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Hausdorff dimension of subsets with proportional fibre frequencies of the general Sierpinski carpet

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Abstract
In this paper we study a class of subsets of the general Sierpinski carpets for which the digits in the expansions lie in two specified horizontal fibres with proportional frequencies. We calculate the Hausdorff dimension of these subsets and give necessary and sufficient conditions for the corresponding Hausdorff measure to be positive and finite.

Mathematics Subject Classification: 28A80, 28A78

1. Introduction

Let T be the expanding endomorphism of the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ given by the matrix $\mathrm{diag}(n, m)$ where $2 \leqslant m < n$ are integers. The simplest invariant sets for T have the form

$$K(T, D) = \left\{ \sum_{k=1}^{\infty} \begin{pmatrix} n^{-1} & 0 \\ 0 & m^{-1} \end{pmatrix}^k d_k \ : \ d_k \in D \text{ for all } k \geqslant 1 \right\},$$

where $D \subseteq I \times J$ is a set of digits with $I = \{0, 1, \ldots, n-1\}$ and $J = \{0, 1, \ldots, m-1\}$. Alternatively, define a map $K_T : (I \times J)^{\mathbb{N}} \longrightarrow \mathbb{T}^2$ by

$$K_T(x) = \sum_{k=1}^{\infty} \begin{pmatrix} n^{-1} & 0 \\ 0 & m^{-1} \end{pmatrix}^k x_k, \qquad x = (x_k)_{k=1}^{\infty} \in (I \times J)^{\mathbb{N}}. \tag{1}$$

Then $K(T, D) = K_T(D^{\mathbb{N}})$. So each element of $K(T, D)$ can be represented as an expansion in base $\mathrm{diag}(n^{-1}, m^{-1})$ with digits in D and $x = (x_k)_{k=1}^{\infty}$ is called a coding of $K_T(x)$.

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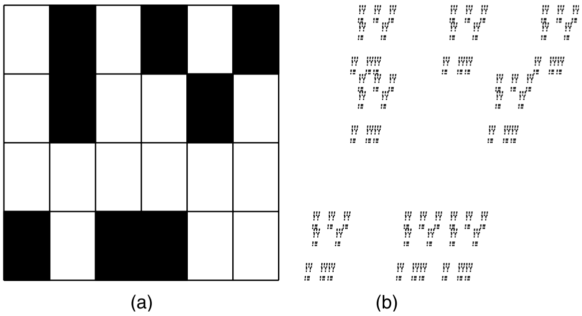

Figure 1. (*a*) A pattern $D = \{(0,0),(2,0),(3,0),(1,2),(4,2),(1,3),(3,3),(5,3)\}$, where $m=4$, $n=6$, $\mathcal{L}=\{b_1,b_2,b_3\}=\{0,2,3\}$, $\Gamma_1=\{(0,0),(2,0),(3,0)\}$, $\Gamma_2=\{(1,2),(4,2)\}$ and $\Gamma_3=\{(1,3),(3,3),(5,3)\}$, $n_1=n_3=3$ and $n_2=2$. (b) The corresponding Sierpinski carpet $K(T,D)$.

The set $K(T,D)$, called the *general Sierpinski carpet*, was first studied by McMullen [9] and Bedford [2], independently, to determine its Hausdorff and box-counting dimensions. From then on, some further problems related to the general Sierpinski carpet $K(T,D)$ were proposed and considered by many authors. Peres [12, 13] studied its packing and Hausdorff measures. Kenyon and Peres [6, 7] extended the results of McMullen [9] and Bedford [2] to the compact subsets of the 2-torus corresponding to shifts of the finite type or sofic shifts and to the Sierpinski sponges. The singular spectrum was studied by King [8] for the general Sierpinski carpet, and later by Olsen [11] for the Sierpinski sponges. As we know, lots of interesting results have been established for certain subsets of self-similar sets by way of multifractal analysis. Some detailed description on this topic and recent developments are included in [1]. Unfortunately, less analogous results have been revealed for the general self-affine sets. However, for the general Sierpinski carpet, a special class of self-affine sets, some analogous results for certain subsets have been established by many authors.

Let σ denote the projection of $\mathbb{R}^2$ onto its second coordinate. Let $\mathcal{L}=\sigma(D):=\{b_1,b_2,\ldots,b_\ell\}$. Set

$$\Gamma_i=\{d\in D:\sigma(d)=b_i\},\qquad i=1,2,\ldots,\ell.$$

Then the Γ_i are the *horizontal fibres* of D and form a partition $\Gamma:=\{\Gamma_1,\ldots,\Gamma_\ell\}$ of D, i.e. $D=\bigcup_{i=1}^{\ell}\Gamma_i$ with disjoint union. Put

$$n_i=\#\Gamma_i,\qquad 1\leqslant i\leqslant\ell\ \text{ and }\ \alpha=\log_n m,$$

where and throughout this paper we use $\#A$ to denote the cardinality of a finite set A. D is said to have *uniform horizontal fibres* if $n_i=n_j$ for all $1\leqslant i,j\leqslant\ell$. A pattern D and corresponding $K(T,D)$ are shown in figure 1 for the readers' understanding.

For a probability vector $\boldsymbol{p}=(p_d)_{d\in D}$ on D, i.e. $\sum_{d\in D}p_d=1$ with each $p_d\in(0,1)$, let

$$\Xi_{\boldsymbol{p}}=\left\{x=(x_i)_{i=1}^{\infty}\in D^{\mathbb{N}}:\ \lim_{k\to\infty}\frac{\#\{1\leqslant j\leqslant k:x_j=d\}}{k}=p_d,\ d\in D\right\}. \tag{2}$$

Then $\Xi_{\boldsymbol{p}}$ is a subset of $D^{\mathbb{N}}$ such that the occurrence of each digit $d\in D$ in each of its elements has a prescribed frequency p_d. Thus $K_T(\Xi_{\boldsymbol{p}})$ is a subset of $K(T,D)$ whose elements have prescribed digit frequencies in their codings. Nielsen [10] gave an overall investigation on $K_T(\Xi_{\boldsymbol{p}})$, obtaining its box, packing and Hausdorff dimensions as well as sufficient and necessary conditions for the packing and Hausdorff measures in their dimensions to be positive and finite. The approach used in [10] also works for the study of subsets with prescribed horizontal fibre frequencies. For any $x = (x_j)_{j=1}^{\infty} \in D^{\mathbb{N}}$ and $1 \leqslant i \leqslant \ell$, set

$$N_k(x, \Gamma_i) = \#\{1 \leqslant j \leqslant k : x_j \in \Gamma_i\}.$$

Whenever there exists the limit

$$\zeta(x, \Gamma_i) := \lim_{k\to\infty} \frac{N_k(x, \Gamma_i)}{k}, \tag{3}$$

it is called the *frequency* of the horizontal fibre Γ_i in the coding of x. When we write the symbols $\zeta(x, \Gamma_i)$, we are already assuming the existence of the limit in (3). Let $\boldsymbol{c} = (c_1, c_2, \cdots, c_\ell)$ be a probability vector, i.e. $\sum_{j=1}^{\ell} c_j = 1$ with $c_j > 0$. Let

$$\Omega(\Gamma, \boldsymbol{c}) = \left\{d = (d_i)_{i=1}^{\infty} \in D^{\mathbb{N}} : \zeta(d, \Gamma_j) = c_j,\ \ j = 1, 2, \cdots, \ell\right\}.$$

