Separation properties for \( MW \)-fractals

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Abstract

Corresponding to the irreducible 0−1 matrix \((a_{ij})_{n \times n}\), take similitude contraction mappings \( \varphi_{ij} \) for each \( a_{ij} = 1 \), in \( \mathbb{R}^d \) with ratio \( 0 < r_{ij} < 1 \). There exist unique nonempty compact sets \( F_1, \cdots, F_n \) satisfying for each \( 1 \leq i \leq n \)

\[
 F_i = \bigcup_{1 \leq j \leq n, a_{ij} = 1} \varphi_{ij}(F_j).
\]

We prove that open set condition holds if and only if \( F_i \) is an \( s \)-set for some \( 1 \leq i \leq n \), where \( s \) is such that the spectral radius of matrix \( (r_{ij}^s)_{n \times n} \) is 1.

Keywords: Open set condition; \( s \)-set; \( MW \)-fractal.

1 Introduction

The self-similar fractal is defined as the unique compact set \( F \) such that \( F = \bigcup_{i=1}^n \varphi_i(F) \) where, for each \( 1 \leq i \leq n \), \( \varphi_i \) is a similitude contraction mapping in \( \mathbb{R}^d \) with similitude ratio \( 0 < r_i < 1 \). For self-similar fractal, A.Schief [1] has proved the following three conditions are equivalent:

(i) \( \{\varphi_i : 1 \leq i \leq n\} \) satisfies the open set condition;
(ii) \( \{\varphi_i : 1 \leq i \leq n\} \) satisfies the strong open set condition;
(iii) \( F \) is a \( t \)-set, where \( t \) is such that \( \sum_{i=1}^n r_i^t = 1 \).

Now let us consider the generalization of self-similar fractal, sometimes called \( MW \)-fractal. Let \( A_{n \times n} = (a_{ij})_{n \times n} \) be an irreducible 0−1 matrix. Let \( \mathcal{F} = \{\varphi_{ij} : 1 \leq i, j \leq n, a_{ij} = 1\} \) be a collection of similitude contraction mappings in \( \mathbb{R}^d \) with \( 0 < r_{ij} < 1 \) as the similitude ratio of \( \varphi_{ij} \). Then there exist unique nonempty compact sets \( F_1, \cdots, F_n \) such that for each \( 1 \leq i \leq n \)

\[
 F_i = \bigcup_{1 \leq j \leq n, a_{ij} = 1} \varphi_{ij}(F_j).
\]
We say $\mathcal{F}$ satisfies the open set condition if there are nonempty bounded open sets $O_1, \ldots, O_n$ such that for each $1 \leq i \leq n$ 
\[ O_i \supset \bigcup_{1 \leq j \leq n, a_{ij} = 1} \varphi_{ij}(O_j) \quad \text{and} \quad \varphi_{ij_1}(O_{j_1}) \cap \varphi_{ij_2}(O_{j_2}) = \emptyset \quad \text{if} \quad j_1 \neq j_2 \]
and $\mathcal{F}$ satisfies the strong open set condition if furthermore $F_i \cap O_i \neq \emptyset$ for some $1 \leq i \leq n$. Obviously when $\mathcal{F}$ satisfies the strong open set condition, $F_i \cap O_i \neq \emptyset$ for all $1 \leq i \leq n$ by (1).

It is known that if $\mathcal{F}$ satisfies the open set condition, then $F_i$ are s-set for all $1 \leq i \leq n$, where $s$ is such that the spectral radius of matrix $(r_{ij}^s)_{n \times n}$ is 1 (we adopt the convention that the $(i, j)$-th entry of matrix $(r_{ij}^s)_{n \times n}$ is 0 if $a_{ij} = 0$) (see [2][3]). It is worth to notice that from the irreducibility of $A_{n \times n}$ it follows that number 1 is the eigenvalue with the largest absolute value of $(r_{ij}^s)_{n \times n}$ which has strictly positive left and right eigenvectors (see Theorem 0.16 in [4]).

In this paper, we will show that the similar results hold for $MW$-fractals, i.e. the following three conditions are equivalent:

(i) $\mathcal{F}$ satisfies the open set condition;

(ii) $\mathcal{F}$ satisfies the strong open set condition;

(iii) $F_i$ is an $s$-set for some $1 \leq i \leq n$.

