])\} \qquad \text{and} \qquad N_{\Gamma_\beta,1}(F_1([0,1])) = \emptyset.$$

When $t_1 = 0$ we have $2\beta - 1 < t < 1 - 2\beta$ and so

$$N_{\Gamma_\beta,1}(F_0([0,1])) = \{S_0([t,t+1])\} \qquad \text{and} \qquad N_{\Gamma_\beta,1}(F_1([0,1])) = \{S_1([t,t+1])\}.$$

Thus (9) is true when $k = 1$.

Suppose that (9) is true for $k = n$. Arbitrarily fix a $J = (j_i)_{i=1}^{n+1} \in \{0,1\}^{n+1}$.

Case 1. $J|n \notin \prod_{i=1}^{n} D_i$. Then $N_{\Gamma_\beta,n}(F_{J|n}([0,1])) = \emptyset$ by inductive hypothesis and so $N_{\Gamma_\beta,n+1}(F_J([0,1])) = \emptyset$.

Case 2. $J|n \in \prod_{i=1}^{n} D_i$ (and so $J|n - (t_i)_{i=1}^{n} \in \{0,1\}^n$). Let $L = (\ell_i)_{i=1}^{n+1}$ be such that

$$L|n = (\ell_i)_{i=1}^{n} = J|n - (t_i)_{i=1}^{n} \text{ and } \ell_{n+1} = j_{n+1} - y_{n+1}.$$

(A) Suppose $t_{n+1} = 1$. Then (7) shows that

$$\sum_{i=1}^{n+1} j_i\beta^{i-1}(1-\beta) - \left(\sum_{i=1}^{n}(j_i - t_i)\beta^{i-1}(1-\beta) + (j_{n+1} - y_{n+1})\beta^n(1-\beta) + t\right) < -\beta^{n+1}. \tag{10}$$

At this moment, we have

$$D_{n+1} = \{1\},\ y_{n+1} = 0,\ \ell_{n+1} = j_{n+1} \text{ and so } L \in \{0,1\}^{n+1}.$$

From (10) it follows that

$$F_J(0) - S_L(t) < -\beta^{n+1}. \tag{11}$$

By taking $j_{n+1} = 0$ we obtain that

$$N_{\Gamma_\beta,n+1}(F_{(J|n)0}([0,1])) = \emptyset.$$

Taking $j_{n+1} = 1$, (11) gives

$$F_{(J|n)1}([0,1]) \cap S_{(J|n-(t_i)_{i=1}^{n})1}([t,t+1]) = \emptyset.$$

By inductive hypothesis we know $N_{\Gamma_\beta,n}(F_{J|n}([0,1])) \neq \emptyset$, which implies that

$$N_{\Gamma_\beta,n+1}(F_{(J|n)1}([0,1])) = \{S_{(J|n-(t_i)_{i=1}^{n})0}([t,t+1])\},$$

where the condition $1/3 < \beta < 1/2$ is used. Thus, (9) is true when $k = n+1$ and $t_{n+1} = 1$.

(B) Suppose $t_{n+1} = -1$. (9) can be verified by the same argument as above.

(C) Suppose $t_{n+1} = 0$. Then (7) shows that

$$\begin{cases} \left|\sum_{i=1}^{n+1} j_i\beta^{i-1}(1-\beta) - \left(\sum_{i=1}^{n}(j_i - t_i)\beta^{i-1}(1-\beta) + (j_{n+1} - 1)\beta^n(1-\beta) + t\right)\right| > \beta^{n+1}, \\ \left|\sum_{i=1}^{n+1} j_i\beta^{i-1}(1-\beta) - \left(\sum_{i=1}^{n}(j_i - t_i)\beta^{i-1}(1-\beta) + (j_{n+1} + 1)\beta^n(1-\beta) + t\right)\right| > \beta^{n+1}. \end{cases}$$

