The Dimensions Of Self-Similar Sets

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Abstract

For self-similar set $F$ we prove that $\dim_H F = \dim_B F = \dim_P F$ using different method from Fa[4] and give implicitly the dimension value even if the open set condition isn’t satisfied.

1 Introduction

Let $\phi_i$ be similar contraction mappings in $\mathbb{R}^d$ with ratios $c_i, 1 \leq i \leq n$. Hu[5] proved that there exists unique compact set $F \subset \mathbb{R}^d$ such that

$$F = \bigcup_{i=1}^{n} \phi_i(F).$$

Further $\dim_H F = \dim_B F = \dim_P F = s$ and $F$ is an $s$-set where $s$ is such that

$$\sum_{i=1}^{n} c_i^s = 1,$$

if $\phi_i$’s satisfy the open set condition, i.e., there is a bounded nonempty open set $O$ such that

$$\bigcup_{i=1}^{n} \phi_i(O) \subset O$$

with the left hand is disjoint union. Recently Sh[10] proved that $F$ is an $s$-set here $\sum_{i=1}^{n} c_i^s = 1$ if and only if $\phi_i$’s satisfy the open condition.

Now for $\epsilon > 0$ write

$$\Omega(\epsilon) = \{\sigma \in S^* \mid c_\sigma \leq \epsilon \text{ and } c_{\sigma|[\sigma|-1]} > \epsilon\},$$

where $S^* = \bigcup_{i=1}^{\infty} \{1, 2, \cdots, n\}^i$ and $c_\sigma = c_{\sigma(1)}c_{\sigma(2)} \cdots c_{\sigma(k)}$ for $\sigma = (\sigma(1), \sigma(2), \cdots, \sigma(k)) \in S^*$. And for $\sigma \in S^*$, $|\sigma|$ denotes the length of $\sigma$ and $\sigma|k = (\sigma(1), \cdots, \sigma(k))$ for $k \leq |\sigma|$. Let $A \subset \mathbb{R}^d$ be a bounded open set with $A \supset F$. It is easy to see that $c_0 \epsilon < c_\sigma \leq \epsilon$ for
any \( \sigma \in \Omega(\epsilon) \) where \( c_0 = \min_{1 \leq i \leq n} c_i \). We introduce nonnegative real numbers \( \alpha_0(A) \) and \( \beta_0(A) \) as follows

\[
\alpha_0(A) = \sup\{\alpha | \lim_{\epsilon \to 0} \frac{\epsilon^{-d} m_d(\bigcup_{\sigma \in \Omega(\epsilon)} \phi_{\sigma}(A))}{\sum_{\sigma \in \Omega(\epsilon)} c_{\sigma}^{1-\alpha}} = \infty\},
\]

\[
\beta_0(A) = \sup\{\beta | \lim_{\epsilon \to 0} \frac{\epsilon^{-d} m_d(\bigcup_{\sigma \in \Omega(\epsilon)} \phi_{\sigma}(A))}{\sum_{\sigma \in \Omega(\epsilon)} c_{\sigma}^{1-\beta}} = \infty\},
\]

where \( \phi_{\sigma} = \phi_{\sigma(1)} \circ \phi_{\sigma(2)} \circ \cdots \circ \phi_{\sigma(k)} \) for \( \sigma = (\sigma(1), \sigma(2), \cdots, \sigma(k)) \in S^* \) and \( m_d(B) \) is the Lebesgue measure of \( B \subset \mathbb{R}^d \).

In this paper we prove

(i) \( \alpha_0(A) \) and \( \beta_0(A) \) are independent of the choice of \( A \) and \( \alpha_0(A) = \beta_0(A) \). We denote the common value by \( \alpha_0 \);

(ii) \( \dim_H F = \dim_B F = \dim_P F = \alpha_0 \); (For self-similar set \( F \) Fa[4] has proved that its Hausdorff dimension, Box dimension and Packing dimension are equal)

(iii) \( \mathcal{H}^{\alpha_0}(F) < \infty \) iff \( \lim_{\epsilon \to 0} \frac{\epsilon^{-d} m_d(\bigcup_{\sigma \in \Omega(\epsilon)} \phi_{\sigma}(A))}{\sum_{\sigma \in \Omega(\epsilon)} c_{\sigma}^{1-\alpha_0}} < \infty \);

(iv) If \( \mathcal{H}^{\alpha_0}(F) > 0 \) then \( \lim_{\epsilon \to 0} \frac{\epsilon^{-d} m_d(\bigcup_{\sigma \in \Omega(\epsilon)} \phi_{\sigma}(A))}{\sum_{\sigma \in \Omega(\epsilon)} c_{\sigma}^{1-\alpha_0}} > 0 \);

(v) We generalize this dimension results into the cases of MW-construction (Ma & Wi[9]) and recurrent sets (De[2], Be[1] and Wen[11]).