Now let us introduce some notations and definitions as follows.

1) $\Omega = \{\sigma = (\sigma(1), \sigma(2), \ldots) : a_{\sigma(|\sigma|)}, \sigma(|\sigma|) = 1 \quad \text{for} \quad k \in \mathbb{N}\}; \Omega_k = \{\sigma = (\sigma(1), \ldots, \sigma(k)) : a_{\sigma(i)}, \sigma(|\sigma|) = 1 \quad \text{for} \quad 1 \leq i \leq k \leq k - 1\}$ and $\Omega^* = \bigcup_{k \geq 2} \Omega_k$. For $\sigma \in \Omega^*$, $|\sigma|$ denotes the word length and the $\sigma(k)$, $1 \leq k \leq |\sigma|$, the $k$-th component of $\sigma$. When $k \leq |\sigma|$, we denote $\sigma(k) = (\sigma(1), \ldots, \sigma(k))$. Let $\Omega^*_k = \{\sigma \in \Omega^* : \sigma(1) = k\}$ for $1 \leq k \leq n$, and $\Omega^* = \prod_{k=1}^n \Omega^*_k$. $\tilde{\sigma}$ denotes the element of $\tilde{\Omega}^*$ with $\sigma_k \in \Omega^*_k$, $1 \leq k \leq n$, being the $k$-th component of $\tilde{\sigma}$.

2) For any $\sigma \in \Omega^*_k$, $k \geq 2$, $r_{|\sigma|} = \prod_{i=1}^{k-1} r_{\sigma(i), \sigma(i+1)}$; $\varphi_{\sigma} = \varphi_{\sigma(1), \sigma(2)} \circ \varphi_{\sigma(2), \sigma(3)} \circ \cdots \circ \varphi_{\sigma(k-1), \sigma(k)}$; $F_{\sigma} = \varphi_{\sigma}(F_{\sigma(k)})$.

3) For any $\sigma, \tau \in \Omega^*$, $\sigma$ and $\tau$ are called to be connected if and only if $|\sigma| = \tau(1)$, and define $\sigma \tau = (\sigma(1), \ldots, \sigma(|\sigma|), \tau(2), \ldots, \tau(|\sigma|))$. It is easy to see that for connected $\sigma$ and $\tau$, $|\sigma \tau| = |\sigma| + |\tau| - 1$, $r_{\sigma \tau} = r_{\sigma} r_{\tau}$, $\varphi_{\sigma \tau} = \varphi_{\sigma} \circ \varphi_{\tau}$ and $F_{\sigma \tau} = \varphi_{\sigma}(F_{\tau})$.

4) For any two different $\sigma, \tau \in \Omega^*$, we say $\sigma$ and $\tau$ are incomparable if there exists no $\omega \in \Omega^*$ such that $\sigma = \tau \omega$ or $\tau = \sigma \omega$. We adopt the convention that $\sigma$ and $\tau$ are comparable. For any $\tilde{\sigma}, \tilde{\tau} \in \tilde{\Omega}^*$, we say $\tilde{\sigma}$ and $\tilde{\tau}$ are incomparable if $\sigma_k$ and $\tau_k$ are incomparable for all $1 \leq k \leq n$. For $A \subset \tilde{\Omega}^*$, $A$ is termed as incomparable subset of $\tilde{\Omega}^*$ if for any different $\tilde{\sigma}, \tilde{\tau} \in A$, $\tilde{\sigma}$ and $\tilde{\tau}$ are incomparable.

5) For any $\tilde{\sigma}, \tilde{\tau} \in \tilde{\Omega}^*$, define $\tilde{\sigma} \tilde{\tau} = \tilde{\omega}$ where $\omega_k = \sigma_k \tau_{\sigma_k(1)}$. Then for any $\tilde{\sigma}, \tilde{\tau} \in \tilde{\Omega}^*$, the notation $\tilde{\sigma} \tilde{\tau}$ will work, but not for any $\sigma, \tau \in \Omega^*$. Define $r_{\tilde{\sigma}} = (r_{\sigma(1)}, \ldots, r_{\sigma(n)})'$ for any $\tilde{\sigma} \in \tilde{\Omega}^*$.