Then $\Omega(\Gamma, \boldsymbol{c})$ is a subset of $D^{\mathbb{N}}$ such that the occurrence of digits from each horizontal fibre Γ_i in each of its elements has a prescribed frequency c_i. Thus $K_T(\Omega(\Gamma, \boldsymbol{c}))$ is a subset of $K(T, D)$ whose elements have prescribed horizontal fibre frequencies in their codings. From the definition of $\Xi_{\boldsymbol{p}}$ defined by (2), it follows that $K_T(\Xi_{\boldsymbol{p}}) \subseteq K_T(\Omega(\Gamma, \boldsymbol{c}))$ if taking $c_j = \sum_{d\in\Gamma_j} p_d$ for $1 \leqslant j \leqslant \ell$. Each digit $d \in D$ is required to occur as an entry of elements of $\Xi_{\boldsymbol{p}}$ with a prescribed frequency p_d, while its occurrence as an entry of elements of $\Omega(\Gamma, \boldsymbol{c})$ is relatively more free.

For any Borel subset E of $\mathbb{R}^2$, let $\dim_{\mathrm{B}} E$, $\dim_{\mathrm{P}} E$ and $\dim_{\mathrm{H}} E$, respectively, denote its box, packing and Hausdorff dimensions, $\mathcal{P}^{\gamma}(E)$ and $\mathcal{H}^{\gamma}(E)$ denote its γ-dimensional packing and Hausdorff measures. Gui and Li [5] obtained the following results (see [5, theorems 1.1 and 1.2]).

(R1) $\dim_{\mathrm{H}} K_T(\Omega(\Gamma, \boldsymbol{c})) = \dim_{\mathrm{P}} K_T(\Omega(\Gamma, \boldsymbol{c})) = \sum_{j=1}^{\ell} c_j(\log_m n_j^{\alpha} - \log_m c_j)$;

(R2) $\dim_{\mathrm{H}} K_T(\Omega(\Gamma, \boldsymbol{c})) = \dim_{\mathrm{H}} K(T, D)$ if and only if $c_j = \frac{n_j^{\alpha}}{\sum_{i=1}^{\ell} n_i^{\alpha}}$, $j = 1, 2, \cdots, \ell$;

(R3) $\dim_{\mathrm{P}} K_T(\Omega(\Gamma, \boldsymbol{c})) = \dim_{\mathrm{P}} K(T, D)$ if and only if $c_i = c_j$ and $n_i = n_j$ for all $1 \leqslant i, j \leqslant \ell$;

(R4) Let $\gamma = \dim_{\mathrm{H}} K_T(\Omega(\Gamma, \boldsymbol{c})) = \dim_{\mathrm{P}} K_T(\Omega(\Gamma, \boldsymbol{c})) = \sum_{j=1}^{\ell} c_j(\log_m n_j^{\alpha} - \log_m c_j)$.

(a) $0 < \mathcal{H}^{\gamma}(K_T(\Omega(\Gamma, \boldsymbol{c}))) \leqslant \mathcal{P}^{\gamma}(K_T(\Omega(\Gamma, \boldsymbol{c}))) < \infty$ if and only if $c_i = c_j$ and $n_i = n_j$ for all $1 \leqslant i, j \leqslant \ell$;

(b) If there exist some $1 \leqslant i \neq j \leqslant \ell$ such that $c_i \neq c_j$ or $n_i \neq n_j$, then $\mathcal{H}^{\gamma}(K_T(\Omega(\Gamma, \boldsymbol{c}))) = \mathcal{P}^{\gamma}(K_T(\Omega(\Gamma, \boldsymbol{c}))) = +\infty$.

In this paper, we investigate another class of subsets of the general Sierpinski carpet. We assume that $\#\mathcal{L} = \ell > 2$ in the following discussion. Recall that $\mathcal{L} = \sigma(D)$ where σ is the projection of $\mathbb{R}^2$ onto its second coordinate. Thus, when $\#\mathcal{L} = \ell = 2$, i.e. D has just two horizontal fibres, the set $\Omega(s, t, \beta)$ defined below is identical to $\Omega(\Gamma, \boldsymbol{c})$ for $\boldsymbol{c} = ((1+\beta)^{-1}, \beta(1+\beta)^{-1})$ which reduces to the case considered in [5]. For any two distinct horizontal fibres Γ_s, Γ_t and $\beta > 0$, we now consider the set

$$\Omega(s, t, \beta) = \left\{x = (x_i)_{i=1}^{\infty} \in D^{\mathbb{N}} : \zeta(x, \Gamma_s) = \beta\zeta(x, \Gamma_t) > 0\right\}. \tag{4}$$

Then $\Omega(s, t, \beta)$ is a subset of $D^{\mathbb{N}}$ such that the frequency of the horizontal fibre Γ_s in x is proportional to that of Γ_t. And so $K_T(\Omega(s, t, \beta))$ is the subset of $K(T, D)$ for which the digits in the expansions of their elements lie in two specified horizontal fibres with proportional frequencies. Clearly, $K_T(\Omega(s, t, \beta))$ is T-invariant, dense in $K(T, D)$ but not compact in general. Thus

$$\dim_{\mathrm{B}} K_T(\Omega(s, t, \beta)) = \dim_{\mathrm{B}} K(T, D) = (1-\alpha)\log_m \#\mathcal{L} + \alpha \log_m \#D,$$

where the box dimension of $K(T, D)$ was established by McMullen [9] and Bedford [2], independently. Let

$$\Sigma = \left\{ \boldsymbol{p} = (p_i)_{i=1}^{\ell} : p_i \in (0,1) \text{ and } \sum_{i=1}^{\ell} p_i = 1 \text{ and } p_s = \beta p_t \right\}. \tag{5}$$

It is easy to see that

$$K_T(\Omega(s,t,\beta)) \supset \bigcup_{\boldsymbol{p}\in\Sigma} K_T(\Omega(\Gamma, \boldsymbol{p})). \tag{6}$$