### 2 Dimensions of self-similar set

It is easy to get the following

**Proposition 2.1**

\[
\alpha_0(A) = \inf\{\alpha | \lim_{\epsilon \to 0} \frac{\epsilon^{-d} m_d(\bigcup_{\sigma \in \Omega(\epsilon)} \phi_{\sigma}(A))}{\sum_{\sigma \in \Omega(\epsilon)} c_{\sigma}^{1-\alpha}} = 0\},
\]

\[
\beta_0(A) = \inf\{\beta | \lim_{\epsilon \to 0} \frac{\epsilon^{-d} m_d(\bigcup_{\sigma \in \Omega(\epsilon)} \phi_{\sigma}(A))}{\sum_{\sigma \in \Omega(\epsilon)} c_{\sigma}^{1-\beta}} = 0\}.
\]

**Proposition 2.2** \( 0 \leq \alpha_0(A) \leq 1; 0 \leq \beta_0(A) \leq 1 \).

**Proof** Note that \( \sum_{\sigma \in \Omega(\epsilon)} c_{\sigma} = 1 \). Taking \( \alpha = 0 \) then

\[
\lim_{\epsilon \to 0} \frac{\epsilon^{-d} m_d(\bigcup_{\sigma \in \Omega(\epsilon)} \phi_{\sigma}(A))}{\sum_{\sigma \in \Omega(\epsilon)} c_{\sigma}^{1-\alpha}} = \lim_{\epsilon \to 0} \frac{\epsilon^{-d} m_d(\bigcup_{\sigma \in \Omega(\epsilon)} \phi_{\sigma}(A))}{\sum_{\sigma \in \Omega(\epsilon)} c_{\sigma}^{1-\beta}} \geq c
\]
for some positive constant $c$. Thus $\alpha_0(A) \geq 0$. On the other hand, taking $\alpha = 1$, we have

$$\lim_{\epsilon \to 0} \frac{\epsilon^{-d}m_d(\bigcup_{\sigma \in \Omega(\epsilon)} \phi_\sigma(A))}{\text{Card}(\Omega(\epsilon))} \leq c$$

for some constant $c$. Thus $\alpha_0(A) \leq 1$.

$0 \leq \beta_0(A) \leq 1$ can be proved by the same method. QED

**Theorem 2.3**

(i) $\alpha_0(A)$ and $\beta_0(A)$ are independent of the choice of $A$ and $\alpha_0(A) = \beta_0(A)$, denoting the common value by $\alpha_0$;

(ii) $\dim_H F = \dim_B F = \dim_F F = \alpha_0$;

(iii) $\mathcal{H}^{\alpha_0}(F) < \infty$ iff $\lim_{\epsilon \to 0} \frac{\epsilon^{-d}m_d\left(\bigcup_{\sigma \in \Omega(\epsilon)} \phi_\sigma(A)\right)}{\sum_{\sigma \in \Omega(\epsilon)} c_{\sigma}^{s(1-\alpha)}} < \infty$;

(iv) If $\mathcal{H}^{\alpha_0}(F) > 0$ then $\lim_{\epsilon \to 0} \frac{\epsilon^{-d}m_d\left(\bigcup_{\sigma \in \Omega(\epsilon)} \phi_\sigma(A)\right)}{\sum_{\sigma \in \Omega(\epsilon)} c_{\sigma}^{s(1-\alpha)}} > 0$.

**Proof** (i) For $B \subset \mathbb{R}^d$ and $\epsilon > 0$ let

$$B^\epsilon = \{ x \in \mathbb{R}^d : \text{there exists } y \in B \text{ such that } \rho(x, y) < \epsilon \}$$

where $\rho(x, y)$ is the Euclidean distance between $x$ and $y$. Since $A$ is a bounded open set containing set $F$, there are positive number $\delta_1$ and $\delta_2$ such that $F^{\delta_1} \subset A \subset F^{\delta_2}$ which means $\alpha_0(F^{\delta_1}) \leq \alpha_0(A) \leq \alpha_0(F^{\delta_2})$ and $\beta_0(F^{\delta_1}) \leq \beta_0(A) \leq \beta_0(F^{\delta_2})$. Thus it suffices to prove $\alpha_0(F^{\delta_1})$ and $\beta_0(F^{\delta_1})$ are independent of the choice of positive number $\delta$ and $\alpha_0(F^{\delta}) = \beta_0(F^{\delta})$, which follows from the proof of (ii).

(ii) Fixing $x \in F$ and denoting the diameter of $A$ by $|A|$ we choose subfamily $\Omega^*(\epsilon)$ from $\Omega(\epsilon)$ such that

1. for any different $\sigma, \tau \in \Omega^*(\epsilon)$, $\rho(\phi_\sigma(x), \phi_\tau(x)) > 4|A|\epsilon$;
2. if $\sigma \in \Omega(\epsilon) \setminus \Omega^*(\epsilon)$ there exists $\tau \in \Omega^*(\epsilon)$ such that $\rho(\phi_\sigma(x), \phi_\tau(x)) \leq 4|A|\epsilon$.