6) For $(a_1, \ldots, a_n)'$, $(b_1, \ldots, b_n)' \in \mathbb{R}^n$, we say $(a_1, \ldots, a_n)' \leq (b_1, \ldots, b_n)'$ if and only if $a_i \leq b_i$ for all $1 \leq i \leq n$, and $(a_1, \ldots, a_n)' < (b_1, \ldots, b_n)'$ if and only if $a_i < b_i$ for all $1 \leq i \leq n$.

7) We use $d(\cdot, \cdot)$ to denote the Euclidean metric in $\mathbb{R}^d$, and $D(\cdot, \cdot)$ the Hausdorff metric in the complete metric space $\mathcal{H}(\mathbb{R}^d)$ of nonempty compact subsets of $\mathbb{R}^d$. For positive
real number \( r \) and set \( E \), write \( B(r, E) = \{ y \in \mathbb{R}^d : d(y, E) < r \} \). Thus \( B(r, \{x\}) \) is an open ball with centre at \( x \) and radius \( r \).

8) In the following, denote \( h = \min_{1 \leq i \leq n} \mathcal{H}^s(F_i) \) and \( r_0 = \min_{a_{ij}=1} r_{ij} \).

## 2 Main results

**Proposition 2.1** If \( \mathcal{H}(F_i) > 0 \) for some \( 1 \leq i \leq n \), then \( \mathcal{H}(F_i) > 0 \) for all \( 1 \leq i \leq n \) and

\[
(r_{ij}^s)_{n \times n} \left( \begin{array}{c}
\mathcal{H}(F_1) \\
\vdots \\
\mathcal{H}(F_n)
\end{array} \right) = \left( \begin{array}{c}
\mathcal{H}(F_1) \\
\vdots \\
\mathcal{H}(F_n)
\end{array} \right).
\]

**Proof** It is easy to get that \( \mathcal{H}(F_i) > 0 \) for all \( 1 \leq i \leq n \) and

\[
\left( \begin{array}{c}
\mathcal{H}(F_1) \\
\vdots \\
\mathcal{H}(F_n)
\end{array} \right) \leq (r_{ij}^s)_{n \times n} \left( \begin{array}{c}
\mathcal{H}(F_1) \\
\vdots \\
\mathcal{H}(F_n)
\end{array} \right) \quad \text{(3)}
\]

by (1). Now let \((\beta_1, \cdots, \beta_n)\) be a strictly positive left eigenvector of matrix \((r_{ij}^s)_{n \times n}\) relative to the eigenvalue 1. Therefore

\[
(\beta_1, \cdots, \beta_n) \left( \begin{array}{c}
\mathcal{H}(F_1) \\
\vdots \\
\mathcal{H}(F_n)
\end{array} \right) \leq (\beta_1, \cdots, \beta_n) (r_{ij}^s)_{n \times n} \left( \begin{array}{c}
\mathcal{H}(F_1) \\
\vdots \\
\mathcal{H}(F_n)
\end{array} \right) = (\beta_1, \cdots, \beta_n) \left( \begin{array}{c}
\mathcal{H}(F_1) \\
\vdots \\
\mathcal{H}(F_n)
\end{array} \right)
\]

which implies (3) takes equality. **QED**

By Proposition 2.1 we can get the following Corollary 2.2 immediately.

**Corollary 2.2** For any incomparable \( \sigma, \tau \in \Omega_k^*, 1 \leq k \leq n \), we have \( \mathcal{H}(F_\sigma \cap F_\tau) = 0 \). **QED**

Now for any \( B \subset \mathbb{R}^d \), let

\[
\mu(B) = \inf \{ \sum_{i \in I} |U_i|^* : U_i \text{ open set and } \bigcup_{i \in I} U_i \supset B \},
\]

where \( |A| \) denotes the diameter of set \( A \). The \( \mu(\cdot) \) is an outer measure (see [5]). Similar to the Proposition 3 of [6], we have