We emphasize that the inclusion is proper since $K_T(\Omega(s,t,\beta))$ contains points for which $\zeta(x, \Gamma_i), i \neq s, t$ are not well defined if $\#\mathcal{L} = \ell \geqslant 4$. We define a function on Σ by

$$f(\boldsymbol{p}) = \sum_{j=1}^{\ell} (p_j \log_m n_j^{\alpha} - p_j \log_m p_j), \tag{7}$$

where $n_j = \#\Gamma_j,\ j = 1, 2, \ldots, \ell$. Note that the function $f(\boldsymbol{p})$ can be continuously extended to $\mathrm{cl}(\Sigma)$ (the closure of Σ) by interpreting $0 \log_m 0$ as 0. Then $f(\boldsymbol{p})$ can obtain its maximum $f_{\max}$ on $\mathrm{cl}(\Sigma)$. In fact, the maximum $f_{\max}$ cannot be reached on the boundary of $\mathrm{cl}(\Sigma)$, and there exists a unique point $\boldsymbol{p}^* = (p_i^*)_{i=1}^{\ell} \in \Sigma$ such that $f(\boldsymbol{p}^*) = f_{\max} = \max_{\boldsymbol{p}\in\mathrm{cl}(\Sigma)} f(\boldsymbol{p}) = \max_{\boldsymbol{p}\in\Sigma} f(\boldsymbol{p})$. This fact is shown in the following section as proposition 2.3. Throughout this paper, the notation $\boldsymbol{p}^* = (p_i^*)_{i=1}^{\ell}$ is always assumed to be the unique maximum point of $f(\boldsymbol{p})$ whenever it occurs. More precisely, as we can see in proposition 2.3, $\boldsymbol{p}^* = (p_i^*)_{i=1}^{\ell}$ is given by (12) and so by (7)

$$f(\boldsymbol{p}^*) = \log_m \left((1+\beta) \left(\beta^{-\beta} n_s^{\alpha\beta} n_t^{\alpha} \right)^{\frac{1}{1+\beta}} + \sum_{j\in\mathcal{I}} n_j^{\alpha} \right), \tag{8}$$

where $\mathcal{I} := \{1, 2, \ldots, \ell\} \backslash \{s, t\}$. Its verification is left for the readers. Therefore, we can obtain a lower bound for the Hausdorff dimension of $K_T(\Omega(s,t,\beta))$:

$$\dim_{\mathrm{H}} K_T(\Omega(s,t,\beta))) \geqslant \log_m \left((1+\beta) \left(\beta^{-\beta} n_s^{\alpha\beta} n_t^{\alpha} \right)^{\frac{1}{1+\beta}} + \sum_{j\in\mathcal{I}} n_j^{\alpha} \right)$$

by (6), (R1) and (8). However, our main result shows that the opposite inequality also holds. In this paper, we obtain the following results.

Theorem 1.1. *Let* $\alpha = \log_n m$ *and* $n_j = \#\Gamma_j,\ j = 1, 2, \ldots, \ell$. *Let* $\Omega(s,t,\beta)$ *be defined as* (4). *Then*

$$\dim_{\mathrm{H}} K_T(\Omega(s,t,\beta)) = \log_m \left((1+\beta) \left(\beta^{-\beta} n_s^{\alpha\beta} n_t^{\alpha} \right)^{\frac{1}{1+\beta}} + \sum_{j\in\mathcal{I}} n_j^{\alpha} \right),$$

where $\mathcal{I} = \{1, 2, \ldots, \ell\} \backslash \{s, t\}$. *In addition,* $\dim_{\mathrm{H}} K_T(\Omega(s,t,\beta)) = \dim_{\mathrm{P}} K_T(\Omega(s,t,\beta))$ *when* $D \backslash (\Gamma_s \cup \Gamma_t)$ *has uniform horizontal fibres, i.e. all* n_j *are equal for* $j \in \mathcal{I}$.

Theorem 1.2. *Let* $\gamma = \log_m \left((1+\beta) \left(\beta^{-\beta} n_s^{\alpha\beta} n_t^{\alpha} \right)^{\frac{1}{1+\beta}} + \sum_{j\in\mathcal{I}} n_j^{\alpha} \right)$ *where* α, n_j *and* $\mathcal{I}$ *are the same as in theorem 1.1.*

(I) *If* $\beta \neq 1$, *then* $\mathcal{H}^{\gamma}(K_T(\Omega(s,t,\beta))) = \mathcal{P}^{\gamma}(K_T(\Omega(s,t,\beta))) = +\infty$;

(II) *If* $\beta = 1$, *then*

(a) $0 < \mathcal{H}^{\gamma}(K_T(\Omega(s,t,\beta))), \mathcal{P}^{\gamma}(K_T(\Omega(s,t,\beta))) < +\infty$ *when* D *has uniform horizontal fibres;*

(b) $\mathcal{H}^{\gamma}(K_T(\Omega(s,t,\beta))) = \mathcal{P}^{\gamma}(K_T(\Omega(s,t,\beta))) = +\infty$ *when* D *does not have uniform horizontal fibres.*

Example. Take $D = I \times J = \{0, 1, \ldots, n-1\} \times \{0, 1, \ldots, m-1\}$. Then $\mathcal{L} = \sigma(D) = \{b_1, b_2, \ldots, b_m\} = \{0, 1, \ldots, m-1\}$ (recall that σ denotes the projection of $\mathbb{R}^2$ onto its second coordinate) and the horizontal fibres of D are $\Gamma_i = \{(0, i-1), (1, i-1), \ldots, (n-1, i-1)\}$ for $i = 1, 2, \ldots, \ell = m$ (so all $n_j = n$ for $1 \leqslant j \leqslant m$). For this case, the general Sierpinski carpet degenerates to the unit square, i.e. $K(T, D) = K(T, I \times J) = [0, 1]^2$. We denote by σ_1 the projection of $\mathbb{R}^2$ onto its first coordinate. Then, from (1) it follows that for each $x = (x_i)_{i=1}^{\infty} \in (I \times J)^{\mathbb{N}}$

$$K_T(x) = \left(\sum_{k=1}^{\infty} \frac{\sigma_1(x_k)}{n^k}, \sum_{k=1}^{\infty} \frac{\sigma(x_k)}{m^k} \right) \in [0, 1]^2,$$

i.e. $K_T(x)$ has its first and second coordinates represented as the n-adic and m-adic expansions, respectively. Thus, by (4) we have that for $1 \leqslant s \neq t \leqslant m$ and $\beta > 0$