This implies that

$F_{(J|n)1}([0,1]) \cap S_{(J|n-(t_i)_{i=1}^n)0}([t,t+1]) =$ and

$$F_{(J|n)0}([0,1]) \cap S_{(J|n-(t_i)_{i=1}^n)1}([t,t+1]) = \emptyset.$$

Therefore, by inductive hypothesis $N_{\Gamma_\beta,n}(F_{J|n}([0,1])) \neq \emptyset$ and $1/3 < \beta < 1/2$ we have

$$N_{\Gamma_\beta,n+1}(F_{(J|n)0}([0,1])) = \{S_{(J|n-(t_i)_{i=1}^n)0}([t,t+1])\}$$

and

$$N_{\Gamma_\beta,n+1}(F_{(J|n)1}([0,1])) = \{S_{(J|n-(t_i)_{i=1}^n)1}([t,t+1])\},$$

i.e. (9) is true when $k = n+1$ and $t_{n+1} = 0$.

(ii) $\Longrightarrow$ (i). Suppose that $N_{\Gamma_\beta,k}(F_J([0,1])) = \{S_{I0}([t,t+1]), S_{I1}([t,t+1])\}$ for some $J = (j_i)_{i=1}^k \in \{0,1\}^k$, $I \in \{0,1\}^{k-1}$ with $k \in \mathbb{N}$. Let $\bar{j}_k = 1 - j_k$ and $\bar{J} = (j_i)_{i=1}^{k-1}\bar{j}_k$. Then $\#N_{\Gamma_\beta,k}(F_{\bar{J}}([0,1])) = 1$. Without loss of generality, assume $j_k = 0$. Then

$$N_{\Gamma_\beta,k}(F_{\bar{J}}([0,1])) = \{S_{I1}([t,t+1])\}.$$

Note that $\Gamma_\beta - \Gamma_\beta = [-1,1]$ means that $\Gamma_\beta \cap (\Gamma_\beta + t) \neq \emptyset$ for any $t \in [-1,1]$. Thus we have

$$F_J([0,1]) \cap S_{I1}([t,t+1]) \cap \Gamma_\beta \cap (\Gamma_\beta + t) \neq \emptyset$$

and

$$F_{\bar{J}}([0,1]) \cap S_{I1}([t,t+1]) \cap \Gamma_\beta \cap (\Gamma_\beta + t) \neq \emptyset.$$

Take $x \in F_J([0,1]) \cap S_{I1}([t,t+1]) \cap \Gamma_\beta \cap (\Gamma_\beta + t)$, $y \in F_{\bar{J}}([0,1]) \cap S_{I1}([t,t+1]) \cap \Gamma_\beta \cap (\Gamma_\beta + t)$. Let $\tilde{x} = (x_\ell)_{\ell=1}^\infty$, $\tilde{y} = (y_\ell)_{\ell=1}^\infty$ be the unique $\{0,1\}$-code of x and y, respectively. By $x^* = \Pi((x_\ell^*)_{\ell=1}^\infty)$, $y^* = \Pi((y_\ell^*)_{\ell=1}^\infty)$ we denote the accompanying points of x and y, respectively. Thus

$$x = x^* + t, \quad x_k = 0, \quad x_k^* = 1 \qquad \text{and} \qquad y = y^* + t, \quad y_k = 1, \quad y_k^* = 1.$$

Therefore, t has two distinct $\{-1,0,1\}$-codes. □

3. Proof of theorem 1.2

In this section, we give the proof of theorem 1.2. The following lemma (cf [2] [13, lemma 2.2]) gives a description of a strong periodic infinite string.

