Separation properties for MW-fractals

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Abstract

Corresponding to the irreducible $0-1$ matrix $(a_{ij})_{n\times n}$, take similitude contraction mappings φ_{ij} for each $a_{ij}=1$, in $\mathbf{R}^d$ with ratio $0<r_{ij}<1$. There exist unique nonempty compact sets $F_1,\cdots,F_n$ satisfying for each $1\le i\le n$

$$F_i=\bigcup_{1\le j\le n, a_{ij}=1}\varphi_{ij}(F_j).$$

We prove that open set condition holds if and only if F_i is an s-set for some $1\le i\le n$, where s is such that the spectral radius of matrix $(r_{ij}^s)_{n\times n}$ is 1.
Keywords: Open set condition; s-set; MW-fractal.

1 Introduction

The self-similar fractal is defined as the unique compact set F such that $F=\bigcup_{i=1}^n\varphi_i(F)$ where, for each $1\le i\le n$, φ_i is a similitude contraction mapping in $\mathbf{R}^d$ with similitude ratio $0<r_i<1$. For self-similar fractal, A.Schief [1] has proved the following three conditions are equivalent:

(i) $\{\varphi_i:1\le i\le n\}$ satisfies the open set condition;

(ii) $\{\varphi_i:1\le i\le n\}$ satisfies the strong open set condition;

(iii) F is a t-set, where t is such that $\sum_{i=1}^n r_i^t=1$.

Now let us consider the generalization of self-similar fractal, sometimes called MW-fractal. Let $A_{n\times n}=(a_{ij})_{n\times n}$ be an irreducible $0-1$ matrix. Let $\mathcal{F}=\{\varphi_{ij}:1\le i,j\le n, a_{ij}=1\}$ be a collection of similitude contraction mappings in $\mathbf{R}^d$ with $0<r_{ij}<1$ as the similitude ratio of φ_{ij}. Then there exist unique nonempty compact sets $F_1,\cdots,F_n$ such that for each $1\le i\le n$

$$F_i=\bigcup_{1\le j\le n, a_{ij}=1}\varphi_{ij}(F_j). \tag{1}$$

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We say $\mathcal{F}$ satisfies the open set condition if there are nonempty bounded open sets $O_1, \cdots, O_n$ such that for each $1 \le i \le n$

$$O_i \supset \bigcup_{1 \le j \le n, a_{ij}=1} \varphi_{ij}(O_j) \text{ and } \varphi_{ij_1}(O_{j_1}) \bigcap \varphi_{ij_2}(O_{j_2}) = \emptyset \text{ if } j_1 \ne j_2$$

and $\mathcal{F}$ satisfies the strong open set condition if furthermore $F_i \bigcap O_i \ne \emptyset$ for some $1 \le i \le n$. Obviously when $\mathcal{F}$ satisfies the strong open set condition, $F_i \bigcap O_i \ne \emptyset$ for all $1 \le i \le n$ by (1).

It is known that if $\mathcal{F}$ satisfies the open set condition, then F_i are s-set for all $1 \le i \le n$, where s is such that the spectral radius of matrix $(r_{ij}^s)_{n\times n}$ is 1 (we adopt the convention that the (i,j)-th entry of matrix $(r_{ij}^s)_{n\times n}$ is 0 if $a_{ij} = 0$) (see [2][3]). It is worth to notice that from the irreducibility of $A_{n\times n}$ it follows that number 1 is the eigenvalue with the largest absolute value of $(r_{ij}^s)_{n\times n}$ which has strictly positive left and right eigenvectors (see Theorem 0.16 in [4]).

In this paper, we will show that the similar results hold for MW-fractals, i.e. the following three conditions are equivalent:

(i) $\mathcal{F}$ satisfies the open set condition;

(ii) $\mathcal{F}$ satisfies the strong open set condition;

(iii) F_i is an s-set for some $1 \le i \le n$.

Now let us introduce some notations and definitions as follows.