$$\begin{aligned}\Omega(s,t,\beta) &= \left\{ x = (x_i)_{i=1}^{\infty} \in (I \times J)^{\mathbb{N}} : \zeta(x, \Gamma_s) = \beta\zeta(x, \Gamma_t) > 0 \right\} \\ &= \left\{ x = (x_i)_{i=1}^{\infty} \in (I \times J)^{\mathbb{N}} : \lim_{k\to\infty} \frac{\#\{1 \leqslant j \leqslant k : \sigma(x_j) = s-1\}}{k} \right. \\ &\qquad \left. = \beta \lim_{k\to\infty} \frac{\#\{1 \leqslant j \leqslant k : \sigma(x_j) = t-1\}}{k} > 0 \right\}\end{aligned}$$

and so

$$\begin{aligned}K_T(\Omega(s,t,\beta)) = [0,1] \times &\left\{ \sum_{i=1}^{\infty} \frac{y_i}{m^i} : (y_i)_{i=1}^{\infty} \in J^{\mathbb{N}} \text{ satisfying } \lim_{k\to\infty} \frac{\#\{1 \leqslant j \leqslant k : y_j = s-1\}}{k} \right. \\ &\left. = \beta \lim_{k\to\infty} \frac{\#\{1 \leqslant j \leqslant k : y_j = t-1\}}{k} > 0 \right\} := [0,1] \times \triangle.\end{aligned}$$

Therefore, by theorem 1.1 we have

$$1 + \dim_{\mathrm{H}} \triangle = \dim_{\mathrm{H}} K_T(\Omega(s,t,\beta)) = 1 + \log_m \left((1+\beta)\beta^{-\frac{\beta}{1+\beta}} + m - 2 \right).$$

The above equality also shows that

$$\dim_{\mathrm{H}} \triangle = \log_m \left((1+\beta)\beta^{-\frac{\beta}{1+\beta}} + m - 2 \right),$$

which was previously obtained in [1, theorem 2].

Below we give a more complicated example. Let

$$\begin{aligned}V = &\left\{ (x, y) \in [0,1]^2 : x = \sum_{k=1}^{\infty} \frac{x_k}{6^k} \text{ with } x_k \in \{0,1,2,3,4,5\}, \right. \\ &\quad y = \sum_{k=1}^{\infty} \frac{y_k}{4^k} \text{ with } y_k \in \{0,1,2,3\}, \lim_{k\to\infty} \frac{\#\{1 \leqslant i \leqslant k : y_i = 1\}}{k} \\ &\quad \left. = \beta \lim_{k\to\infty} \frac{\#\{1 \leqslant i \leqslant k : y_i = 2\}}{k} > 0 \text{ and } x_k + y_k = \text{odd number} \right\}.\end{aligned}$$

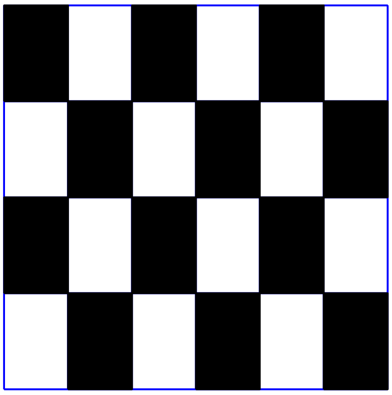

Figure 2. A pattern D with uniform horizontal fibres for $n = 6$ and $m = 4$.

Then, with D illustrated as in figure 2 we have $V = K_T(\Omega(2, 3, \beta))$ and by theorem 1.1

$$\dim_H V = \dim_P V = \log_4\left((1+\beta)\beta^{-\frac{\beta}{1+\beta}} + 2\right) + \log_6 3.$$

In addition, theorem 1.2 shows that the Hausdorff and packing measures of V in its dimension are positive finite when $\beta = 1$, infinite when $\beta \neq 1$.

The rest of this paper is organized as follows. In section 2, some basic facts and known results needed in the proof of our theorems are described. Proofs of theorems 1.1 and 1.2 are arranged in section 3.

2. Preliminaries

As in [9,10,12,13], a class of approximate squares are used to calculate the various dimensions of the general Sierpinski carpets and its subsets. For each $x = (x_j)_{j=1}^{\infty} \in (I \times J)^{\mathbb{N}}$ and each positive integer k, let

$$Q_k(x) = \{K_T(y) : y = (y_j)_{j=1}^{\infty} \in (I \times J)^{\mathbb{N}},\ y_j = x_j \text{ for } 1 \leqslant j \leqslant [\alpha k]$$
$$\text{and } \sigma(y_j) = \sigma(x_j) \text{ for } [\alpha k] + 1 \leqslant j \leqslant k\},$$

where, as usual, $[x]$ with $x \in \mathbb{R}$ denote the greatest integer function. The sets $Q_k(x)$ are approximate squares in $[0, 1]^2$, whose sizes have length $n^{-[\alpha k]}$ and m^{-k}. Note that the ratio of the sizes of $Q_k(x)$ is at most n, and their diameters $\operatorname{diam} Q_k(x)$ satisfy

$$\sqrt{2}m^{-k} \leqslant \operatorname{diam} Q_k(x) \leqslant \sqrt{2}nm^{-k}.$$

So in the definition of Hausdorff measure, we can restrict attention to covers by such approximate squares since any set of diameter less than m^{-k} can be covered by a bounded number of approximate squares $Q_k(x)$. The following lemma and related remark appear in [10] in which the approximate square $Q_k(x)$ plays the same role as the ball does in the classical density theorems. The following lemma 2.1, involved in Hausdorff measure, is just a reformulation of the Rogers–Taylor density theorem as stated by Peres in section 2 of [13]. The proof for results in remark 2.2, involved in packing measure, is given by Nielsen [10] as lemma 5 in section 2.

Lemma 2.1. ([10, lemma 4]) *Suppose that δ is a positive number, that μ is a finite Borel measure on $[0,1]^2$ and that E is a subset of $(I \times J)^{\mathbb{N}}$ such that $K_T(E)$ is a Borel subset of $[0,1]^2$, and $\mu(K_T(E)) > 0$, put*

$$A(x) = \limsup_{k \longrightarrow \infty}(k\delta + \log_m \mu(Q_k(x))).$$

for each point $x \in E$.

(1) If $A(x) = -\infty$ for all $x \in E$, then $\mathcal{H}^\delta(K_T(E)) = +\infty$.
(2) If $A(x) = +\infty$ for all $x \in E$, then $\mathcal{H}^\delta(K_T(E)) = 0$.
(3) If there are numbers a and b such that $a \leqslant A(x) \leqslant b$ for all $x \in E$, then $0 < \mathcal{H}^\delta(K_T(E)) < +\infty$.