Lemma 3.1. *Let $(x_k)_{k=1}^\infty \in \{0,1\}^{\mathbb{N}}$. If there exists a positive integer q such that $x_{k+q} \geqslant x_k$ for all $k \in \mathbb{N}$, then $(x_k)_{k=1}^\infty$ is strong periodic.*

Lemma 3.2. *Let Γ be defined as in (5). If Γ is a self-similar set, then the γ^* is strong periodic.*

Proof. When Γ is a singleton, we have $\gamma^* = \bar{0}$ and so γ^* is strong periodic. Suppose that Γ is not a singleton. Let Γ be generated by an IFS$\{f_i(x) = r_i x + b_i\}_{i=1}^N$ where $r_i \in (0,1)$ and $0 = b_1 \leqslant b_2 \leqslant \cdots \leqslant b_N$. Take $m \in \mathbb{N} \cup \{0\}$ such that $\beta^m(1-\beta) \in \Gamma$. Then

$$\Gamma \ni f_1(\beta^m(1-\beta)) = \beta^m r_1(1-\beta) = \sum_{k=1}^\infty x_k \beta^{k-1}(1-\beta) \qquad \text{with } x_k \leqslant \gamma_k^*, \quad k \in \mathbb{N}.$$

Since $r_1 < 1$, we have $x_k = 0$ if $k \leqslant m+1$. Thus there exists a positive integer q such that

$$r_1 = \beta^q + \sum_{i>q} x_i \beta^i, \ \ x_i \in \{0,1\}.$$

If $k \in \mathbb{N}$ is such that $\gamma_k^* = 1$, then $\beta^{k-1}(1-\beta) \in \Gamma$ by (6) and so

$$f_1(\beta^{k-1}(1-\beta)) = \beta^{q+k-1}(1-\beta) + \sum_{i>q} x_i \beta^{i+k-1}(1-\beta) \in \Gamma,$$

implying $\gamma_{k+q}^* = 1$. Hence $\gamma_{k+q}^* \geqslant \gamma_k^*$ for all k. The desired result is then obtained by lemma 3.1. □

Proof of theorem 1.2. The necessity part follows directly from lemma 3.2 since $\Gamma_\beta \cap (\Gamma_\beta + t) = \Gamma + \gamma$.

To prove sufficiency, we restate an alternative representation of a self-similar set. Let $D = \{d_1, d_2, \ldots, d_N\}$ be a finite set of real numbers. Let $f_i(x) = r(x + d_i)$, $1 \leqslant i \leqslant N$ and $|r| < 1$. Then the self-similar set, denoted by $T(r, D)$, generated by IFS $(f_i)_{i=1}^N$ can be represented as

$$T(r, D) = \left\{ \sum_{k=1}^{\infty} d_k r^k : d_k \in D \right\}.$$

Since γ^* is strong periodic, it can be written as $\gamma^* = I\overline{I+J} \in \{0,1\}^{\mathbb{N}}$ for some $p \in \mathbb{N}$ and some $I = i_1 i_2 \ldots i_p$, $J = j_1 j_2 \ldots j_p \in \{0,1\}^p$ satisfying $I + J := (i_1 + j_1)(i_2 + j_2) \ldots (i_p + j_p) \in \{0,1\}^p$. Take a finite set of real numbers

$$\begin{aligned} D &= \left\{ \sum_{k=1}^{p} \sigma_k \beta^{k-1-p}(1-\beta) + \sum_{k=1}^{p} \tau_k \beta^{k-1}(1-\beta) : \sigma_1 \ldots \sigma_p \preccurlyeq I, \tau_1 \ldots \tau_p \preccurlyeq J \right\} \\ &= \left\{ \beta^{-p} \sum_{k=1}^{2p} \sigma_k \beta^{k-1}(1-\beta) : \sigma_1 \ldots \sigma_p \sigma_{p+1} \ldots \sigma_{2p} \preccurlyeq IJ \right\}. \end{aligned} \tag{12}$$