Let $J(\epsilon) = \text{Card}(\Omega(\epsilon))$. Thus

$$\bigcup_{\sigma \in \Omega^*(\epsilon)} B(\phi_\sigma(x), 5|A|\epsilon) \supset \bigcup_{\sigma \in \Omega(\epsilon)} B(\phi_\sigma(x), |A|\epsilon) \supset \bigcup_{\sigma \in \Omega(\epsilon)} \phi_\sigma(A)$$

where $B(x, r)$ denotes a ball in $\mathbb{R}^d$ with center at $x$ and radius $r$. Thus

$$J(\epsilon)m_d B(\phi_\sigma(x), 5|A|\epsilon) \geq m_d \bigcup_{\sigma \in \Omega(\epsilon)} \phi_\sigma(A).$$

Therefore for any nonnegative real number $\alpha$

$$J(\epsilon)C^s \geq \frac{c|A|^{-d}m_d\left(\bigcup_{\sigma \in \Omega(\epsilon)} \phi_\sigma(A)\right)}{\sum_{\sigma \in \Omega(\epsilon)} c_{\sigma}^{s(1-\alpha)}},$$

(6)
where \( c \) is a positive constant. First we prove \( \dim_H F \geq \alpha_0(A)s \). It is clear when \( \alpha_0(A) = 0 \). Suppose \( \alpha_0(A) > 0 \) and take \( 0 < \alpha < \alpha_0(A) \). Thus by the definition of \( \alpha_0(A) \) and (6) we can take \( \epsilon_1 > 0 \) such that

\[
J(\epsilon_1)\epsilon_1^\alpha \geq 2c_0^{\alpha s}.
\]

Therefore by (7) we have

\[
\sum_i |V_i|^\alpha \geq (c_0\epsilon_1)^{2\alpha s} |A|^\alpha;
\]

\[
\sum_{V_i \in \mathcal{V}_{\lambda_1}} |V_i|^\alpha = min_{\sigma \in \Omega^*}(\epsilon_1) \sum_{V_i \in \mathcal{V}_{\sigma}} |V_i|^\alpha.
\]

Now taking \( \epsilon_1 \rightarrow 0 \) we have

\[
\sum_i |V_i|^\alpha \geq J(\epsilon_1) \sum_{V_i \in \mathcal{V}_{\lambda_1}} |V_i|^\alpha \geq 2c_0^{-\alpha} \epsilon_1^{1-\alpha} \sum_{V_i \in \mathcal{V}_{\lambda_1}} |V_i|^\alpha
\]

\[
= 2(c_\lambda c_0^{-\alpha} \epsilon_1^{1-\alpha})^\alpha \sum_{V_i \in \mathcal{V}_{\lambda_1}} |\varphi_{\lambda_1}^{-1} V_i|^\alpha \geq 2 \sum_{V_i \in \mathcal{V}_{\lambda_1}} |\varphi_{\lambda_1}^{-1} V_i|^\alpha.
\]

Since \( \mathcal{V}_{\lambda_1} \) is a covering of \( \varphi_{\lambda_1}(F) \), \( \varphi_{\lambda_1}^{-1} \mathcal{V}_{\lambda_1} = \{ \varphi_{\lambda_1}^{-1}(V_i) : V_i \in \mathcal{V}_{\lambda_1} \} \) is a finite open \( c_0\epsilon_1 |A| \)-covering of \( F \). As above we have

(a) if there exists some \( V_i \) such that \( |V_i| \geq (c_0\epsilon_1)^{2}|A| \) then (8) holds by (9);

(b) otherwise otherwise denote \( \varphi_{\lambda_1}^{-1} \mathcal{V}_{\lambda_1} \) by \( \{ U_i \} \). Repeating the above step for the covering \( \{ U_i \} \) of \( F \) and noticing that \( \text{Card}\{ V_i \} \) is finite, thus (8) holds after finite steps. Consequently \( \dim_H F \geq \alpha s \) which means \( \dim_H F \geq \alpha_0(A)s \).

Now taking \( \delta_1 > 0 \) we prove that \( \dim_H F \leq \dim_B F \leq \alpha_0(F^{\delta_1})s \). Letting \( \alpha > \alpha_0(F^{\delta_1}) \) there exists sequence \( \epsilon_n \) such that

\[
\frac{\epsilon_n^{-d} \text{md}(\bigcup_{\sigma \in \Omega^*(\epsilon_n)} \mathcal{V}_{\sigma} F^{\delta_1}))}{\sum_{\sigma \in \Omega^*(\epsilon_n)} c_\sigma^{s(1-\alpha)}} \leq 1.
\]

Thus

\[
\epsilon_n^{-d} \text{md}(\bigcup_{\sigma \in \Omega^*(\epsilon_n)} \mathcal{V}_{\sigma} F^{\delta_1})) \leq \sum_{\sigma \in \Omega^*(\epsilon_n)} c_\sigma^{s(1-\alpha)} \leq (c_0\epsilon_n)^{-sa},
\]

\[
(c_0\epsilon_n)^{d-sa} \geq \epsilon_n^{-d} \text{md}(\bigcup_{\sigma \in \Omega^*(\epsilon_n)} \mathcal{V}_{\sigma} F^{\delta_1})) \geq \epsilon_n^{-d} \text{md}(F^{c_0\epsilon_n\delta_1}),
\]

\[
d - sa \leq \frac{\log(\epsilon_n^{-d} \text{md}(F^{c_0\epsilon_n\delta_1}))}{\log(c_0\epsilon_n)}.
\]

which implies \( \dim_B F \leq \alpha_0(F^{\delta_1}) \).
Repeating the above procedure of proof with $\beta_0(A)$ instead of $\alpha_0(A)$ we can attain $\dim_H F \geq \beta_0(A)$s and $\lim \dim_H F \leq \beta_0(\delta^1)$s for any given $\delta_1 > 0$. As a result, we get $\dim_H F = \dim_B F = \dim_P F = \alpha_0(\delta^1)s = \beta_0(\delta^1)s$ for any given $\delta_1 > 0$ which indicates $\alpha_0(\delta^1)$ and $\beta_0(\delta^1)$ are independent of the choice of $\delta_1 > 0$ and $\alpha_0(\delta^1) = \beta_0(\delta^1)$. Furthermore $\alpha_0(A) = \beta_0(A)$ and they are independent of the choice of open set $A$ by (i).