Remark 2.2. ([10, lemma 5]) *Lemma 2.1 works for $\mathcal{P}^\delta(K_T(E))$ if the* lim sup *is replaced by* lim inf *in the definition of $A(x)$.*

The Borel measures on $[0,1]^2$ to which the above lemma will be applied are constructed as follows. Let $\boldsymbol{q} = (q_d)_{d\in D}$ be a probability vector on D, i.e. $\sum_{d\in D} q_d = 1$ with each $q_d \in (0,1)$. Then $\boldsymbol{q}$ determines a unique infinite product Borel probability measure, denoted by $\mu_{\boldsymbol{q}}$, on $D^{\mathbb{N}}$. For any finite sequence $(x_1, x_2, \cdots, x_k) \in D^k$

$$\mu_{\boldsymbol{q}}(\mathbb{C}(x_1, x_2, \cdots, x_k)) = \prod_{j=1}^{k} q_{x_j}, \tag{9}$$

where $\mathbb{C}(x_1, x_2, \cdots, x_k) := \{d = (d_j)_{j=1}^\infty \in D^{\mathbb{N}} : d_j = x_j \text{ for } 1 \leqslant j \leqslant k\}$ is a cylinder set of $D^{\mathbb{N}}$ with base $(x_1, x_2, \cdots, x_k)$. Let $\widetilde{\mu}_{\boldsymbol{q}}$ be the Borel probability measure on $K_T(D^{\mathbb{N}})$ which is the image measure of $\mu_{\boldsymbol{q}}$ under K_T, i.e. $\widetilde{\mu}_{\boldsymbol{q}}(A) = \mu_{\boldsymbol{q}}(K_T^{-1}A)$ for a Borel set $A \subseteq \mathbb{R}^2$. From the fact that each approximate square $Q_k(x)$ is an image of a finite union of cylinder sets under K_T, it follows that for any $x = (x_j)_{j=1}^\infty \in D^{\mathbb{N}}$ (cf formula (4) in [10], also formula (4.4) in [4])

$$\widetilde{\mu}_{\boldsymbol{q}}(Q_k(x)) = \prod_{j=1}^{[\alpha k]} q_{x_j} \times \prod_{j=[\alpha k]+1}^{k} \sum_{d\in D, \sigma(d)=\sigma(x_j)} q_d. \tag{10}$$

From the Kolmogorov strong law of large numbers (cf [3, corollary 2 in section 5.2]) it follows that $\mu_{\boldsymbol{q}}(\Xi_{\boldsymbol{q}}) = 1$ where, for a probability vector $\boldsymbol{q} = (q_d)_{d\in D}$ on D, $\mu_{\boldsymbol{q}}$ and $\Xi_{\boldsymbol{q}}$ are defined by (9) and (2), respectively. Therefore,

$$\widetilde{\mu}_{\boldsymbol{q}}(K_T(\Omega(s,t,\beta))) = 1, \qquad \text{if } \sum_{d\in\Gamma_s} q_d = \beta \sum_{d\in\Gamma_t} q_d. \tag{11}$$

Proposition 2.3. *Let $f(\boldsymbol{p})$ be defined by (7) with $\boldsymbol{p} \in \mathrm{cl}(\Sigma)$ (recall that Σ is defined by (5)). Then the maximum value $\max_{\boldsymbol{p}\in \mathrm{cl}(\Sigma)} f(\boldsymbol{p})$ is uniquely reached at $\boldsymbol{p}^* = (p_j^*)_{j=1}^\ell \in \Sigma$ where, with $\mathcal{I} = \{1, 2, \ldots, \ell\} \backslash \{s, t\}$*

$$\begin{cases} p_s^* = \beta p_t^* \\ p_j^* = \left(\beta^\beta n_s^{-\alpha\beta} n_t^{-\alpha}\right)^{\frac{1}{1+\beta}} n_j^\alpha p_t^*, \qquad j \in \mathcal{I} \\ p_t^* = \left(1 + \beta + \left(\beta^\beta n_s^{-\alpha\beta} n_t^{-\alpha}\right)^{\frac{1}{1+\beta}} \sum_{j\in\mathcal{I}} n_j^\alpha\right)^{-1}. \end{cases} \tag{12}$$

Proof. Clearly, $f(\boldsymbol{p})$ can obtain its maximum on $\mathrm{cl}(\Sigma)$ since $f(\boldsymbol{p})$ is continuous and $\mathrm{cl}(\Sigma)$ is compact. We first show that the maximum point is unique. Note that $f(\boldsymbol{p})$ is a strictly concave function in $\boldsymbol{p}$. In fact, the second summand of $f(\boldsymbol{p})$ is strictly concave and the first is concave.

On the other hand, $\mathrm{cl}(\Sigma)$ is convex, the constraint inequalities (i.e. $0 \leqslant p_i \leqslant 1, 1 \leqslant i \leqslant \ell$) are both convex and concave and its constraint equalities (i.e. $\sum_{i=1}^{\ell} p_i = 1$ and $p_s = \beta p_t$ in (5)) are all linear. By a well-known property of convex programming, there exists a unique $\boldsymbol{p}^* \in \mathrm{cl}(\Sigma)$ such that $f(\boldsymbol{p})$ attains its maximum at the point $\boldsymbol{p}^*$.

We now show that the maximum of $f(\boldsymbol{p})$ is obtained in Σ, equivalently, that $\boldsymbol{p}^* \in \Sigma$. Suppose $\boldsymbol{p}^* = (p_j^*)_{j=1}^{\ell} \in \mathrm{cl}(\Sigma) \setminus \Sigma$. Let $D_1 = \{1 \leqslant j \leqslant \ell : p_j^* = 0\}$ and $D_2 = \{1, 2, \ldots, \ell\} \setminus D_1$. Then both D_1 and D_2 are nonempty ($D_1 \neq \emptyset$ derives from the fact that $(p_j^*)_{j=1}^{\ell} \in \mathrm{cl}(\Sigma)\setminus\Sigma$ if and only if $p_j^* = 0$ for some j. $D_2 \neq \emptyset$ since $\sum_{j=1}^{\ell} p_j^* = 1$ and $\ell > 2$.). Take $\widetilde{\boldsymbol{p}} = (\widetilde{p}_j)_{j=1}^{\ell} \in \Sigma$. Let $\boldsymbol{p}_t = t\widetilde{\boldsymbol{p}} + (1-t)\boldsymbol{p}^* = (t\widetilde{p}_j + (1-t)p_j^*)_{j=1}^{\ell}$, $t \in [0, 1]$. Then $\boldsymbol{p}_t \in \Sigma$ for $t \in (0, 1]$ and $\boldsymbol{p}_0 = \boldsymbol{p}^*$. Note that

$$f'(\boldsymbol{p}_t) = \frac{\mathrm{d}}{\mathrm{d}t} f(\boldsymbol{p}_t) = -\sum_{j=1}^{\ell} (\widetilde{p}_j - p_j^*) \log_m (t\widetilde{p}_j + (1-t)p_j^*) + \sum_{j=1}^{\ell} (\widetilde{p}_j - p_j^*) \log_m n_j^{\alpha}$$

$$= -\sum_{j \in D_1} \widetilde{p}_j \log_m (t\widetilde{p}_j) - \sum_{j \in D_2} (\widetilde{p}_j - p_j^*) \log_m (t\widetilde{p}_j + (1-t)p_j^*) + \sum_{j=1}^{\ell} (\widetilde{p}_j - p_j^*) \log_m n_j^{\alpha}.$$