We shall show $\Gamma = T(\beta^p, D)$. Now arbitrarily fix an $x \in \Gamma$ with $\{0,1\}$-code $\tilde{x} = x_1 x_2 \ldots$. Then for each $k \in \mathbb{N}$, $x_{kp+1} x_{kp+2} \ldots x_{kp+p}$ can be uniquely represented as

$$x_{kp+1} x_{kp+2} \cdots x_{kp+p} = y_{kp+1} y_{kp+2} \cdots y_{kp+p} + z_{kp+1} z_{kp+2} \cdots z_{kp+p},$$

where $y_{kp+1} y_{kp+2} \ldots y_{kp+p} \preccurlyeq I$ and $z_{kp+1} z_{kp+2} \ldots z_{kp+p} \preccurlyeq J$. Hence (note that $x_1 x_2 \ldots x_p \preccurlyeq I$)

$$\begin{aligned} x &= \sum_{k=1}^{\infty} x_k \beta^{k-1}(1-\beta) = \beta^p \sum_{k=1}^{p} x_k \beta^{k-1-p}(1-\beta) + \beta^{2p} \sum_{k=p+1}^{2p} (y_k + z_k)\beta^{k-1-2p}(1-\beta) \\ &\quad + \beta^{3p} \sum_{k=2p+1}^{3p} (y_k + z_k)\beta^{k-1-3p}(1-\beta) + \cdots \\ &= \beta^p \left(\sum_{k=1}^{p} x_k \beta^{k-1-p}(1-\beta) + \sum_{k=p+1}^{2p} z_k \beta^{k-1-p}(1-\beta) \right) \\ &\quad + \beta^{2p} \left(\sum_{k=p+1}^{2p} y_k \beta^{k-1-2p}(1-\beta) + \sum_{k=2p+1}^{3p} z_k \beta^{k-1-2p}(1-\beta) \right) + \cdots \in T(\beta^p, D). \end{aligned}$$

Thus, $\Gamma \subseteq T(\beta^p, D)$. The inverse inclusion is left for the readers. □

From the above proof it follows that if $\gamma^* = I\overline{I+J} \in \{0,1\}^{\mathbb{N}}$ with $I = i_1 i_2 \ldots i_p$, $J = j_1 j_2 \ldots j_p \in \{0,1\}^p$, then Γ can be generated by the IFS $\{f_i(x) = \beta^p(x + d_i) : d \in D\}$ where D is determined by (12). More exactly,

$$\Gamma = \bigcup_{K \preccurlyeq IJ} (\beta^p \Gamma + \Pi(K\bar{0})). \tag{13}$$

We claim that the right side of (13) is a disjoint union, and so the resulting IFS satisfies the SSC. In fact, for a given $K = k_1 \dots k_p k_{p+1} \dots k_{2p} \preccurlyeq IJ$ we have

$$\beta^p\Gamma + \Pi(K\bar{0}) = \left\{\beta^p x + \sum_{j=1}^{2p} k_j\beta^{j-1}(1-\beta) : x \in \Gamma\right\}$$
$$= \left\{\sum_{j=1}^{p} k_j\beta^{j-1}(1-\beta) + \sum_{j=p+1}^{2p}(k_j + x_{j-p})\beta^{j-1}(1-\beta) + \sum_{j>2p} x_{j-p}\beta^{j-1}(1-\beta) : (x_k)_{k=1}^{\infty} \preccurlyeq \gamma^*\right\}.$$

Thus each point of $\beta^p\Gamma + \Pi(K\bar{0})$ has a unique $\{0, 1\}$-code of the form $k_1 \dots k_p(k_{p+1} + x_1) \dots (k_{2p} + x_p)x_{p+1}x_{p+2} \dots$ for some $(x_k)_{k=1}^{\infty} \preccurlyeq \gamma^*$.