(iii) Now we prove $\mathcal{H}^{\alpha_0}(F) < \infty$ iff $\lim_{\epsilon \rightarrow 0} \frac{-d m_d(\bigcup_{\sigma \in \Omega(\epsilon)} \phi_\sigma(A))}{\sum_{\sigma \in \Omega(\epsilon)} c_\sigma^{a_0}} < \infty$.

Suppose that $\lim_{\epsilon \rightarrow 0} \frac{-d m_d(\bigcup_{\sigma \in \Omega(\epsilon)} \phi_\sigma(A))}{\sum_{\sigma \in \Omega(\epsilon)} c_\sigma^{a_0}} = \infty$. Then we can take $\epsilon_1 > 0$ such that (7) holds with $\alpha_0$ instead of $\alpha$. For any $k \in N$ and for any finite open $(c_0 \epsilon_1)^k |A|$-covering $\{V_i\}$ of $F$, repeating $k-1$ time steps of proof of the above we can get

$$\sum_i |V_i|^{a_0} \geq 2^{k-1} \sum_j |U_j|^{a_0},$$

where $\{U_j\}$ is a finite open $c_0 \epsilon_1 |A|$-covering of $F$. According to the same mahtod of (ii) after finite steps, saying $l$ steps, we get

$$\sum_j |U_j|^{a_0} \geq 2^l (c_0 \epsilon_1)^{2a_0} |A|^{a_0}, \quad \sum_j |V_j|^{a_0} \geq 2^{l+k-1} (c_0 \epsilon_1)^{2a_0} |A|^{a_0},$$

which means $\mathcal{H}^{\alpha_0}(F) = \infty$ if letting $k$ tends to $\infty$.

Suppose $\mathcal{H}^{\alpha_0}(F) = \infty$. Thus for any $M > 0$ there exists $\epsilon_0$ such that for any $\epsilon_0$-covering $\{V_i\}$ of $F$

$$\sum_i |V_i|^{a_0} > M.$$

On the other hand, for any $\epsilon > 0$

$$J(\epsilon)(\epsilon \delta_1)^d \leq \text{const.} m_d(\bigcup_{\sigma \in \Omega^*(\epsilon)} \phi_\sigma(A)),$$

since $\bigcup_{\sigma \in \Omega^*(\epsilon)} B(\phi_\sigma(x), c_0 \epsilon \delta_1) \subset \bigcup_{\sigma \in \Omega(\epsilon)} \phi_\sigma(A)$ where $\delta_1$ is such that $F^{\delta_1} \subset A$. Thus

$$J(\epsilon)|\phi_\sigma(A)| \leq \text{const.} \frac{\epsilon^{-d} m_d(\bigcup_{\sigma \in \Omega(\epsilon)} \phi_\sigma(A))}{\sum_{\sigma \in \Omega(\epsilon)} c_\sigma^{a_0}}.$$}

Now taking $\epsilon$ such that $10 \epsilon |A| < \epsilon_0$ and considering the covering $\{B(\phi_\sigma(x), 5 |A| \epsilon), \sigma \in \Omega^*(\epsilon)\}$ of $F$ which is an $\epsilon_0$-covering of $F$, we have

$$\sum_{\sigma \in \Omega^*(\epsilon)} (10 |A| \epsilon)^{a_0} = \text{const.} J(\epsilon)^{\alpha_0} \geq M.$$

Therefore

$$\frac{\epsilon^{-d} m_d(\bigcup_{\sigma \in \Omega(\epsilon)} \phi_\sigma(A))}{\sum_{\sigma \in \Omega(\epsilon)} c_\sigma^{a_0}} \geq \text{const.} M,$$

for $\epsilon < (10 |A|)^{-1} \epsilon_0$ which indicates

$$\lim_{\epsilon \rightarrow 0} \frac{\epsilon^{-d} m_d(\bigcup_{\sigma \in \Omega(\epsilon)} \phi_\sigma(A))}{\sum_{\sigma \in \Omega(\epsilon)} c_\sigma^{a_0}} = \infty.$$
(iv) Suppose \( \lim_{\varepsilon \to 0} \frac{\varepsilon^{-d} m_d(\bigcup_{\sigma \in \Omega(\varepsilon)} \phi_\sigma(A))}{\sum_{\sigma \in \Omega(\varepsilon)} c_\sigma^{(1-\alpha)}} = 0 \). Then for any \( h > 0 \) there exist sequence \( \varepsilon_n \downarrow 0 \) such that

\[
\varepsilon_n^{-d} m_d\left( \bigcup_{\sigma \in \Omega(\varepsilon_n)} \phi_\sigma(A) \right) < h \sum_{\sigma \in \Omega(\varepsilon_n)} c_\sigma^{(1-\alpha)} \leq h c_0^{-\alpha} \varepsilon_n^{-\alpha s}.
\]