Thus we have $\lim_{t\to 0+} f'(\boldsymbol{p}_t) = +\infty$. Note that $\lim_{t\to 0+} f(\boldsymbol{p}_t) = f(\boldsymbol{p}^*)$. Thus, $f(\boldsymbol{p}_t) > f(\boldsymbol{p}^*) = f_{\max}$ when t is small enough, leading to a contradiction. Now let

$$L(\boldsymbol{p}, \lambda_1, \lambda_2) = \sum_{j=1}^{\ell} (p_j \log_m n_j^{\alpha} - p_j \log_m p_j) + \frac{\lambda_1}{\log m}(p_s - \beta p_t) + \frac{\lambda_2}{\log m}\left(\sum_{j=1}^{\ell} p_j - 1\right),$$

where, and throughout this paper, log denotes the natural logarithm. Since $\boldsymbol{p}^* = (p_j^*)_{j=1}^{\ell} \in \Sigma$ is the unique point such that $f(\boldsymbol{p}^*) = \max_{\boldsymbol{p}\in\Sigma} f(\boldsymbol{p})$ and $f(\boldsymbol{p})$ is a strictly concave function in $\boldsymbol{p}$, $\boldsymbol{p}^*$ is uniquely solved by the (method of Lagrange multipliers)

$$\begin{cases} \dfrac{\partial L}{\partial p_j} = 0, & 1 \leqslant j \leqslant \ell, \\ \dfrac{\partial L}{\partial \lambda_i} = 0, & i = 1, 2, \end{cases}$$

i.e.

$$\begin{cases} \alpha \log n_j - \log p_j - 1 + \lambda_2 = 0, & 1 \leqslant j \leqslant \ell, \quad j \neq s, t, \\ \alpha \log n_s - \log p_s - 1 + \lambda_1 + \lambda_2 = 0, \\ \alpha \log n_t - \log p_t - 1 - \beta\lambda_1 + \lambda_2 = 0, \\ p_s - \beta p_t = 0, \\ \displaystyle\sum_{j=1}^{\ell} p_j - 1 = 0. \end{cases}$$

This yields (12). □

3. Proofs

In this section, we give the proofs of theorems 1.1 and 1.2. These will be based on lemma 2.1, remark 2.2, (R1) and (R4).

Proof of theorem 1.1. It suffices to show that

$$\dim_{\mathrm{H}} K_T(\Omega(s, t, \beta)) \leqslant \log_m \left((1+\beta)\left(\beta^{-\beta} n_s^{\alpha\beta} n_t^{\alpha}\right)^{\frac{1}{1+\beta}} + \sum_{j\in\mathcal{I}} n_j^{\alpha} \right) := \gamma.$$

For any $x = (x_i)_{i=1}^{\infty} \in \Omega(s, t, \beta)$ and any positive integer k, denote

$$S_k(x) = \sum_{j \in \mathcal{I}} N_k(x, \Gamma_j) \log_m n_j. \tag{13}$$

Let $\boldsymbol{p}^* = (p_j^*)_{j=1}^{\ell}$ be given by (12). Take $\boldsymbol{q} = (q_d)_{d \in D}$ where $q_d = \frac{p_j^*}{n_j}$ for $d \in \Gamma_j$. Then $\boldsymbol{q} = (q_d)_{d \in D}$ is a probability vector on D and $\widetilde{\mu}_{\boldsymbol{q}}(K_T(\Omega(s, t, \beta))) = 1$ by (11). From (10) and (12) it follows:

$$\begin{aligned}
\log_m \widetilde{\mu}_{\boldsymbol{q}}(Q_k(x)) &= \sum_{j=1}^{[\alpha k]} \log_m q_{x_j} + \sum_{j=[\alpha k]+1}^{k} \log_m \sum_{d \in D, \sigma(d) = \sigma(x_j)} q_d \\
&= \sum_{j=1}^{\ell} N_{[\alpha k]}(x, \Gamma_j) \log_m \frac{p_j^*}{n_j} + \sum_{j=1}^{\ell} (N_k(x, \Gamma_j) - N_{[\alpha k]}(x, \Gamma_j)) \log_m p_j^* \\
&= \sum_{j=1}^{\ell} N_k(x, \Gamma_j) \log_m p_j^* - \sum_{j=1}^{\ell} N_{[\alpha k]}(x, \Gamma_j) \log_m n_j \\
&= \sum_{j \in \mathcal{I}} N_k(x, \Gamma_j) \left(\log_m \left(\beta^{\beta} n_s^{-\alpha\beta} n_t^{-\alpha} \right)^{\frac{1}{1+\beta}} + \log_m n_j^{\alpha} + \log_m p_t^* \right) + N_k(x, \Gamma_s) \\
&\quad \times \left(\log_m \beta + \log_m p_t^* \right) + N_k(x, \Gamma_t) \log_m p_t^* - \sum_{j=1}^{\ell} N_{[\alpha k]}(x, \Gamma_j) \log_m n_j \\
&= \sum_{j \in \mathcal{I}} N_k(x, \Gamma_j) \log_m \left(\beta^{\beta} n_s^{-\alpha\beta} n_t^{-\alpha} \right)^{\frac{1}{1+\beta}} + N_k(x, \Gamma_s) \log_m \beta + k \log_m p_t^* \\
&\quad - N_{[\alpha k]}(x, \Gamma_s) \log_m n_s - N_{[\alpha k]}(x, \Gamma_t) \log_m n_t + \alpha S_k(x) - S_{[\alpha k]}(x).
\end{aligned} \tag{14}$$

Therefore, for all $x \in \Omega(s, t, \beta)$

$$\begin{aligned}
\limsup_{k \to \infty} \frac{1}{k} \log_m \widetilde{\mu}_{\boldsymbol{q}}(Q_k(x)) &= (1 - (1+\beta) p_t^*) \log_m \left(\beta^{\beta} n_s^{-\alpha\beta} n_t^{-\alpha} \right)^{\frac{1}{1+\beta}} + p_t^* \beta \log_m \beta + \log_m p_t^* \\
&\quad - p_t^* \alpha\beta \log_m n_s - p_t^* \alpha \log_m n_t + \alpha \limsup_{k \to \infty} \left(\frac{S_k(x)}{k} - \frac{S_{[\alpha k]}(x)}{\alpha k} \right).
\end{aligned}$$