1) $\Omega = \{\sigma = (\sigma(1), \sigma(2), \cdots) : a_{\sigma(k),\sigma(k+1)} = 1 \text{ for } k \in \mathbf{N}\}$; $\Omega_k = \{\sigma = (\sigma(1), \cdots, \sigma(k)) : a_{\sigma(l),\sigma(l+1)} = 1 \text{ for } 1 \le l \le k-1\}$ and $\Omega^* = \bigcup_{k\ge 2} \Omega_k$. For $\sigma \in \Omega^*$, $|\sigma|$ denotes the word length and the $\sigma(k)$, $1 \le k \le |\sigma|$, the k-th component of σ. When $k \le |\sigma|$, we denote $\sigma|k = (\sigma(1), \cdots, \sigma(k))$. Let $\Omega_k^* = \{\sigma \in \Omega^* : \sigma(1) = k\}$ for $1 \le k \le n$, and $\tilde{\Omega}^* = \prod_{k=1}^n \Omega_k^*$. $\tilde{\sigma}$ denotes the element of $\tilde{\Omega}^*$ with $\sigma_k \in \Omega_k^*, 1 \le k \le n$, being the k-th component of $\tilde{\sigma}$.

2) For any $\sigma \in \Omega_k, k \ge 2$, $r_\sigma = \prod_{l=1}^{k-1} r_{\sigma(l),\sigma(l+1)}$; $\varphi_\sigma = \varphi_{\sigma(1),\sigma(2)} \circ \varphi_{\sigma(2),\sigma(3)} \circ \cdots \circ \varphi_{\sigma(k-1),\sigma(k)}$; $F_\sigma = \varphi_\sigma(F_{\sigma(k)})$.

3) For any $\sigma, \tau \in \Omega^*$, σ and τ are called to be connected if and only if $\sigma(|\sigma|) = \tau(1)$, and define $\sigma\tau = (\sigma(1), \cdots, \sigma(|\sigma|), \tau(2), \cdots, \tau(|\tau|))$. It is easy to see that for connected σ and τ, $|\sigma\tau| = |\sigma| + |\tau| - 1, r_{\sigma\tau} = r_\sigma r_\tau$, $\varphi_{\sigma\tau} = \varphi_\sigma \circ \varphi_\tau$ and $F_{\sigma\tau} = \varphi_\sigma(F_\tau)$.

4) For any two different $\sigma, \tau \in \Omega^*$, we say σ and τ are incomparable if there exists no $\omega \in \Omega^*$ such that $\sigma = \tau\omega$ or $\tau = \sigma\omega$. We adopt the convention that σ and σ are comparable. For any $\tilde{\sigma}, \tilde{\tau} \in \tilde{\Omega}^*$, we say $\tilde{\sigma}$ and $\tilde{\tau}$ are incomparable if σ_k and τ_k are incomparable for all $1 \le k \le n$. For $A \subset \tilde{\Omega}^*$, A is termed as incomparable subset of $\tilde{\Omega}^*$ if for any different $\tilde{\sigma}, \tilde{\tau} \in A$, $\tilde{\sigma}$ and $\tilde{\tau}$ are incomparable.

5) For any $\tilde{\sigma}, \tilde{\tau} \in \tilde{\Omega}^*$, define $\tilde{\sigma}\tilde{\tau} = \tilde{\omega}$ where $\omega_k = \sigma_k \tau_{\sigma_k(|\sigma_k|)}$. Then for any $\tilde{\sigma}, \tilde{\tau} \in \tilde{\Omega}^*$, the notation $\tilde{\sigma}\tilde{\tau}$ will work, but not for any $\sigma, \tau \in \Omega^*$. Define $r_{\tilde{\sigma}} = (r_{\sigma_1}, \cdots, r_{\sigma_n})'$ for any $\tilde{\sigma} \in \tilde{\Omega}^*$.

6) For $(a_1, \cdots, a_n)', (b_1, \cdots, b_n)' \in \mathbf{R}^n$, we say $(a_1, \cdots, a_n)' \le (b_1, \cdots, b_n)'$ if and only if $a_i \le b_i$ for all $1 \le i \le n$, and $(a_1, \cdots, a_n)' < (b_1, \cdots, b_n)'$ if and only if $a_i < b_i$ for all $1 \le i \le n$.