We consider the covering \( \{B(\phi_\sigma(x), 5\varepsilon_n |A|), \sigma \in \Omega^*(\varepsilon_n)\} \) of \( F \). Since

\[
\bigcup_{\sigma \in \Omega^*(\varepsilon_n)} B(\phi_\sigma(x), c_0 \varepsilon_n \delta_1) \subset \bigcup_{\sigma \in \Omega(\varepsilon_n)} \phi_\sigma(A)
\]

where \( \delta_1 \) is such that \( F^{\delta_1} \subset A \), then

\[
J(\varepsilon_n) m_d(B(\phi_\sigma(x), c_0 \varepsilon_n \delta_1)) \leq m_d\left( \bigcup_{\sigma \in \Omega(\varepsilon_n)} \phi_\sigma(A) \right),
\]

(10)

\[
J(\varepsilon) \leq \text{const.} \varepsilon_n^{-d} m_d\left( \bigcup_{\sigma \in \Omega(\varepsilon_n)} \phi_\sigma(A) \right) \leq \text{const.} h \varepsilon_n^{-\alpha s}.
\]

Therefore we have

\[
\sum_{\sigma \in \Omega^*(\varepsilon_n)} |B(\phi_\sigma(x), 5\varepsilon_n |A|)|^{\alpha s} = J(\varepsilon_n)(10 |A| \varepsilon_n)^{\alpha s} \leq \text{const.} h,
\]

which indicates \( \mathcal{H}^{\alpha s}(F) = 0 \). As a result we get \( \mathcal{H}^{\alpha s}(F) > 0 \) implies \( \lim_{\varepsilon \to 0} \frac{\varepsilon^{-d} m_d(\bigcup_{\sigma \in \Omega(\varepsilon)} \phi_\sigma(A))}{\sum_{\sigma \in \Omega(\varepsilon)} c_\sigma^{(1-\alpha)}} > 0 \).

\( \text{QED} \)

**Conjecture:** If \( \lim_{\varepsilon \to 0} \frac{\varepsilon^{-d} m_d(\bigcup_{\sigma \in \Omega(\varepsilon)} \phi_\sigma(A))}{\sum_{\sigma \in \Omega(\varepsilon)} c_\sigma^{(1-\alpha)}} > 0 \) then \( \mathcal{H}^{\alpha s}(F) > 0 \).

**Corollary 2.4** If \( \phi_i \)’s satisfy the open set condition then \( \dim_H F = \dim_B F = \dim_P F = s \).

**Proof** Let bounded nonempty open set \( O \) make \( \phi_i \)’s satisfy the open set condition. Taking \( A = O^1 \) thus

\[
\text{const.} \geq \frac{\varepsilon^{-d} m_d(\bigcup_{\sigma \in \Omega(\varepsilon)} \phi_\sigma(O^1))}{\text{Card}\Omega(\varepsilon)} \geq \frac{\varepsilon^{-d} m_d(\bigcup_{\sigma \in \Omega(\varepsilon)} \phi_\sigma(O))}{\text{Card}\Omega(\varepsilon)} \geq \text{const.} > 0,
\]

which means \( \alpha_0 = 1 \). Therefore \( \dim_H F = \dim_B F = \dim_P F = s \) by Theorem 2.3. \( \text{QED} \)

**Remark 2.5** If the above **Conjecture** holds then it is easy to get

(a) \( F \) is a \( \alpha_0 s \)-set iff \( 0 < \lim_{\varepsilon \to 0} \frac{\varepsilon^{-d} m_d(\bigcup_{\sigma \in \Omega(\varepsilon)} \phi_\sigma(A))}{\sum_{\sigma \in \Omega(\varepsilon)} c_\sigma^{(1-\alpha)}} < \infty; \)

(b) \( \phi_i \)’s satisfy the open set condition iff \( \lim_{\varepsilon \to 0} \frac{\varepsilon^{-d} m_d(\bigcup_{\sigma \in \Omega(\varepsilon)} \phi_\sigma(A))}{\text{Card}\Omega(\varepsilon)} > 0; \)

(c) \( \mathcal{H}^s(F) = 0 \) iff \( \lim_{\varepsilon \to 0} \frac{\varepsilon^{-d} m_d(\bigcup_{\sigma \in \Omega(\varepsilon)} \phi_\sigma(A))}{\text{Card}\Omega(\varepsilon)} = 0. \)
3 Generalization to MW-construction and generalized recurrent set