Take distinct $K = k_1 \dots k_p k_{p+1} \dots k_{2p} \preccurlyeq IJ$ and $K^* = k_1^* \dots k_p^* k_{p+1}^* \dots k_{2p}^* \preccurlyeq IJ$. If $k_1 \dots k_p \neq k_1^* \dots k_p^*$, then $(\beta^p\Gamma + \Pi(K\bar{0})) \cap (\beta^p\Gamma + \Pi(K^*\bar{0})) = \emptyset$ follows directly from the above arguments. If $k_1 \dots k_p = k_1^* \dots k_p^*$, we have, without loss of generality, $k_{p+1} = 1$ and $k_{p+1}^* = 0$. Hence, $\gamma_1^* = 0$ and so each $(x_k)_{k=1}^{\infty} \preccurlyeq \gamma^*$ has $x_1 = 0$. $(\beta^p\Gamma + \Pi(K\bar{0})) \cap (\beta^p\Gamma + \Pi(K^*\bar{0})) = \emptyset$ then follows from the above arguments. Note that

$$\#D = \#\{K \in \{0, 1\}^{2p} : K \preccurlyeq IJ\} = \prod_{k=1}^{p}(i_k + 1)(j_k + 1) = \prod_{k=1}^{p}(i_k + j_k + 1) = 2^{\sum_{k=1}^{p}(i_k + j_k)}.$$

Consequently, we have

$$\dim_H \Gamma_\beta \cap (\Gamma_\beta + t) = \dim_H \Gamma = \frac{\log \#D}{-p\log\beta} = \frac{\sum_{k=1}^{p}(i_k + j_k)\log 2}{-p\log\beta}.$$

This extends the results in [11] for the case $\beta \in (1/3, 1/2)$ and t has a unique $\{-1, 0, 1\}$-code.

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References

[1] Davis G J and Hu T Y 1995 On the structure of the intersection of two middle third Cantor sets *Publ. Math.* **39** 43–60

[2] Deng G T, Heng X G and Wen Z X 2008 Self-similar structure on intersection of triadic Cantor sets *J. Math. Anal. Appl.* **337** 617–31

[3] Falconer K J 1990 Fractal geometry *Mathematical Foundations and Applications* (Chichester: Wiley)

[4] Hutchinson J E 1981 Fractal and self similarity *Indian Univ. Math. J.* **30** 713–47

[5] Igudesman K B 2003 Lacunary self-similar fractal sets and intersection of cantor sets *Lobachevskii J. Math.* **12** 41–5

[6] Kraft R L 1992 Intersection of thick Cantor sets *Mem. A.M.S.* **97**

[7] Kraft R L 1994 What's the difference between Cantor sets? *Am. Math. Mon.* **101** 640–50

[8] Kraft R L 1999 Random intersections of thick Cantor sets *Trans. Am. Math. Soc.* **352** 1315–28

[9] Kenyon R and Peres Y 1991 Intersecting random translates of invariant Cantor sets *Invent. Math.* **104** 601–29

[10] Li W and Xiao D 1999 Intersections of translations of cantor triadic sets *Acta. Math. Sci.* **19** 214–9
[11] Li W and Xiao D 1998 On the intersection of translation of middle-α Cantor sets *Fractals and Beyond-Complexities in the Sciences (Malta, October 1998)* (Singapore: World Scientific) pp 137–48
[12] Li W and Xiao D 2000 Dimensions of subsets of a class of Cantor-type sets *Acta Math. Sin.* **43** 225–32
[13] Li W, Yao Y and Zhang Y Self-similar structure of homogeneous symmetric Cantor sets *Math. Nachr.* at press
[14] Nekka F and Li J 2002 Intersection of triadic Cantor sets with their translates: I. Fundamental properties *Chaos Solitons Fractals* **13** 1807–17
[15] Williams R F 1991 How big is the intersection of two thick Cantor sets? *Continuum Theory and Dynamical Systems: Proc. 1989 Joint Summer Research Conf. on Continua and Dynamics* (Arcata, CA, 1989) ed M Brown (Providence, RI: American Mathematical Society) MR 92f:58116
[16] Zou Y, Li W and Yan C Intersecting nonhomogeneous Cantor sets with their translations unpublished