By means of (12), it is easy to check that

$$\begin{aligned}
&(1 - (1+\beta) p_t^*) \log_m \left(\beta^{\beta} n_s^{-\alpha\beta} n_t^{-\alpha} \right)^{\frac{1}{1+\beta}} + p_t^* \beta \log_m \beta + \log_m p_t^* \\
&\quad - p_t^* \alpha\beta \log_m n_s - p_t^* \alpha \log_m n_t = -\gamma.
\end{aligned}$$

In the following, we show that for every point $x \in \Omega(s, t, \beta)$,

$$\limsup_{k \to \infty} \left(\frac{S_k(x)}{k} - \frac{S_{[\alpha k]}(x)}{\alpha k} \right) \geqslant 0. \tag{15}$$

This essentially can be derived from lemma 4.1 in [6]. Obviously, for every point $x \in \Omega(s, t, \beta)$ and any $k \in \boldsymbol{N}$, from (13) we have

$$\sup_k |S_{k+1}(x) - S_k(x)| < \infty. \tag{16}$$

For a fixed $x = (x_j)_{j=1}^{\infty} \in \Omega(s, t, \beta)$, let $Y(k) = S_k(x)$. We extend Y to $[1, +\infty)$ by piecewise linear interpolation. Then Y is a Lipschitz function by (16). Now define $g : [0, \infty) \to \mathbb{R}$ by

$$g(z) = \mathrm{e}^{-z} Y(e^z).$$

We claim that $g(z)$ is bounded and uniformly continuous on $[0,\infty)$. In fact, for $z\in(0,+\infty)$ (the following is also true when $z=0$)

$$\begin{aligned}|g(z)| &\leqslant |g(0)|\mathrm{e}^{-z}+|g(z)-g(0)\mathrm{e}^{-z}|\leqslant |Y(1)|+\mathrm{e}^{-z}|Y(\mathrm{e}^{z})-Y(1)|\\ &\leqslant |Y(1)|+\frac{|Y(\mathrm{e}^{z})-Y(1)|}{\mathrm{e}^{z}-1}\leqslant |Y(1)|+\sup_{u,v\in[1,\infty),u\neq v}\frac{|Y(u)-Y(v)|}{|u-v|}\\ &=|Y(1)|+\mathrm{Lip}Y,\end{aligned}\tag{17}$$

and for any $\delta>0$

$$\begin{aligned}|g(z+\delta)-g(z)| &= |\mathrm{e}^{-(z+\delta)}Y(\mathrm{e}^{z+\delta})-\mathrm{e}^{-z}Y(\mathrm{e}^{z})|\\ &\leqslant \mathrm{e}^{-(z+\delta)}|Y(\mathrm{e}^{z+\delta})-Y(\mathrm{e}^{z})|+|g(z)|(1-\mathrm{e}^{-\delta})\\ &=\frac{|Y(\mathrm{e}^{z+\delta})-Y(\mathrm{e}^{z})|}{\mathrm{e}^{z+\delta}-\mathrm{e}^{z}}\times\frac{\mathrm{e}^{z+\delta}-\mathrm{e}^{z}}{\mathrm{e}^{z+\delta}}+|g(z)|(1-\mathrm{e}^{-\delta})\\ &\leqslant (1-\mathrm{e}^{-\delta})\mathrm{Lip}Y+(1-\mathrm{e}^{-\delta})(|Y(1)|+\mathrm{Lip}Y),\end{aligned}$$

by (17). Now for any $v>-\log\alpha$, we have

$$\begin{aligned}\left|\int_{-\log\alpha}^{v}(g(z)-g(z+\log\alpha))\,\mathrm{d}z\right| &= \left|\int_{-\log\alpha}^{v}g(z)\,\mathrm{d}z-\int_{-\log\alpha}^{v}g(z+\log\alpha)\,\mathrm{d}z\right|\\ &=\left|\int_{-\log\alpha}^{v}g(z)\,\mathrm{d}z-\int_{0}^{v+\log\alpha}g(z)\,\mathrm{d}z\right|\\ &=\left|\int_{0}^{-\log\alpha}g(z)\,\mathrm{d}z+\int_{v}^{v+\log\alpha}g(z)\,\mathrm{d}z\right|\\ &\leqslant\left|\int_{0}^{-\log\alpha}g(z)\,\mathrm{d}z\right|+\left|\int_{v}^{v+\log\alpha}g(z)\,\mathrm{d}z\right|\\ &\leqslant 2(|Y(1)|+\mathrm{Lip}Y)|\log\alpha|,\end{aligned}$$

by (17). Therefore,

$$\limsup_{z\to+\infty}(g(z)-g(z+\log\alpha))\geqslant 0.$$

Otherwise, $\left|\int_{-\log\alpha}^{v}(g(z)-g(z+\log\alpha))\,\mathrm{d}z\right|\to+\infty$ as $v\to+\infty$. By letting $z=\log t$, this gives

$$\limsup_{t\to+\infty}\left(\frac{Y(t)}{t}-\frac{Y(\alpha t)}{\alpha t}\right)\geqslant 0.$$

Note that

$$\begin{aligned}\frac{Y(t)}{t}-\frac{Y(\alpha t)}{\alpha t} &= \left(\frac{Y(t)-Y([t])}{t}-\frac{Y(\alpha t)-Y([\alpha t])}{\alpha t}\right)+\frac{Y([t])}{[t]}\left(\frac{[t]}{t}-1\right)\\ &\quad+\left(\frac{S_{[\alpha[t]]}(x)}{\alpha[t]}-\frac{S_{[\alpha t]}(x)}{\alpha t}\right)+\left(\frac{S_{[t]}(x)}{[t]}-\frac{S_{[\alpha[t]]}(x)}{\alpha[t]}\right),\end{aligned}\tag{18}$$

where, as before, $[t]$ with $t\in\mathbb{R}$ denotes the greatest integer function. However, the first three terms in the right side of (18) tend to zero as $t\to+\infty$ by the facts that both functions $|Y(t)-Y([t])|$ and $g(z)$ are bounded, and $g(z)$ is uniformly continuous. Hence (15) holds. Therefore, for every $x=(x_j)_{j=1}^{\infty}\in\Omega(s,t,\beta)$ we have

$$\limsup_{k\to\infty}\frac{1}{k}\log_m\widetilde{\mu}_q(Q_k(x))\geqslant-\gamma,$$

which leads to

$$\limsup_{k\to\infty}(k\delta+\log_m\widetilde{\mu}_{\boldsymbol{q}}(Q_k(x)))=\limsup_{k\to\infty}k\left(\delta+\frac{1}{k}\log_m\widetilde{\mu}_{\boldsymbol{q}}(Q_k(x))\right)=+\infty,$$

for any $\delta>\gamma$. Now lemma 2.1 (2) implies that $\dim_{\mathrm{H}} K_T(\Omega(s,t,\beta))\leqslant\gamma$.