Let $A_{n \times n} = (a_{ij})_{n \times n}$ be an irreducible $0 - 1$ matrix. \{\phi_{ij} : a_{ij} = 1\} is a family of similar maps in $\mathbb{R}^d$ with the ratio $c_{ij}$ for $\phi_{ij}$. Let $s$ be such that the spectral radius of $(a_{ij}c_{ij}^s)_{n \times n}$ is 1 where we take $a_{ij}c_{ij}^s = 0$ when $a_{ij} = 0$. Write

$$\Omega_A = \{\sigma \in \prod_{i=1}^{\infty} \{1,2,\ldots, n\} : \sigma = (\sigma(1), \sigma(2), \ldots), a_{\sigma(l), \sigma(l+1)} = 1, l \in \mathbb{N}\},$$

$$\Omega^*_A = \{\sigma \in \bigcup_{i=2}^{\infty} \{1,2,\ldots, n\}^i : \sigma = (\sigma(1), \ldots, \sigma(k)), a_{\sigma(l), \sigma(l+1)} = 1, 1 \leq l \leq k - 1\}.$$ 

There exist unique compact sets $F_1, F_2, \ldots, F_n$ which sometimes is called MW-construction such that

$$F_i = \bigcup_{\{j:a_{ij}=1\}} \phi_{ij}(F_j), \quad 1 \leq i \leq n. \quad (11)$$

It is well-known that when \{\phi_{ij} : a_{ij} = 1\} satisfy the open condition, i.e., there is nonempty bounded open sets $O_1, O_2, \ldots, O_n$ such that

$$O_i \supset \bigcup_{\{j:a_{ij}=1\}} \phi_{ij}(O_j), \quad 1 \leq i \leq n,$$

with the right hand being disjoint union, we have

$$\dim_H F_i = \dim_B F_i = \dim_P F_i = s, \quad 1 \leq i \leq n,$$

and $F_i$ are all $s$-set.

Furthermore in Li[6] we prove that

**Proposition 3.1** \{\phi_{ij} : a_{ij} = 1\} satisfies the open set condition iff $F_i$ is an $s$-set for some $1 \leq i \leq n$ where $s$ is given above. \quad QED

Now for $1 \leq i \leq n$ let

$$\alpha_i = \sup\{ \alpha : \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^{-d}m_d(\bigcup_{\sigma \in \Omega_i(\epsilon)} \phi_{\sigma}(A_{\sigma(|\sigma|)})} {\sum_{\sigma \in \Omega_i(\epsilon)} c_{\sigma}^s(1-\alpha)} = \infty\}, \quad (12)$$

$$\beta_i = \sup\{ \beta : \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^{-d}m_d(\bigcup_{\sigma \in \Omega_i(\epsilon)} \phi_{\sigma}(A_{\sigma(|\sigma|)})} {\sum_{\sigma \in \Omega_i(\epsilon)} c_{\sigma}^s(1-\beta)} = \infty\},$$

where $A_i \supset F_i$ are bounded open sets; $|\sigma|$ denotes the length of $\sigma$; $\Omega_i(\epsilon) = \{\sigma \in \Omega^*_A : \sigma(1) = i, \ c_\sigma \leq \epsilon \ \text{and} \ c_\sigma(|\sigma|-1) > \epsilon\}$; \(c_\sigma = c_{\sigma(1),\sigma(2)}c_{\sigma(2),\sigma(3)}\cdots c_{\sigma(|\sigma|-1),\sigma(|\sigma|)}\); \(\phi_{\sigma} = \phi_{\sigma(1),\sigma(2)} \circ \phi_{\sigma(2),\sigma(3)} \circ \cdots \circ \phi_{\sigma(|\sigma|-1),\sigma(|\sigma|)}\). Write $c_0 = \min_{a_{ij} = 1} c_{ij}$.

In usual, we always take some bounded open set $A$ with $A \supset \bigcup_i F_i$ instead of $A_i$'s in (12). Similarly it is easy to get
Proposition 3.2  (1) 0 ≤ α_i ≤ β_i ≤ 1 for 1 ≤ i ≤ n;  (2) When \{φ_{ij} : a_{ij} = 1\} satisfies the open set condition, we have α_i = β_i = 1 for all 1 ≤ i ≤ n.  QED

Similar to Theorem 2.3 we have

Theorem 3.3  (I) All α_i and β_i are equal, denoting by α_0 the common value. And dim_H F_i = dim_B F_i = dim_P F_i = α_0 s for 1 ≤ i ≤ n.

(II) H^{α_s}(F_i) < ∞ for some 1 ≤ i ≤ n if \(\lim_{ε → 0} \frac{ε^{-d} m_d (\bigcup_{σ \in Ω_i(ε)} φ_σ (A_σ(σ|σ|)))}{\sum_{σ \in Ω_i(ε)} ε^s (1 - α)} < ∞\) for some 1 ≤ i ≤ n. And if H^{α_s}(F_i) > 0 for some 1 ≤ i ≤ n then \(\lim_{ε → 0} \frac{ε^{-d} m_d (\bigcup_{σ \in Ω_i(ε)} φ_σ (A_σ(σ|σ|)))}{\sum_{σ \in Ω_i(ε)} ε^s (1 - α)} > 0\)

for all 1 ≤ i ≤ n.

Proof  (I) Without loss of generality we suppose that α_1 = min_{1 ≤ i ≤ n} α_i, β_1 = min_{1 ≤ i ≤ n} β_i, β_n = max_{1 ≤ i ≤ n} β_i.