7) We use $d(\cdot,\cdot)$ to denote the Euclidean metric in $\mathbf{R}^d$, and $D(\cdot,\cdot)$ the Hausdorff metric in the complete metric space $\mathcal{H}(\mathbf{R}^d)$ of nonempty compact subsets of $\mathbf{R}^d$. For positive

real number r and set E, write $B(r,E)=\{y\in\mathbf{R}^d: d(y,E)<r\}$. Thus $B(r,\{x\})$ is an open ball with centre at x and radius r.

8) In the following, denote $h=\min_{1\le i\le n}\mathcal{H}^s(F_i)/\max_{1\le i\le n}\mathcal{H}^s(F_i)$ and $r_0=\min_{a_{ij}=1}r_{ij}$.

2 Main results

Proposition 2.1 *If $\mathcal{H}^s(F_i)>0$ for some $1\le i\le n$, then $\mathcal{H}^s(F_i)>0$ for all $1\le i\le n$ and*

$$(r_{ij}^s)_{n\times n}\begin{pmatrix}\mathcal{H}^s(F_1)\\ \vdots\\ \mathcal{H}^s(F_n)\end{pmatrix}=\begin{pmatrix}\mathcal{H}^s(F_1)\\ \vdots\\ \mathcal{H}^s(F_n)\end{pmatrix}. \tag{2}$$

Proof It is easy to get that $\mathcal{H}^s(F_i)>0$ for all $1\le i\le n$ and

$$\begin{pmatrix}\mathcal{H}^s(F_1)\\ \vdots\\ \mathcal{H}^s(F_n)\end{pmatrix}\le(r_{ij}^s)_{n\times n}\begin{pmatrix}\mathcal{H}^s(F_1)\\ \vdots\\ \mathcal{H}^s(F_n)\end{pmatrix} \tag{3}$$

by (1). Now let $(\beta_1,\cdots,\beta_n)$ be a strictly positive left eigenvector of matrix $(r_{ij}^s)_{n\times n}$ relative to the eigenvalue 1. Therefore

$$(\beta_1,\cdots,\beta_n)\begin{pmatrix}\mathcal{H}^s(F_1)\\ \vdots\\ \mathcal{H}^s(F_n)\end{pmatrix}\le(\beta_1,\cdots,\beta_n)(r_{ij}^s)_{n\times n}\begin{pmatrix}\mathcal{H}^s(F_1)\\ \vdots\\ \mathcal{H}^s(F_n)\end{pmatrix}=(\beta_1,\cdots,\beta_n)\begin{pmatrix}\mathcal{H}^s(F_1)\\ \vdots\\ \mathcal{H}^s(F_n)\end{pmatrix}$$

which implies (3) takes equality. **QED**

By Proposition 2.1 we can get the following Corollary 2.2 immediately.

Corollary 2.2 *For any incomparable $\sigma,\tau\in\Omega_k^*, 1\le k\le n$, we have $\mathcal{H}^s(F_\sigma\bigcap F_\tau)=0$.* **QED**

Now for any $B\subset\mathbf{R}^d$, let

$$\mu(B)=\inf\{\sum_{i\in I}|U_i|^s: U_i \text{ open set and } \bigcup_{i\in I}U_i\supset B\},$$

where $|A|$ denotes the diameter of set A. The $\mu(\cdot)$ is an outer measure (see [5]). Similar to the Proposition 3 of [6], we have

Proposition 2.3 *For any Borel measurable set $B_i\subset F_i, 1\le i\le n$, we have $\mathcal{H}^s(B_i)=\mu(B_i)$.*

