The Dimensions Of Self-Similar Sets *

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Abstract

For self-similar set F we prove that $\dim_H F = \dim_B F = \dim_P F$ using different method from Fa[4] and give implicitly the dimension value even if the open set condition isn't satisfied.

1 Introduction

Let ϕ_i be similar contraction mappings in $\mathbf{R}^d$ with ratios c_i ,$1 \leq i \leq n$.Hu[5] proved that there exists unique compact set $F \subset \mathbf{R}^d$ such that

$$F = \bigcup_{i=1}^{n} \phi_i(F). \tag{1}$$

Further $\dim_H F = \dim_B F = \dim_P F = s$ and F is an s-set where s is such that

$$\sum_{i=1}^{n} c_i^s = 1, \tag{2}$$

if ϕ_i's satisfy the open set condition, i.e., there is a bounded nonempty open set O such that

$$\bigcup_{i=1}^{n} \phi_i(O) \subset O \tag{3}$$

with the left hand is disjoint union. Recently Sh[10] proved that F is an s-set here $\sum_{i=1}^{n} c_i^s = 1$ if and only if ϕ_i's satisfy the open condition.

Now for $\epsilon > 0$ write

$$\Omega(\epsilon) = \{\sigma \in S^* \mid c_\sigma \leq \epsilon \text{ and } c_{\sigma|(|\sigma|-1)} > \epsilon\},$$

where $S^* = \bigcup_{i=1}^{\infty}\{1, 2, \cdots, n\}^i$ and $c_\sigma = c_{\sigma(1)}c_{\sigma(2)} \cdots c_{\sigma(k)}$ for $\sigma = (\sigma(1), \sigma(2), \cdots, \sigma(k)) \in S^*$. And for $\sigma \in S^*$, $|\sigma|$ denotes the length of σ and $\sigma|k = (\sigma(1), \cdots, \sigma(k))$ for $k \leq |\sigma|$. Let $A \subset \mathbf{R}^d$ be a bounded open set with $A \supset F$.It is easy to see that $c_0\epsilon < c_\sigma \leq \epsilon$ for

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any $\sigma \in \Omega(\epsilon)$ where $c_0 = \min_{1\le i\le n} c_i$. We introduce nonnegative real numbers $\alpha_0(A)$ and $\beta_0(A)$ as follows

$$\alpha_0(A) = \sup\{\alpha \mid \underline{\lim}_{\epsilon\to 0} \frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon)} \phi_\sigma(A))}{\sum_{\sigma\in\Omega(\epsilon)} c_\sigma^{s(1-\alpha)}} = \infty\}, \tag{4}$$

$$\beta_0(A) = \sup\{\beta \mid \overline{\lim}_{\epsilon\to 0} \frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon)} \phi_\sigma(A))}{\sum_{\sigma\in\Omega(\epsilon)} c_\sigma^{s(1-\beta)}} = \infty\}, \tag{5}$$

where $\phi_\sigma = \phi_{\sigma(1)} \circ \phi_{\sigma(2)} \circ \cdots \circ \phi_{\sigma(k)}$ for $\sigma = (\sigma(1), \sigma(2), \cdots, \sigma(k)) \in S^*$ and $\mathrm{m_d}(B)$ is the Lebesque measure of $B \subset \mathbf{R}^d$.

In this paper we prove

(i) $\alpha_0(A)$ and $\beta_0(A)$ are independent of the choice of A and $\alpha_0(A) = \beta_0(A)$.We denote the common value by α_0;

(ii) $\dim_H F = \dim_B F = \dim_P F = \alpha_0 s$; (For self-similar set F Fa[4] has proved that its Hausdorff dimension,Box dimension and Packing dimension are equal)

(iii) $\mathcal{H}^{\alpha_0 s}(F) < \infty$ iff $\underline{\lim}_{\epsilon\to 0} \frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon)} \phi_\sigma(A))}{\sum_{\sigma\in\Omega(\epsilon)} c_\sigma^{s(1-\alpha_0)}} < \infty$;

(iv) If $\mathcal{H}^{\alpha_0 s}(F) > 0$ then $\underline{\lim}_{\epsilon\to 0} \frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon)} \phi_\sigma(A))}{\sum_{\sigma\in\Omega(\epsilon)} c_\sigma^{s(1-\alpha_0)}} > 0$;

(v) We generalize this dimension results into the cases of MW-construction (Ma & Wi[9]) and recurrent sets (De[2],Be[1] and Wen[11]).