Fix some j, 1 ≤ j ≤ n. First step we prove dim_H F_j ≥ α_1 s. Taking x_i ∈ F_i and writing δ = max_i |F_i|. We choose the subfamily Ω_i(ε) from Ω_i(ε) such that

(1) for any σ, τ ∈ Ω_i(ε) and σ ≠ τ

\[ \rho(φ_σ(x_{σ(σ|σ|)}), φ_τ(x_{τ(τ|τ)})) > 4δε; \]

(2) if σ ∈ Ω_i(ε) \ Ω_i(ε) there exists τ ∈ Ω_i(ε) such that

\[ \rho(φ_σ(x_{σ(σ|σ|)}), φ_τ(x_{τ(τ|τ)})) ≤ 4δε. \]

Let J_i(ε) = CardΩ_i(ε). Thus

\[ \bigcup_{σ \in Ω_i(ε)} B(φ_σ(x_{σ(σ|σ|)}), 5δε) ⊇ \bigcup_{σ \in Ω_ε(ε)} B(φ_σ(x_{σ(σ|σ|)}), δε) ⊇ \bigcup_{σ \in Ω_i(ε)} φ_σ (A_σ(σ|σ|)). \]

Therefore we have

\[ J_i(ε)m_d B(φ_σ(x_{σ(σ|σ|)}), 5δε) ≥ m_d (\bigcup_{σ \in Ω_i(ε)} φ_σ (A_σ(σ|σ|))), \]

\[ J_i(ε)ε^s ≥ \frac{ε^{-d} m_d (\bigcup_{σ \in Ω_i(ε)} φ_σ (A_σ(σ|σ|)))}{\sum_{σ \in Ω_i(ε)} ε^s (1 - α)} (\sum_{σ \in Ω_i(ε)} ε^s (1 - α)) δ^{-d} \text{const.} ε^s. \]

Now let (m_1, ⋯, m_n) be the strictly positive right eigenvector responding to the eigenvalue 1. Then

\[ (c^s_{ij} a_{ij})_{n×n} = \begin{pmatrix} m_1 \\ ⋮ \\ m_n \end{pmatrix}, \]

Therefore

\[ \left[ \frac{\text{min} m_i}{\text{max} m_i} \right]^2 ≤ \sum_{σ \in Ω_i(ε)} c^s_σ ≤ \left[ \frac{\text{max} m_i}{\text{min} m_i} \right]^2. \]
In addition
\[ 1 \leq (\epsilon c_{\sigma}^{-1})^{\alpha s} \leq c_{0}^{-\alpha s}. \]

Therefore
\[ J_{i}(\epsilon)^{\alpha s} \geq \frac{e^{-dM_{d}(\sum_{\sigma \in \Omega_{i}(\epsilon)} A_{\sigma}(\lambda_{\sigma}))}}{\sum_{\sigma \in \Omega_{i}(\epsilon)} c_{\sigma}^{\alpha s}} \delta^{-d} \text{const}. \quad (13) \]

If \( \alpha_{1} = 0 \), it is trivial. We assume \( \alpha_{1} > 0 \) and take \( 0 < \alpha < \alpha_{1} \). Thus we have
\[ \lim_{\epsilon \to 0} J_{i}(\epsilon)^{\alpha s} = \infty, \]
by (13) for \( 1 \leq i \leq n \). Take \( \epsilon_{1} > 0 \) such that \( J_{i}(\epsilon_{1}) \epsilon_{1}^{\alpha s} c_{0}^{\alpha s} \geq 2 \) for all \( 1 \leq i \leq n \).

Considering the arbitrary finite open \( c_{0} \epsilon_{1} \delta \)-covering \( \{V_{i}\} \) of \( F_{j} \), thus

(a) if there exists some \( V_{i} \) with \( |V_{i}| \geq (c_{0} \epsilon_{1})^{2} \delta \) then
\[
\sum_{i} |V_{i}|^{\alpha s} \geq (c_{0} \epsilon_{1})^{2 \alpha s} \delta^{\alpha s}; \quad (14)
\]

(b) otherwise we have
\[
\sum_{i} |V_{i}|^{\alpha s} = \epsilon_{1}^{\alpha s} \sum_{i} |\epsilon_{1}^{-1} V_{i}|^{\alpha s}.
\]

For each \( \sigma \in \Omega_{j}^{*}(\epsilon_{1}) \), let \( \mathcal{V}_{\sigma} = \{ V_{i} : V_{i} \cap B(\phi_{\sigma}(x_{\sigma}(|\sigma|)), \epsilon_{1} \delta) \neq \emptyset \} \). Thus \( \mathcal{V}_{\sigma} \) is a covering of \( \phi_{\sigma}(F_{\sigma}(|\sigma|)) \) and for any \( \sigma, \tau \in \Omega_{j}^{*}(\epsilon_{1}) \), \( \sigma \neq \tau \), we have \( \mathcal{V}_{\sigma} \cap \mathcal{V}_{\tau} = \emptyset \). Take \( \lambda_{1} \in \Omega_{j}^{*}(\epsilon_{1}) \) such that
\[
\sum_{V_{i} \in \mathcal{V}_{\lambda_{1}}} |V_{i}|^{\alpha s} = \min_{\sigma \in \Omega_{j}^{*}(\epsilon_{1})} \sum_{V_{i} \in \mathcal{V}_{\sigma}} |V_{i}|^{\alpha s}.
\]