Proof By the definition of $\mu(\cdot)$ and $\mathcal{H}^s(\cdot)$, it is clear that

$$\mu(B_i) \leq \mathcal{H}^s(B_i). \tag{4}$$

Now let us consider the case $B_i = F_i, 1 \leq i \leq n$. We suffice to show that for each open covering $\{U_j(i)\}_j$ of F_i and each $\delta > 0$, there is another covering $\{V_j(i)\}_j$ of F_i with $|V_j(i)| < \delta$ for each j and $\sum_j |V_j(i)|^s = \sum_j |U_j(i)|^s$. To do this, we choose k large enough such that $r_\sigma < \delta |\bigcup_{i=1}^n \bigcup_j U_j(i)|^{-1}$ for each $\sigma \in \Omega_k$. We replace each open set family $\{U_j(i)\}_j$ with $\bigcup_{\sigma\in\Omega_k,\sigma(1)=i} \{\varphi_\sigma(U_j(\sigma(k)))\}_j$, denoted by $\{V_j(i)\}_j$, which covers the F_i. So

$$\begin{pmatrix} \sum_j |V_j(1)|^s \\ \vdots \\ \sum_j |V_j(n)|^s \end{pmatrix} = (r_{ij}^s)_{n\times n}^k \begin{pmatrix} \sum_j |U_j(1)|^s \\ \vdots \\ \sum_j |U_j(n)|^s \end{pmatrix},$$

which leads that

$$(\beta_1, \cdots, \beta_n) \begin{pmatrix} \sum_j |V_j(1)|^s \\ \vdots \\ \sum_j |V_j(n)|^s \end{pmatrix} = (\beta_1, \cdots, \beta_n) \begin{pmatrix} \sum_j |U_j(1)|^s \\ \vdots \\ \sum_j |U_j(n)|^s \end{pmatrix}.$$

Therefore

$$(\beta_1, \cdots, \beta_n) \begin{pmatrix} \mathcal{H}^s(F_1) \\ \vdots \\ \mathcal{H}^s(F_n) \end{pmatrix} \leq (\beta_1, \cdots, \beta_n) \begin{pmatrix} \mu(F_1) \\ \vdots \\ \mu(F_n) \end{pmatrix}.$$

Thus we have $\mu(F_i) = \mathcal{H}^s(F_i)$ for $1 \leq i \leq n$ by (4).

Now for Borel measurable sets $B_i \subset F_i, 1 \leq i \leq n$, we have

$$\begin{aligned} \begin{pmatrix} \mathcal{H}^s(F_1) \\ \vdots \\ \mathcal{H}^s(F_n) \end{pmatrix} &= \begin{pmatrix} \mathcal{H}^s(B_1) \\ \vdots \\ \mathcal{H}^s(B_n) \end{pmatrix} + \begin{pmatrix} \mathcal{H}^s(F_1\backslash B_1) \\ \vdots \\ \mathcal{H}^s(F_n\backslash B_n) \end{pmatrix} \geq \begin{pmatrix} \mu(B_1) \\ \vdots \\ \mu(B_n) \end{pmatrix} + \begin{pmatrix} \mu(F_1\backslash B_1) \\ \vdots \\ \mu(F_n\backslash B_n) \end{pmatrix} \\ &\geq \begin{pmatrix} \mu(F_1) \\ \vdots \\ \mu(F_n) \end{pmatrix} = \begin{pmatrix} \mathcal{H}^s(F_1) \\ \vdots \\ \mathcal{H}^s(F_n) \end{pmatrix} \end{aligned}$$

which complete our proof by (4). **QED**

Theorem 2.4 *If $\mathcal{H}^s(F_i) > 0$ for some $1 \leq i \leq n$, then $\mathcal{F}$ satisfies the strong open set condition.*

Proof First it is easy to see that $\mathcal{H}^s(F_i) > 0$ for some $1 \leq i \leq n$ implies that $0 < \mathcal{H}^s(F_i) < \infty$ for all $1 \leq i \leq n$. On the other hand, it is not difficult to show that $s = 0$ if and only if F_i is singleton for all $1 \leq i \leq n$. Then the strong open set condition holds when $s = 0$. In the following, we assume $s > 0$. Given $0 < p < 1$, let

$$I_p = \{\sigma \in \Omega^* : r_\sigma < p \leq r_{\sigma|(|\sigma|-1)}\}.$$

It is easy to see that any two different elements of I_p are incomparable. We divide the proof into the following four steps.