2 Demensions of self-similar set

It is easy to get the following

Proposition 2.1

$$\alpha_0(A) = \inf\{\alpha \mid \underline{\lim}_{\epsilon\to 0} \frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon)} \phi_\sigma(A))}{\sum_{\sigma\in\Omega(\epsilon)} c_\sigma^{s(1-\alpha)}} = 0\},$$

$$\beta_0(A) = \inf\{\beta \mid \overline{\lim}_{\epsilon\to 0} \frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon)} \phi_\sigma(A))}{\sum_{\sigma\in\Omega(\epsilon)} c_\sigma^{s(1-\beta)}} = 0\}.$$

Proposition 2.2 $0 \le \alpha_0(A) \le 1$; $0 \le \beta_0(A) \le 1$.

Proof Note that $\sum_{\sigma\in\Omega(\epsilon)} c_\sigma^s = 1$. Taking $\alpha = 0$ then

$$\underline{\lim}_{\epsilon\to 0} \frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon)} \phi_\sigma(A))}{\sum_{\sigma\in\Omega(\epsilon)} c_\sigma^{s}} = \underline{\lim}_{\epsilon\to 0}\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon)} \phi_\sigma(A)) \ge c$$

for some positive constant c. Thus $\alpha_0(A) \geq 0$. On the other hand, taking $\alpha = 1$, we have

$$\underline{\lim}_{\epsilon\to 0}\frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon)}\phi_\sigma(A))}{\mathrm{Card}\Omega(\epsilon)} \leq c$$

for some constant c. Thus $\alpha_0(A) \leq 1$.

$0 \leq \beta_0(A) \leq 1$ can be proved by the same method. **QED**

Theorem 2.3

(i) $\alpha_0(A)$ and $\beta_0(A)$ are independent of the choice of A and $\alpha_0(A) = \beta_0(A)$,denoting the common value by α_0;

(ii) $\dim_H F = \dim_B F = \dim_P F = \alpha_0 s$;

(iii) $\mathcal{H}^{\alpha_0 s}(F) < \infty$ iff $\underline{\lim}_{\epsilon\to 0}\frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon)}\phi_\sigma(A))}{\sum_{\sigma\in\Omega(\epsilon)}c_\sigma^{s(1-\alpha_0)}} < \infty$;

(iv) If $\mathcal{H}^{\alpha_0 s}(F) > 0$ then $\underline{\lim}_{\epsilon\to 0}\frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon)}\phi_\sigma(A))}{\sum_{\sigma\in\Omega(\epsilon)}c_\sigma^{s(1-\alpha_0)}} > 0$.

Proof **(i)** For $B \subset \mathbf{R}^d$ and $\epsilon > 0$ let

$$B^\epsilon = \{x \in \mathbf{R}^d : \text{there exists y} \in \text{B such that } \rho(\text{x},\text{y}) < \epsilon\}$$

where $\rho(x, y)$ is the Euclidean distance between x and y . Since A is a bounded open set containing set F, there are positive number δ_1 and δ_2 such that $F^{\delta_1} \subset A \subset F^{\delta_2}$ which means $\alpha_0(F^{\delta_1}) \leq \alpha_0(A) \leq \alpha_0(F^{\delta_2})$ and $\beta_0(F^{\delta_1}) \leq \beta_0(A) \leq \beta_0(F^{\delta_2})$. Thus it suffices to prove $\alpha_0(F^\delta)$ and $\beta_0(F^\delta)$ are independent of the choice of positive number δ and $\alpha_0(F^\delta) = \beta_0(F^\delta)$, which follows from the proof of **(ii)**.

(ii) Fixing $x \in F$ and denoting the diameter of A by $|A|$ we choose subfamily $\Omega^*(\epsilon)$ from $\Omega(\epsilon)$ such that

(1)for any different $\sigma, \tau \in \Omega^*(\epsilon)$, $\rho(\phi_\sigma(x), \phi_\tau(x)) > 4|A|\epsilon$;

(2) if $\sigma \in \Omega(\epsilon) \setminus \Omega^*(\epsilon)$ there exiets $\tau \in \Omega^*(\epsilon)$ such that $\rho(\phi_\sigma(x), \phi_\tau(x)) \leq 4|A|\epsilon$.