**Proposition 2.3** For any Borel measurable set \( B_i \subset F_i, 1 \leq i \leq n \), we have \( \mathcal{H}(B_i) = \mu(B_i) \).
Proof  By the definition of $\mu(\cdot)$ and $\mathcal{H}^s(\cdot)$, it is clear that
\[
\mu(B_i) \leq \mathcal{H}^s(B_i).
\] (4)

Now let us consider the case $B_i = F_i$, $1 \leq i \leq n$. We suffice to show that for each open covering $\{U_j(i)\}_j$ of $F_i$ and each $\delta > 0$, there is another covering $\{V_j(i)\}_j$ of $F_i$ with $|V_j(i)| < \delta$ for each $j$ and $\sum_{j} |V_j(i)|^s = \sum_{j} |U_j(i)|^s$. To do this, we choose $k$ large enough such that $r_{\sigma} < \delta \sum_{j=1}^{n} \cup_{j} U_j(i) - 1$ for each $\sigma \in \Omega_k$. We replace each open set family
$\{U_j(i)\}_j$ with $\cup_{\sigma \in \Omega_k, \sigma(1)=i} \{\varphi_{\sigma}(U_j(\sigma(k)))\}_j$, denoted by $\{V_j(i)\}_j$, which covers the $F_i$. So
\[
\left(\begin{array}{cccc}
\sum_{j} |V_j(1)|^s \\
\vdots \\
\sum_{j} |V_j(n)|^s
\end{array}\right) = \left(\begin{array}{cccc}
r_{ij}^s \\
\vdots \\
\mu(F_n)
\end{array}\right)
\]
which leads that
\[
(\beta_1, \ldots, \beta_n) \left(\begin{array}{cccc}
\sum_{j} |V_j(1)|^s \\
\vdots \\
\sum_{j} |V_j(n)|^s
\end{array}\right) = (\beta_1, \ldots, \beta_n) \left(\begin{array}{cccc}
\sum_{j} |U_j(1)|^s \\
\vdots \\
\sum_{j} |U_j(n)|^s
\end{array}\right).
\]
Therefore
\[
(\beta_1, \ldots, \beta_n) \left(\begin{array}{c}
\mathcal{H}^s(F_1) \\
\vdots \\
\mathcal{H}^s(F_n)
\end{array}\right) \leq (\beta_1, \ldots, \beta_n) \left(\begin{array}{c}
\mu(F_1) \\
\vdots \\
\mu(F_n)
\end{array}\right).
\]
Thus we have $\mu(F_i) = \mathcal{H}^s(F_i)$ for $1 \leq i \leq n$ by (4).

Now for Borel measurable sets $B_i \subset F_i$, $1 \leq i \leq n$, we have
\[
\left(\begin{array}{c}
\mathcal{H}^s(F_1) \\
\vdots \\
\mathcal{H}^s(F_n)
\end{array}\right) = \left(\begin{array}{cccc}
\mathcal{H}^s(B_1) \\
\vdots \\
\mathcal{H}^s(B_n)
\end{array}\right) + \left(\begin{array}{cccc}
\mathcal{H}^s(F_1 \setminus B_1) \\
\vdots \\
\mathcal{H}^s(F_n \setminus B_n)
\end{array}\right) \geq \left(\begin{array}{cccc}
\mu(B_1) \\
\vdots \\
\mu(B_n)
\end{array}\right) + \left(\begin{array}{cccc}
\mu(F_1 \setminus B_1) \\
\vdots \\
\mu(F_n \setminus B_n)
\end{array}\right)
\]
which complete our proof by (4). QED

**Theorem 2.4** If $\mathcal{H}^s(F_i) > 0$ for some $1 \leq i \leq n$, then $\mathcal{F}$ satisfies the strong open set condition.

**Proof**  First it is easy to see that $\mathcal{H}^s(F_i) > 0$ for some $1 \leq i \leq n$ implies that $0 < \mathcal{H}^s(F_i) < \infty$ for all $1 \leq i \leq n$. On the other hand, it is not difficult to show that $s = 0$ if and only if $F_i$ is singleton for all $1 \leq i \leq n$. Then the strong open set condition holds when $s = 0$. In the following, we assume $s > 0$. Given $0 < p < 1$, let
\[
I_p = \{\sigma \in \Omega^*: r_{\sigma} < p \leq r_{\sigma}|(1-s)|\}.
\]
It is easy to see that any two different elements of $I_p$ are incomparable. We divide the proof into the following four steps.