(a) Given $x > 0$, there exists open set families $\{U_j(i)\}_j, 1 \leq i \leq n$, such that $\bigcup_j U_j(i) \supset F_i$ and

$$\begin{pmatrix} \sum_j |U_j(1)|^s \\ \vdots \\ \sum_j |U_j(n)|^s \end{pmatrix} \leq (1+x^s) \begin{pmatrix} \mathcal{H}^s(F_1) \\ \vdots \\ \mathcal{H}^s(F_n) \end{pmatrix}. \tag{5}$$

Writing $V_i = \bigcup_j U_j(i), 1 \leq i \leq n$, let $\delta = \min_{1\leq i\leq n} d(F_i, V_i^c)$. At first, we will show that if $\tilde{\sigma}$ and $\tilde{\tau}$ are incomparable and $h^{\frac{1}{s}} r_{\tilde{\tau}} > x r_{\tilde{\sigma}}$, then

$$\begin{pmatrix} D(F_{\sigma_1}, F_{\tau_1}) \\ \vdots \\ D(F_{\sigma_n}, F_{\tau_n}) \end{pmatrix} \geq \delta \begin{pmatrix} r_{\sigma_1} \\ \vdots \\ r_{\sigma_n} \end{pmatrix}. \tag{6}$$

Otherwise, without loss of generality we assume $D(F_{\sigma_1}, F_{\tau_1}) < \delta r_{\sigma_1}$. Since

$$\begin{pmatrix} d(F_{\sigma_1}, \varphi_{\sigma_1}(V^c_{\sigma_1(|\sigma_1|)})) \\ \vdots \\ d(F_{\sigma_n}, \varphi_{\sigma_n}(V^c_{\sigma_n(|\sigma_n|)})) \end{pmatrix} \geq \delta \begin{pmatrix} r_{\sigma_1} \\ \vdots \\ r_{\sigma_n} \end{pmatrix},$$

we have $F_{\tau_1} \subset \varphi_{\sigma_1}(V_{\sigma_1(|\sigma_1|)})$. Thus by Corollary 2.2, Proposition 2.3 and (5)

$$\begin{aligned} (1+x^s) r^s_{\sigma_1} \mathcal{H}^s(F_{\sigma_1(|\sigma_1|)}) &< r^s_{\sigma_1} \mathcal{H}^s(F_{\sigma_1(|\sigma_1|)}) + r^s_{\tau_1} \mathcal{H}^s(F_{\tau_1(|\tau_1|)}) = \mathcal{H}^s(F_{\sigma_1}) + \mathcal{H}^s(F_{\tau_1}) \\ &= \mathcal{H}^s(F_{\sigma_1} \bigcup F_{\tau_1}) \leq \sum_j |\varphi_{\sigma_1}(U_j(\sigma_1(|\sigma_1|)))|^s \\ &= r^s_{\sigma_1} \sum_j |U_j(\sigma_1(|\sigma_1|))|^s \leq (1+x^s) r^s_{\sigma_1} \mathcal{H}^s(F_{\sigma_1(|\sigma_1|)}) \end{aligned}$$

which leads a contradiction.

(b) Let $0 < \varepsilon < \frac{1}{3}$. For any $\tilde{\sigma} \in \tilde{\Omega}^*$, writing $p_k = |B(\varepsilon r_{\sigma_k}, F_{\sigma_k})|$ and let

$$I^*(\tilde{\sigma}) = \{\tilde{\tau} \in \tilde{\Omega}^* : \tau_k \in I_{p_k},\ 1 \leq k \leq n\},$$

$$I_1(\tilde{\sigma}) = \{\tilde{\tau} \in \tilde{\Omega}^* : \tau_k \in I_{p_k},\ F_{\tau_k} \bigcap B(\varepsilon r_{\sigma_k}, F_{\sigma_k}) \neq \emptyset,\ 1 \leq k \leq n\}.$$