Let $J(\epsilon) = \mathrm{Card}\Omega^*(\epsilon)$. Thus

$$\bigcup_{\sigma\in\Omega^*(\epsilon)} B(\phi_\sigma(x), 5|A|\epsilon) \supset \bigcup_{\sigma\in\Omega(\epsilon)} B(\phi_\sigma(x), |A|\epsilon) \supset \bigcup_{\sigma\in\Omega(\epsilon)} \phi_\sigma(A)$$

where $B(x, r)$ denotes a ball in $\mathbf{R}^d$ with center at x and radiu r. Thus

$$J(\epsilon)\mathrm{m_d}B(\phi_\sigma(x), 5|A|\epsilon) \geq \mathrm{m_d}\bigcup_{\sigma\in\Omega(\epsilon)}\phi_\sigma(A).$$

Therefore for any nonnegative real number α

$$J(\epsilon)\epsilon^{\alpha s} \geq \frac{c|A|^{-d}\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon)}\phi_\sigma(A))}{\sum_{\sigma\in\Omega(\epsilon)}c_\sigma^{s(1-\alpha)}}, \tag{6}$$

where c is a positive constant. First we prove $\dim_H F \geq \alpha_0(A)s$. It is clear when $\alpha_0(A) = 0$. Suppose $\alpha_0(A) > 0$ and take $0 < \alpha < \alpha_0(A)$. Thus by the definition of $\alpha_0(A)$ and (6) we can take $\epsilon_1 > 0$ such that

$$J(\epsilon_1)\epsilon_1^{\alpha s} \geq 2c_0^{-\alpha s}. \tag{7}$$

Considering any finite open $c_0\epsilon_1|A|$-covering $\{V_i\}$ of F , we have

(a) if there exists some V_i such that $|V_i| \geq (c_0\epsilon_1)^2|A|$ then

$$\sum_i |V_i|^{\alpha s} \geq (c_0\epsilon_1)^{2\alpha s}|A|^{\alpha s}; \tag{8}$$

(b) otherwise for each $\sigma \in \Omega^*(\epsilon_1)$ let $\mathcal{V}_\sigma = \{V_i : V_i \cap B(\phi_\sigma(x), \epsilon_1|A|) \neq \emptyset\}$. Then $\mathcal{V}_\sigma$ is a covering of $\phi_\sigma(F)$ and any different $\sigma, \tau \in \Omega^*(\epsilon_1)$, $\mathcal{V}_\sigma \bigcap \mathcal{V}_\tau = \emptyset$. Take $\lambda_1 \in \Omega^*(\epsilon_1)$ such that

$$\sum_{V_i \in \mathcal{V}_{\lambda_1}} |V_i|^{\alpha s} = \min_{\sigma \in \Omega^*(\epsilon_1)} \sum_{V_i \in \mathcal{V}_\sigma} |V_i|^{\alpha s}.$$

Therefore by (7) we have

$$\begin{aligned} \sum_i |V_i|^{\alpha s} &\geq J(\epsilon_1) \sum_{V_i \in \mathcal{V}_{\lambda_1}} |V_i|^{\alpha s} \geq 2c_0^{-\alpha s}\epsilon_1{}^{-\alpha s} \sum_{V_i \in \mathcal{V}_{\lambda_1}} |V_i|^{\alpha s} \\ &= 2(c_{\lambda_1}c_0^{-1}\epsilon_1^{-1})^{\alpha s} \sum_{V_i \in \mathcal{V}_{\lambda_1}} |\phi_{\lambda_1}^{-1}V_i|^{\alpha s} \geq 2 \sum_{V_i \in \mathcal{V}_{\lambda_1}} |\phi_{\lambda_1}^{-1}V_i|^{\alpha s}. \end{aligned} \tag{9}$$

Since $\mathcal{V}_{\lambda_1}$ is a covering of $\phi_{\lambda_1}(F)$, $\phi_{\lambda_1}^{-1}\mathcal{V}_{\lambda_1} = \{\phi_{\lambda_1}^{-1}(V_i) : V_i \in \mathcal{V}_{\lambda_1}\}$ is a finite open $c_0\epsilon_1|A|$-covering of F. As above we have