(a) Given $x > 0$, there exists open set families $\{U_j(i)\}_j$, $1 \leq i \leq n$, such that $\bigcup_j U_j(i) \supset F_i$ and

\[
\begin{pmatrix}
\sum_j |U_j(1)|^s \\
\vdots \\
\sum_j |U_j(n)|^s
\end{pmatrix} \leq (1 + x^s) \begin{pmatrix}
\mathcal{H}^s(F_1) \\
\vdots \\
\mathcal{H}^s(F_n)
\end{pmatrix}.
\]

Writing $V_i = \bigcup_j U_j(i)$, $1 \leq i \leq n$, let $\delta = \min_{1 \leq i \leq n} d(F_i, V_i^c)$. At first, we will show that if $\tilde{\sigma}$ and $\tilde{\tau}$ are incomparable and $h^\frac{1}{s}r_{\tilde{\tau}} > xr_{\tilde{\sigma}}$, then

\[
\begin{pmatrix}
D(F_{\sigma_1}, F_{\tau_1}) \\
\vdots \\
D(F_{\sigma_n}, F_{\tau_n})
\end{pmatrix} \geq \delta \begin{pmatrix}
r_{\sigma_1} \\
\vdots \\
r_{\sigma_n}
\end{pmatrix}.
\]

Otherwise, without loss of generality we assume $D(F_{\sigma_1}, F_{\tau_1}) < \delta r_{\sigma_1}$. Since

\[
\begin{pmatrix}
d(F_{\sigma_1}, \varphi_{\sigma_1}(V_{\sigma_1(\{\sigma_1\})}^c)) \\
\vdots \\
d(F_{\sigma_n}, \varphi_{\sigma_n}(V_{\sigma_n(\{\sigma_n\})}^c))
\end{pmatrix} \geq \delta \begin{pmatrix}
r_{\sigma_1} \\
\vdots \\
r_{\sigma_n}
\end{pmatrix},
\]

we have $F_{\tau_1} \subset \varphi_{\sigma_1}(V_{\sigma_1(\{\sigma_1\})})$. Thus by Corollary 2.2, Proposition 2.3 and (5)

\[
(1 + x^s)r_{\sigma_1}^s \mathcal{H}^s(F_{\sigma_1(\{\sigma_1\})}) < r_{\sigma_1}^s \mathcal{H}^s(F_{\sigma_1(\{\sigma_1\})}) + r_{\tau_1}^s \mathcal{H}^s(F_{\tau_1(\{\tau_1\})}) = \mathcal{H}^s(F_{\sigma_1}) + \mathcal{H}^s(F_{\tau_1})
\]

\[
= \mathcal{H}^s(F_{\sigma_1} \bigcup F_{\tau_1}) \leq \sum_j |\varphi_{\sigma_1}(U_j(\{\sigma_1\}))|^s
\]

\[
= r_{\sigma_1}^s \sum_j |U_j(\{\sigma_1\})|^s \leq (1 + x^s)r_{\sigma_1}^s \mathcal{H}^s(F_{\sigma_1(\{\sigma_1\})})
\]

which leads a contradiction.

(b) Let $0 < \varepsilon < \frac{1}{s}$. For any $\tilde{\sigma} \in \tilde{\Omega}^*$, writing $p_k = |B(\varepsilon r_{\sigma_k}, F_{\sigma_k})|$ and let

$I^*(\tilde{\sigma}) = \{\tilde{\tau} \in \tilde{\Omega}^* : \tau_k \in I_{p_k}, 1 \leq k \leq n\}$. 

$I_1(\tilde{\sigma}) = \{\tilde{\tau} \in \tilde{\Omega}^* : \tau_k \in I_{p_k}, F_{\tau_k} \bigcap B(\varepsilon r_{\sigma_k}, F_{\sigma_k}) \neq \emptyset, 1 \leq k \leq n\}$. 