Note that for any $\tilde{\tau}, \tilde{\omega} \in I^*(\tilde{\sigma})$, $\tilde{\tau}$ and $\tilde{\omega}$ are incomparable if and only if $\tau_k \neq \omega_k$ for all $1 \leq k \leq n$. Let $I(\tilde{\sigma})$ be the incomparable subset of $I_1(\tilde{\sigma})$ which has the largest cardinality. Thus there exists no element of $I_1(\tilde{\sigma})$ which is incomparable with each element of $I(\tilde{\sigma})$. Writing $\gamma = \sup_{\tilde{\sigma}\in\tilde{\Omega}^*} \text{Card } I(\tilde{\sigma})$, we will show that $\gamma < \infty$. Now fix $\tilde{\sigma} \in \tilde{\Omega}^*$. Then for any incomparable $\tilde{\tau}, \tilde{\omega} \in I^*(\tilde{\sigma})$, we have

$$h^{\frac{1}{s}} r_{\tilde{\tau}} > h^{\frac{1}{s}} r_0 r_{\tilde{\omega}}.$$

Taking $x = h^{\frac{1}{s}} r_0$ in (a), we can get the relating δ and

$$(D(F_{\omega_1}, F_{\tau_1}), \cdots, D(F_{\omega_n}, F_{\tau_n}))' \geq \delta r_{\tilde{\omega}}$$

by (6). Therefore there exist $y_{\omega_k} \in F_{\omega_k(|\omega_k|)}$ and $y^*_{\tau_k} \in F_{\tau_k(|\tau_k|)}, 1 \le k \le n$, such that

$$d(\varphi_{\omega_k}(y_{\omega_k}), \varphi_{\tau_k}(y^*_{\tau_k})) \ge \delta r_{\omega_k}. \tag{7}$$

Take $z_{\omega_k} \in F_{\omega_k(|\omega_k|)}$ and $z^*_{\tau_k} \in F_{\tau_k(|\tau_k|)}, 1 \le k \le n$, such that

$$d(y_{\omega_k}, z_{\omega_k}) < \frac{1}{3}\delta r_0, \quad d(y^*_{\tau_k}, z^*_{\tau_k}) < \frac{1}{3}\delta r_0. \tag{8}$$

Then from (7) and (8) it follows that for $1 \le k \le n$

$$d(\varphi_{\omega_k}(z_{\omega_k}), \varphi_{\tau_k}(z^*_{\tau_k})) \ge \frac{\delta}{3} r_{\omega_k}.$$

Now for each $1 \le k \le n$ we fix a finite set $Z_k \subset F_k$ whose $\frac{1}{3}\delta r_0$ neighborhood covers F_k. Therefore for any incomparable $\tilde{\tau}, \tilde{\omega} \in I^*(\tilde{\sigma})$, there exist $z_k \in Z_k, 1 \le k \le n$, such that

$$d(\varphi_{\tau_k}(z_{\tau_k(|\tau_k|)}), \varphi_{\omega_k}(z_{\omega_k(|\omega_k|)})) \ge \frac{\delta}{3} r_{\tau_k} \ge \frac{\delta}{3} p_k r_0.$$

Now fix $z_k \in Z_k, 1 \le k \le n$, choose $I \subset I(\tilde{\sigma})$ such that for any different $\tilde{\tau}, \tilde{\omega} \in I$

$$d(\varphi_{\tau_k}(z_{\tau_k(|\tau_k|)}), \varphi_{\omega_k}(z_{\omega_k(|\omega_k|)})) \ge \frac{\delta}{3} r_{\tau_k} \ge \frac{\delta}{3} p_k r_0, \quad 1 \le k \le n.$$

We observe that, for each $1 \le k \le n$, the collection $H_k = \{B(\frac{\delta}{6} p_k r_0, \varphi_{\tau_k}(z_{\tau_k(|\tau_k|)})), \tilde{\tau} \in I\}$ of open balls are pairwise disjoint and contained in $B(p_k \max_{1\le i\le n} |F_i| + \frac{\delta}{6} p_k r_0, B(\varepsilon r_{\sigma_k}, F_{\sigma_k}))$. Thus

$$(\frac{1}{3}\delta p_k r_0)^d \, \mathrm{Card} H_k \le [2(p_k \max_{1\le i\le n} |F_i| + \frac{\delta}{6} p_k r_0) + p_k]^d.$$