Finally, for the case when $D\backslash(\Gamma_s\cup\Gamma_t)$ has uniform horizontal fibres we have

$$\lim_{k\to\infty}\left(\frac{S_k(x)}{k}-\frac{S_{[\alpha k]}(x)}{\alpha k}\right)=0,$$

by (13). This gives $\dim_{\mathrm{P}} K_T(\Omega(s,t,\beta))\leqslant\gamma$ by remark 2.2.

Proof of theorem 1.2. The desired results (I) and (II)(b) can be derived directly from (R4) (b) by the facts that $K_T(\Omega(s,t,\beta))\supset K_T(\Omega(\Gamma,\boldsymbol{p}^*))$ and $\dim_{\mathrm{H}} K_T(\Omega(s,t,\beta))=\dim_{\mathrm{H}} K_T(\Omega(\Gamma,\boldsymbol{p}^*))=f(\boldsymbol{p}^*)$ by (6), theorem 1.1 and (R1).

When $\beta=1$ and D has uniform horizontal fibres (i.e. $n_j=n_1$ for all $1\leqslant j\leqslant\ell$), we have $p_j^*=\ell^{-1}$ for all $1\leqslant j\leqslant\ell$ by (12). In this case, we have

$$\gamma=f(\boldsymbol{p}^*)=\log_m\ell+\alpha\log_m n_1,$$

and for all $x\in\Omega(s,t,1)$, it follows from the last second equality in (14) that

$$\begin{aligned}\log_m\widetilde{\mu}_{\boldsymbol{q}}(Q_k(x))&=\sum_{j\in\mathcal{I}}N_k(x,\Gamma_j)\left(\log_m\left(\beta^\beta n_s^{-\alpha\beta}n_t^{-\alpha}\right)^{\frac{1}{1+\beta}}+\log_m n_j^\alpha+\log_m p_t^*\right)+N_k(x,\Gamma_s)\\&\quad\times\left(\log_m\beta+\log_m p_t^*\right)+N_k(x,\Gamma_t)\log_m p_t^*-\sum_{j=1}^{\ell}N_{[\alpha k]}(x,\Gamma_j)\log_m n_j\\&=\sum_{j\in\mathcal{I}}N_k(x,\Gamma_j)\left(\log_m n_1^{-\alpha}+\log_m n_1^\alpha+\log_m\ell^{-1}\right)+N_k(x,\Gamma_s)\log_m\ell^{-1}\\&\quad+N_k(x,\Gamma_t)\log_m\ell^{-1}-\sum_{j=1}^{\ell}N_{[\alpha k]}(x,\Gamma_j)\log_m n_1=\log_m\ell^{-1}\sum_{j=1}^{\ell}N_k(x,\Gamma_j)\\&\quad-\log_m n_1\sum_{j=1}^{\ell}N_{[\alpha k]}(x,\Gamma_j)=-k\log_m\ell-[\alpha k]\log_m n_1.\end{aligned}$$

Then

$$A(x)=\limsup_{k\longrightarrow\infty}(kf(\boldsymbol{p}^*)+\log_m\mu_{\boldsymbol{q}}(Q_k(x)))=\limsup_{k\longrightarrow\infty}(\alpha k-[\alpha k])\log_m n_1$$

is bounded on $\Omega(s,t,1)$ and so (II) (a) holds by lemma 2.1 (3).

Finally, as to the results on $\mathcal{P}^\gamma(K_T(\Omega(s,t,\beta)))$ we only need to prove (II) (a). By remark 2.2, this is done by the fact that $\liminf_{k\longrightarrow\infty}(kf(\boldsymbol{p}^*)+\log_m\mu_{\boldsymbol{q}}(Q_k(x)))$ is finite on $\Omega(s,t,1)$.

Remark 3.1. *Let both Γ^* and Γ^{**} be unions of certain horizontal fibres of D. Comparing with* (4) *let*

$$\Omega(\Gamma^*,\Gamma^{**},\beta)=\left\{x=(x_i)_{i=1}^\infty\in D^{\mathbb{N}}:\zeta(x,\Gamma^*)=\beta\zeta(x,\Gamma^{**})>0\right\},\qquad(19)$$

where $\zeta(x,\Gamma^*)$ and $\zeta(x,\Gamma^{**})$ are defined as in (3). *When $\Gamma^*\cap\Gamma^{**}=\emptyset$, an analogue of theorems 1.1 and 1.2 for $K_T(\Omega(\Gamma^*,\Gamma^{**},\beta))$ can be obtained in the same way, even if there are more proportional frequencies in* (19). *In addition, if Γ^* and Γ^{**} are two arbitrary subsets of D, the Hausdorff dimension of $K_T(\Omega(\Gamma^*,\Gamma^{**},\beta))$ can also be determined implicitly in a similar way.*

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References

[1] Barreira L, Saussol B and Schmeling J 2002 Distribution of frequencies of digits via multifractal analysis *J. Number Theory* **97** 410–38

[2] Bedford T 1984 Crinkly curves, Markov partitions and box dimension in self-similar sets *PhD Thesis* University of Warwick

[3] Chow Y S and Teicher H 1978 *Probability Theory* (New York: Springer)

[4] Gatzouras D and Lalley S P 1992 Hausdorff and box dimensions of certain self-affine fractals *Indiana Univ. Math. J.* **41** 533–68

[5] Gui Y X and Li W X 2007 Hausdorff dimension of fiber-coding sub-Sierpinski carpets *J. Math. Anal. Appl.* **331** 62–8

[6] Kenyon R and Peres Y 1996 Measures of full dimension on affine-invariant sets *Ergodic Theory Dyn. Syst.* **16** 307–23

[7] Kenyon R and Peres Y 1996 Hausdorff dimensions of sofic affine-invariant sets *Isr. J. Math.* **122** 540–74

[8] King J 1995 The singularity spectrum for general Sierpinski carpets *Adv. Math.* **116** 1–8

[9] McMullen C 1984 The Hausdorff dimension of general Sierpinski carpets *Nagoya Math. J.* **96** 1–9

[10] Nielsen O A 1999 The Hausdorff and packing dimensions of some sets related to Sierpinski carpets *Can. J. Math.* **51** 1073–88

[11] Olsen L 1998 Self-affine multifractal Sierpinski sponges in $\mathbb{R}^d$ *Pac. J. Math.* **183** 143–99

[12] Peres Y 1994 The packing measure of self-affine carpets *Math. Proc. Camb. Phil. Soc.* **115** 437–50

[13] Peres Y 1994 The self-affine carpets of McMullen and Bedford have infinite Hausdorff measure *Math. Proc. Camb. Phil. Soc.* **116** 513–26