Note that for any $\tilde{\tau}, \tilde{\omega} \in I^*(\tilde{\sigma})$, $\tilde{\tau}$ and $\tilde{\omega}$ are incomparable if and only if $\tau_k \neq \omega_k$ for all $1 \leq k \leq n$. Let $I(\tilde{\sigma})$ be the incomparable subset of $I_1(\tilde{\sigma})$ which has the largest cardinality. Thus there exists no element of $I_1(\tilde{\sigma})$ which is incomparable with each element of $I(\tilde{\sigma})$.

Writing $\gamma = \sup_{\tilde{\sigma} \in \tilde{\Omega}^*} \text{Card } I(\tilde{\sigma})$, we will show that $\gamma < \infty$. Now fix $\tilde{\sigma} \in \tilde{\Omega}^*$. Then for any incomparable $\tilde{\tau}, \tilde{\omega} \in I^*(\tilde{\sigma})$, we have

$h^\frac{1}{s}r_{\tilde{\tau}} > h^\frac{1}{s}r_{\tau_0}r_{\tilde{\omega}}$.

Taking $x = h^\frac{1}{s}r_{\tau_0}$ in (a), we can get the relating $\delta$ and

$(D(F_{\omega_1}, F_{\tau_1}), \ldots, D(F_{\omega_n}, F_{\tau_n}))' \geq \delta r_{\tilde{\omega}}$.
by (6). Therefore there exist \( y_{\omega_k} \in F_{\omega_k(|\omega_k|)} \) and \( y_{\tau_k}^* \in F_{\tau_k(|\tau_k|)} \), \( 1 \leq k \leq n \), such that
\[
d(\varphi_{\omega_k}(y_{\omega_k}), \varphi_{\tau_k}(y_{\tau_k}^*)) \geq \delta r_{\omega_k}.
\] (7)

Take \( z_{\omega_k} \in F_{\omega_k(|\omega_k|)} \) and \( z_{\tau_k}^* \in F_{\tau_k(|\tau_k|)} \), \( 1 \leq k \leq n \), such that
\[
d(y_{\omega_k}, z_{\omega_k}) < \frac{1}{3}\delta r_0, \quad d(y_{\tau_k}^*, z_{\tau_k}^*) < \frac{1}{3}\delta r_0.
\] (8)

Then from (7) and (8) it follows that for \( 1 \leq k \leq n \)
\[
d(\varphi_{\omega_k}(z_{\omega_k}), \varphi_{\tau_k}(z_{\tau_k}^*)) \geq \frac{\delta}{3} r_{\omega_k}.
\]

Now for each \( 1 \leq k \leq n \) we fix a finite set \( Z_k \subset F_k \) whose \( \frac{1}{3}\delta r_0 \) neighborhood covers \( F_k \). Therefore for any incomparable \( \tilde{\tau}, \tilde{\omega} \in I^*(\tilde{\sigma}) \), there exist \( z_k \in Z_k, 1 \leq k \leq n \), such that
\[
d(\varphi_{\tau_k}(z_{\tau_k}(|\tau_k|)), \varphi_{\omega_k}(z_{\omega_k}(|\omega_k|))) \geq \frac{\delta}{3} r_{\tau_k} \geq \frac{\delta}{3} P_k r_0.
\]

Now fix \( z_k \in Z_k, 1 \leq k \leq n \), choose \( I \subset I(\tilde{\sigma}) \) such that for any different \( \tilde{\tau}, \tilde{\omega} \in I \)
\[
d(\varphi_{\tau_k}(z_{\tau_k}(|\tau_k|)), \varphi_{\omega_k}(z_{\omega_k}(|\omega_k|))) \geq \frac{\delta}{3} r_{\tau_k} \geq \frac{\delta}{3} P_k r_0, \quad 1 \leq k \leq n.
\]

We observe that, for each \( 1 \leq k \leq n \), the collection \( H_k = \{ B(\frac{\delta}{6} P_k r_0, \varphi_{\tau_k}(z_{\tau_k}(|\tau_k|)), \tilde{\tau} \in I \} \) of open balls are pairwise disjoint and contained in \( B(p_k \max_{1 \leq i \leq n} |F_i| + \frac{\delta}{6} P_k r_0, B(\varepsilon \sigma_{\tau_k}, F_{\sigma_{\tau_k}})) \). Thus
\[
\left( \frac{1}{3} P_k r_0 \right)^d \text{Card} H_k \leq \left[ 2(p_k \max_{1 \leq i \leq n} |F_i| + \frac{\delta}{6} P_k r_0) + P_k \right]^d.
\]