Hence

$$\mathrm{Card} H_k \le \left(\frac{6 \max_{1\le k\le n} |F_k| + \delta r_0 + 3}{\delta r_0} \right)^d,$$

which implies $\mathrm{Card} I \le \prod_{k=1}^n \mathrm{Card} H_k$ is finite. Thus $\gamma < \infty$.

(c) Choose $\tilde{\sigma}$ such that $\gamma = \mathrm{Card} I(\tilde{\sigma})$. For any $\tilde{\tau} \in \tilde{\Omega}^*$, it is not difficult to find that we can choose

$$I(\tilde{\tau}\tilde{\sigma}) = \{\tilde{\tau}\tilde{\omega} : \tilde{\omega} \in I(\tilde{\sigma})\} \tag{9}$$

by maximality of $I(\tilde{\sigma})$. In the following the set $I(\tilde{\tau}\tilde{\sigma})$ is always chosen as (9) for any $\tilde{\tau} \in \tilde{\Omega}^*$.

(d) For each $1 \le k \le n$, letting

$$V_k = \bigcup_{\tau_k \in \Omega^*_k} B(\frac{\varepsilon}{2} r_{\tau_k \sigma_{\tau_k(|\tau_k|)}}, F_{\tau_k \sigma_{\tau_k(|\tau_k|)}}) = \bigcup_{\tau_k \in \Omega^*_k} \varphi_{\tau_k}(B(\frac{\varepsilon}{2} r_{\sigma_{\tau_k(|\tau_k|)}}, F_{\sigma_{\tau_k(|\tau_k|)}})),$$

we will show that $\mathcal{F}$ satisfies strong open set condition with respect to these V_k. Obviously $V_k \bigcap F_k \ne \emptyset$ for each $1 \le k \le n$. Furthermore

$$\varphi_{ik}(V_k) = \bigcup_{\tau_k \in \Omega^*_k} B(\frac{\varepsilon}{2} r_{(i,k)\tau_k \sigma_{\tau_k(|\tau_k|)}}, F_{(i,k)\tau_k \sigma_{\tau_k(|\tau_k|)}}) \subset V_i.$$

In addition,

$$\varphi_{ik}(V_k)\bigcap\varphi_{ij}(V_j)=\emptyset$$

for each $1\le i\le n$ and $j\ne k$ with $a_{ik}=a_{ij}=1$. Otherwise there exist some $1\le i\le n$ and $\tau_k\in\Omega_k^*,\omega_j\in\Omega_j^*$ with $k\ne j$ such that

$$G=B(\frac{\varepsilon}{2}r_{(i,k)\tau_k\sigma_{\tau_k(|\tau_k|)}},F_{(i,k)\tau_k\sigma_{\tau_k(|\tau_k|)}})\bigcap B(\frac{\varepsilon}{2}r_{(i,j)\omega_j\sigma_{\omega_j(|\omega_j|)}},F_{(i,j)\omega_j\sigma_{\omega_j(|\omega_j|)}})\ne\emptyset.$$

Without loss of generality, we assume $r_{(i,k)\tau_k\sigma_{\tau_k(|\tau_k|)}}\ge r_{(i,j)\omega_j\sigma_{\omega_j(|\omega_j|)}}$. Taking $y\in G$, there exist $y_1\in F_{(i,k)\tau_k\sigma_{\tau_k(|\tau_k|)}}$ and $y_2\in F_{(i,j)\omega_j\sigma_{\omega_j(|\omega_j|)}}$ such that

$$d(y,y_1)<\frac{\varepsilon}{2}r_{(i,k)\tau_k\sigma_{\tau_k(|\tau_k|)}},\quad d(y,y_2)<\frac{\varepsilon}{2}r_{(i,j)\omega_j\sigma_{\omega_j(|\omega_j|)}}.$$