(a') if there exists $\phi_{\lambda_1}^{-1}(V_i) \in \phi_{\lambda_1}^{-1}\mathcal{V}_{\lambda_1}$ such that $|\phi_{\lambda_1}^{-1}(V_i)| \geq (c_0\epsilon_1)^2|A|$ then (8) holds by (9);

(b') othewise denote $\phi_{\lambda_1}^{-1}\mathcal{V}_{\lambda_1}$ by $\{U_i\}$. Repeating the above step for the covering $\{U_i\}$ of F and noticing that Card$\{V_i\}$ is finite, thus (8) holds after finite steps. Consequently $\dim_H F \geq \alpha s$ which means $\dim_H F \geq \alpha_0(A)s$.

Now taking $\delta_1 > 0$ we prove that $\dim_H F \leq \underline{\dim}_B F \leq \alpha_0(F^{\delta_1})s$. Letting $\alpha > \alpha_0(F^{\delta_1})$ there exists sequence $\epsilon_n \searrow 0$ such that

$$\frac{\epsilon_n^{-d}\mathrm{m_d}(\bigcup_{\sigma \in \Omega(\epsilon_n)} \phi_\sigma(F^{\delta_1}))}{\sum_{\sigma \in \Omega(\epsilon_n)} c_\sigma^{s(1-\alpha)}} \leq 1.$$

Thus

$$\epsilon_n^{-d}\mathrm{m_d}(\bigcup_{\sigma \in \Omega(\epsilon_n)} \phi_\sigma(F^{\delta_1})) \leq \sum_{\sigma \in \Omega(\epsilon_n)} c_\sigma^{s(1-\alpha)} \leq (c_0\epsilon_n)^{-s\alpha},$$

$$(c_0\epsilon_n)^{d-s\alpha} \geq c_0^d\mathrm{m_d}(\bigcup_{\sigma \in \Omega(\epsilon_n)} \phi_\sigma(F^{\delta_1})) \geq c_0^d\mathrm{m_d}(F^{c_0\epsilon_n\delta_1}),$$

$$d - s\alpha \leq \frac{\log(\mathrm{m_d}(F^{c_0\epsilon_n\delta_1})c_0^d)}{\log(c_0\epsilon_n)},$$

$$d - s\alpha \leq \overline{\lim}_{n\to 0}\frac{\log[c_0^d\mathrm{m_d}(F^{c_0\epsilon_n\delta_1})]}{\log(c_0\epsilon_n)} \leq \overline{\lim}_{\epsilon\to 0}\frac{\log[\mathrm{m_d}(F^{c_0\epsilon\delta_1})]}{\log(c_0\epsilon\delta_1)},$$

which implies $\underline{\dim}_B F \leq s\alpha$ by the Proposition 3.2 of Fa[3]. Therefore $\underline{\dim}_B F \leq s\alpha_0(F^{\delta_1})$.

Repeating the above procedure of proof with $\beta_0(A)$ instead of $\alpha_0(A)$ we can attain $\dim_H F \geq \beta_0(A)s$ and $\dim_H F \leq \overline{\dim}_B F \leq \beta_0(F^{\delta_1})s$ for any given $\delta_1 > 0$. As a result,we get $\dim_H F = \dim_P F = \dim_B F = \alpha_0(F^{\delta_1})s = \beta_0(F^{\delta_1})s$ for any given $\delta_1 > 0$ which indicates $\alpha_0(F^{\delta_1})$ and $\beta_0(F^{\delta_1})$ are independent of the choice of $\delta_1 > 0$ and $\alpha_0(F^{\delta_1}) = \beta_0(F^{\delta_1})$. Furthermore $\alpha_0(A) = \beta_0(A)$ and they are independent of the choice of open set A by **(i)**.

(iii) Now we prove $\mathcal{H}^{\alpha_0 s}(F) < \infty$ iff $\underline{\lim}_{\epsilon\to 0}\frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon)}\phi_\sigma(A))}{\sum_{\sigma\in\Omega(\epsilon)} c_\sigma^{s(1-\alpha_0)}} < \infty$.