Hence
\[
\text{Card} H_k \leq \left( \frac{6 \max_{1 \leq k \leq n} |F_k| + \delta r_0 + 3}{\delta r_0} \right)^d,
\]

which implies \( \text{Card} I \leq \prod_{k=1}^n \text{Card} H_k \) is finite. Thus \( \gamma < \infty \).

(c) Choose \( \tilde{\sigma} \) such that \( \gamma = \text{Card} I(\tilde{\sigma}) \). For any \( \tilde{\tau} \in \tilde{\Omega}^* \), it is not difficult to find that we can choose
\[
I(\tilde{\tau}\tilde{\sigma}) = \{ \tilde{\tau}\tilde{\omega} : \tilde{\omega} \in I(\tilde{\sigma}) \}
\] (9)

by maximality of \( I(\tilde{\sigma}) \). In the following the set \( I(\tilde{\tau}\tilde{\sigma}) \) is always chosen as (9) for any \( \tilde{\tau} \in \tilde{\Omega}^* \).

(d) For each \( 1 \leq k \leq n \), letting
\[
V_k = \bigcup_{\tau_k \in \Omega_k^*} B(\varepsilon \frac{\delta}{2} r_{\tau_k} \sigma_{\tau_k}(|\tau_k|), F_{\tau_k} \sigma_{\tau_k}(|\tau_k|)) = \bigcup_{\tau_k \in \Omega_k^*} \varphi_{\tau_k}(B(\varepsilon \frac{\delta}{2} r_{\sigma_{\tau_k}}(|\tau_k|), F_{\sigma_{\tau_k}}(|\tau_k|)));
\]
we will show that \( \mathcal{F} \) satisfies strong open set condition with respect to these \( V_k \). Obviously \( V_k \cap F_k \neq \emptyset \) for each \( 1 \leq k \leq n \). Furthermore
\[
\varphi_{i k}(V_k) = \bigcup_{\tau_k \in \Omega_k^*} B(\varepsilon \frac{\delta}{2} r_{i(k)} \tau_k \sigma_{\tau_k}(|\tau_k|), F_{i(k)} \tau_k \sigma_{\tau_k}(|\tau_k|)) \subset V_i.
\]
In addition,
\[ \varphi_{ik}(V_k) \cap \varphi_{ij}(V_j) = \emptyset \]
for each \( 1 \leq i \leq n \) and \( j \neq k \) with \( a_{ik} = a_{ij} = 1 \). Otherwise there exist some \( 1 \leq i \leq n \) and \( \tau_k \in \Omega_{k}, \omega_j \in \Omega_j^* \) with \( k \neq j \) such that
\[
G = B(\frac{\varepsilon}{2} r(i,k) \tau_k \sigma_{\tau_k(\{i\})}) \cap B(\frac{\varepsilon}{2} r(i,j) \omega_j \sigma_{\omega_j(\{i\})} ; F(i,k) \tau_k \sigma_{\tau_k(\{i\})} , F(i,j) \omega_j \sigma_{\omega_j(\{i\})} ) \neq \emptyset.
\]
Without loss of generality, we assume \( r(i,k) \tau_k \sigma_{\tau_k(\{i\})} \geq r(i,j) \omega_j \sigma_{\omega_j(\{i\})} \). Taking \( y \in G \), there exist \( y_1 \in F(i,k) \tau_k \sigma_{\tau_k(\{i\})} \) and \( y_2 \in F(i,j) \omega_j \sigma_{\omega_j(\{i\})} \) such that
\[
d(y, y_1) < \varepsilon \frac{1}{2} r(i,k) \tau_k \sigma_{\tau_k(\{i\})} , \quad d(y, y_2) < \varepsilon \frac{1}{2} r(i,j) \omega_j \sigma_{\omega_j(\{i\})}.
\]
Then we have \( d(y_1, y_2) < \varepsilon r(i,k) \tau_k \sigma_{\tau_k(\{i\})} \) which means
\[
F(i,j) \omega_j \sigma_{\omega_j(\{i\})} \cap B(\varepsilon r(i,k) \tau_k \sigma_{\tau_k(\{i\})} ; F(i,k) \tau_k \sigma_{\tau_k(\{i\})} ) \neq \emptyset.
\] (10)
For each \( 1 \leq m \leq n \) with \( m \neq i \), take \( \alpha_m \in \Omega_m \) with \( \alpha_m(\{|\alpha_m|\}) = i \) and let \( \tilde{e}, \tilde{d} \in \Omega^* \) be such that
\[
eq \begin{cases} (i, j) \omega_j \sigma_{\omega_j(\{i\})} m = i \\
\alpha_m(i, j) \omega_j \sigma_{\omega_j(\{i\})} m \neq i
\end{cases}
\]
d\neq \begin{cases} (i, k) \tau_k m = i \\
\alpha_m(i, k) \tau_k m \neq i
\end{cases}