Then we have $d(y_1,y_2)<\varepsilon r_{(i,k)\tau_k\sigma_{\tau_k(|\tau_k|)}}$ which means

$$F_{(i,j)\omega_j\sigma_{\omega_j(|\omega_j|)}}\bigcap B(\varepsilon r_{(i,k)\tau_k\sigma_{\tau_k(|\tau_k|)}},F_{(i,k)\tau_k\sigma_{\tau_k(|\tau_k|)}})\ne\emptyset. \tag{10}$$

For each $1\le m\le n$ with $m\ne i$, take $\alpha_m\in\Omega_m^*$ with $\alpha_m(|\alpha_m|)=i$ and let $\tilde e,\tilde d\in\tilde\Omega^*$ be such that

$$e_m=\begin{cases}(i,j)\omega_j\sigma_{\omega_j(|\omega_j|)} & m=i\\ \alpha_m(i,j)\omega_j\sigma_{\omega_j(|\omega_j|)} & m\ne i\end{cases}\quad\text{and}\quad d_m=\begin{cases}(i,k)\tau_k & m=i\\ \alpha_m(i,k)\tau_k & m\ne i.\end{cases}$$

Thus we have

$$F_{e_m}\bigcap B(\varepsilon r_{d_m\sigma_{\tau_k(|\tau_k|)}},F_{d_m\sigma_{\tau_k(|\tau_k|)}})\ne\emptyset,\qquad 1\le m\le n$$

by (10). Therefore we can get a $\tilde\beta\in\tilde\Omega^*$ with $e_m=\beta_m\theta_m$ for all $1\le m\le n$ such that $\tilde\beta\in I_1(\tilde d\tilde\sigma)$ and $\tilde\beta$ is incomparable with each element of $I(\tilde d\tilde\sigma)$ by (9) which is in contradiction with the maximality of $I(\tilde d\tilde\sigma)$. **QED**

Now for each $1\le i\le n$, we endow $\Omega(i)=\{\sigma\in\Omega:\sigma(1)=i\}$ with the product σ-field and the canonical product probability measure $\tilde\mu_i$ such that for each cylinder set $C(\tau)=\{\sigma\in\Omega(i):\sigma||\tau|=\tau\}$ where $\tau\in\Omega_i^*$

$$\tilde\mu_i(C(\tau))=\alpha_i^{-1}\alpha_{\tau(|\tau|)}\prod_{k=1}^{|\tau|-1}r^s_{\tau(k),\tau(k+1)},$$

where $(\alpha_1,\cdots,\alpha_n)'$ is some strictly positive right eigenvector of matrix $(r^s_{ij})_{n\times n}$ corresponding to eigenvalue 1.

The natural probability measure μ_i of F_i are defined as the image measure of $\tilde\mu_i$ under the continuous surjective mapping $\pi_i:\Omega(i)\to F_i$ with $\pi_i(\sigma)=\lim_{k\to\infty}\varphi_{\sigma|k}(0)\in F_i$. Notice that if we omit the restriction of $0<\varepsilon<1/3$ in step (b) of the above proof, the proof will still work although, at this time, $I(\tilde\sigma)=\emptyset$ possibly for some $\tilde\sigma\in\tilde\Omega^*$. In addition, by the definition of open set V_k in step (d) it is easy to prove that $\mu_k(V_k)=1$. Hence we can get the following conditions are equivalent actually:

(1) $\mathcal{F}$ satisfies the strong open set conddition with open sets V_k such that $\mu_k(V_k)=1,1\le k\le n$;

(2) $\mathcal{F}$ satisfies the strong open set conddition;
(3) $\mathcal{F}$ satisfies the open set conddition;
(4) F_k is a s-set for some $1 \leq k \leq n$;
(5) F_k are s-sets for all $1 \leq k \leq n$;
(6) For each $\varepsilon > 0$, $\gamma < \infty$ holds;
(7) There exists $\varepsilon > 0$ such that $\gamma < \infty$.

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