Therefore
\[
\sum_{i} |V_{i}|^{\alpha s} \geq J(\epsilon_{1}) \sum_{V_{i} \in \mathcal{V}_{\lambda_{1}}} |V_{i}|^{\alpha s} \geq J(\epsilon_{1}) \epsilon_{1}^{\alpha s} \sum_{V_{i} \in \mathcal{V}_{\lambda_{1}}} |\epsilon_{1}^{-1} V_{i}|^{\alpha s} \geq 2 c_{0}^{-\alpha s} \sum_{V_{i} \in \mathcal{V}_{\lambda_{1}}} |\epsilon_{1}^{-1} V_{i}|^{\alpha s}
\]
\[
= 2(c_{0} \epsilon_{1}^{-1} \epsilon_{1}^{-1})^{\alpha s} \sum_{V_{i} \in \mathcal{V}_{\lambda_{1}}} |\phi_{\lambda_{1}}^{-1} V_{i}|^{\alpha s} \geq 2 \sum_{V_{i} \in \mathcal{V}_{\lambda_{1}}} |\phi_{\lambda_{1}}^{-1} V_{i}|^{\alpha s}. \quad (15)
\]

Since \( \mathcal{V}_{\lambda_{1}} \) is a covering of \( \phi_{\lambda_{1}}(F_{\lambda_{1}(\lambda_{1}))} \), \( \phi_{\lambda_{1}}^{-1} \mathcal{V}_{\lambda_{1}} \) is a finite open \( c_{0} \epsilon_{1} \delta \)-covering of \( F_{\lambda_{1}(\lambda_{1}))} \). Denoting \( \phi_{\lambda_{1}}^{-1} \mathcal{V}_{\lambda_{1}} \) by \( \{ u_{i} \} \) as above we have

(a') if there exists \( u_{i} \in \phi_{\lambda_{1}}^{-1} \mathcal{V}_{\lambda_{1}} \) such that \( |u_{i}| \geq (c_{0} \epsilon_{1})^{2} \delta \) then (14) holds by (15).

(b') otherwise repeating the above step and considering Card\( \{V_{i}\} \) finite, thus (14) holds after finite steps. Therefore \( \dim H F_{j} \geq \alpha s \) for any \( 0 < \alpha < \alpha_{1} \) which means \( \dim H F_{j} \geq \alpha_{1} s \).

Similar to the proof of Theorem 2.3 we also get \( \alpha_{1} s \leq \dim H F_{j} \leq \dim B F_{j} \leq \alpha_{1} s \) and \( \beta_{1} s \leq \dim H F_{j} \leq \overline{\dim} F_{j} \leq \beta_{1} s \) and \( \underline{\dim} F_{j} = \beta_{1} s \). Thus we complete the proof. In addition it is easy to find that all \( \alpha_{i}'s \) and \( \beta_{i}'s \) are equal and independent of the choice of \( A_{i} \).

(II) Finally using the same method as those in proof of Theorem 2.3 (III) and (IV) we can complete the proof of (II). \textbf{QED}
Corollary 3.4 When \( \{ \phi_{ij}: a_{ij} = 1 \} \) satisfies the open set condition, we have for every \( 0 \leq i \leq n \), \( \dim_B F_i = \dim_B F_i = \dim_P F_i = s \). \( \text{QED} \)

Conjecture: if
\[
\lim_{\epsilon \to 0} \frac{\epsilon^{-d} m_d(\bigcup_{\sigma \in \Omega_i(\epsilon)} \phi_\sigma(A_{\sigma(|\sigma|)}))}{\sum_{\sigma \in \Omega_i(\epsilon)} C_{\sigma} s(1-\alpha)} > 0
\]
for some \( 1 \leq i \leq n \), then \( H^\alpha(F_i) > 0 \) for all \( 1 \leq i \leq n \).

Remark 3.5 (1) Since the recurrent set (Dekking [2]) and the generalized recurrent set (Li [8]) are all the special cases of MW-construction (Bedford [1] & Li [7]) the Theorem 3.3 also works there. Thus our Theorem 3.3 actually improves the main results of [11] [12] which discussed the lower bound of Hausdorff dimension of recurrent sets and self-similar sets.

(2) If the above conjecture is true, it is easy to get

(a) \( F_i \) is an \( \alpha s \)-set for some \( 1 \leq i \leq n \) iff for some \( 1 \leq i \leq n \)
\[
0 < \lim_{\epsilon \to 0} \frac{\epsilon^{-d} m_d(\bigcup_{\sigma \in \Omega_i(\epsilon)} \phi_\sigma(A_{\sigma(|\sigma|)}))}{\sum_{\sigma \in \Omega_i(\epsilon)} C_{\sigma} s(1-\alpha)} < \infty;
\]

(b) \( F_i \)'s satisfy the open set condition iff for some \( 1 \leq i \leq n \)
\[
\lim_{\epsilon \to 0} \frac{\epsilon^{-d} m_d(\bigcup_{\sigma \in \Omega_i(\epsilon)} \phi_\sigma(A_{\sigma(|\sigma|)}))}{\text{Card}\Omega_i(\epsilon)} > 0.
\]

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References