Thus we have
\[
F_{e_m} \cap B(\varepsilon d_m \sigma_{\tau_k(\{i\})} ; F_{d_m} \sigma_{\tau_k(\{i\})} ) \neq \emptyset, \quad 1 \leq m \leq n
\]
by (10). Therefore we can get a \( \tilde{\beta} \in \Omega^* \) with \( e_m = \beta_m \theta_m \) for all \( 1 \leq m \leq n \) such that \( \tilde{\beta} \in I_1(\tilde{d}\tilde{\sigma}) \) and \( \tilde{\beta} \) is incomparable with each element of \( I(\tilde{d}\tilde{\sigma}) \) by (9) which is in contradiction with the maximality of \( I(\tilde{d}\tilde{\sigma}) \). QED

Now for each \( 1 \leq i \leq n \), we endow \( \Omega(i) = \{ \sigma \in \Omega : \sigma(1) = i \} \) with the product \( \sigma \)-field and the canonical product probability measure \( \tilde{\mu}_i \) such that for each cylinder set \( C(\tau) = \{ \sigma \in \Omega(i) : \sigma(\{\tau\}) = \tau \} \) where \( \tau \in \Omega_i^* \)
\[
\tilde{\mu}_i(C(\tau)) = \alpha_{i}^{-1} \alpha_{\tau(\{\tau\})} \prod_{k=1}^{\tau - 1} r_{\tau(k), \tau(k+1)},
\]
where \( (\alpha_1, \ldots, \alpha_n)' \) is some strictly positive right eigenvector of matrix \( (r_{ij})_{n \times n} \) corresponding to eigenvalue 1.

The natural probability measure \( \mu_i \) of \( F_i \) are defined as the image measure of \( \tilde{\mu}_i \) under the continuous surjective mapping \( \pi_i : \Omega(i) \rightarrow F_i \) with \( \pi_i(\sigma) = \lim_{k \rightarrow \infty} \varphi_{\sigma|k}(0) \in F_i \). Notice that if we omit the restriction of \( 0 < \varepsilon < 1/3 \) in step (b) of the above proof, the proof will still work although, at this time, \( I(\tilde{\sigma}) = \emptyset \) possibly for some \( \tilde{\sigma} \in \Omega^* \). In addition, by the definition of open set \( V_k \) in step (d) it is easy to prove that \( \mu_k(V_k) = 1 \). Hence we can get the following conditions are equivalent actually:

(1) \( F \) satisfies the strong open set condition with open sets \( V_k \) such that \( \mu_k(V_k) = 1, 1 \leq k \leq n; \)

(2) $\mathcal{F}$ satisfies the strong open set condition;
(3) $\mathcal{F}$ satisfies the open set condition;
(4) $F_k$ is a $s$-set for some $1 \leq k \leq n$;
(5) $F_k$ are $s$-sets for all $1 \leq k \leq n$;
(6) For each $\varepsilon > 0$, $\gamma < \infty$ holds;
(7) There exists $\varepsilon > 0$ such that $\gamma < \infty$.

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References