Suppose that $\underline{\lim}_{\epsilon\to 0}\frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon)}\phi_\sigma(A))}{\sum_{\sigma\in\Omega(\epsilon)} c_\sigma^{s(1-\alpha_0)}} = \infty$. Then we can take $\epsilon_1 > 0$ such that (7) holds with α_0 instead of α. For any $k \in \mathbf{N}$ and for any finite open $(c_0\epsilon_1)^k|A|$-covering $\{V_i\}$ of F, repeating $k-1$ time steps of proof of the above we can get

$$\sum_i |V_i|^{\alpha_0 s} \geq 2^{k-1}\sum_j |U_j|^{\alpha_0 s},$$

where $\{U_j\}$ is a finite open $c_0\epsilon_1|A|$-covering of F. According to the same mahtod of **(ii)** after finite steps, saying l steps, we get

$$\sum_j |U_j|^{\alpha_0 s} \geq 2^l (c_0\epsilon_1)^{2\alpha_0 s}|A|^{\alpha_0 s}, \quad \sum_j |V_j|^{\alpha_0 s} \geq 2^{l+k-1}(c_0\epsilon_1)^{2\alpha_0 s}|A|^{\alpha_0 s},$$

which means $\mathcal{H}^{\alpha_0 s}(F) = \infty$ if letting k tends to ∞.

Suppose $\mathcal{H}^{\alpha_0 s}(F) = \infty$. Thus for any $M > 0$ there exists ϵ_0 such that for any ϵ_0-covering $\{V_i\}$ of F

$$\sum_i |V_i|^{\alpha_0 s} > M.$$

On the other hand, for any $\epsilon > 0$

$$J(\epsilon)(\epsilon\delta_1)^d \leq \mathrm{const.m_d}(\cup_{\sigma\in\Omega(\epsilon)}\phi_\sigma(A)),$$

since $\bigcup_{\sigma\in\Omega^*(\epsilon)} B(\phi_\sigma(x), c_0\epsilon\delta_1) \subset \bigcup_{\sigma\in\Omega(\epsilon)}\phi_\sigma(A)$ where δ_1 is such that $F^{\delta_1} \subset A$. Thus

$$J(\epsilon)\epsilon^{\alpha_0 s} \leq \mathrm{const.}\frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon)}\phi_\sigma(A))}{\sum_{\sigma\in\Omega(\epsilon)} c_\sigma^{s(1-\alpha_0)}}.$$

Now taking ϵ such that $10\epsilon|A| < \epsilon_0$ and considering the covering $\{B(\phi_\sigma(x), 5|A|\epsilon), \sigma \in \Omega^*(\epsilon)\}$ of F which is a ϵ_0-covering of F, we have

$$\sum_{\sigma\in\Omega^*(\epsilon)} (10|A|\epsilon)^{\alpha_0 s} = \mathrm{const.}J(\epsilon)\epsilon^{\alpha_0 s} \geq M.$$

Therefore

$$\frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon)}\phi_\sigma(A))}{\sum_{\sigma\in\Omega(\epsilon)} c_\sigma^{s(1-\alpha_0)}} \geq \mathrm{const.}M,$$

for $\epsilon < (10|A|)^{-1}\epsilon_0$ which indicates

$$\lim_{\epsilon\to 0}\frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon)}\phi_\sigma(A))}{\sum_{\sigma\in\Omega(\epsilon)} c_\sigma^{s(1-\alpha_0)}} = \infty.$$

(iv) Suppose $\underline{\lim}_{\epsilon\to 0}\frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon)}\phi_\sigma(A))}{\sum_{\sigma\in\Omega(\epsilon)}c_\sigma^{s(1-\alpha_0)}}=0$. Then for any $h>0$ there exist sequence $\epsilon_n\searrow 0$ such that

$$\epsilon_n^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon_n)}\phi_\sigma(A))<h\sum_{\sigma\in\Omega(\epsilon_n)}c_\sigma^{s(1-\alpha_0)}\le hc_0^{-\alpha_0 s}\epsilon_n^{-\alpha_0 s}.$$

We consider the covering $\{B(\phi_\sigma(x),5\epsilon_n|A|),\sigma\in\Omega^*(\epsilon_n)\}$ of F. Since

$$\bigcup_{\sigma\in\Omega^*(\epsilon_n)}B(\phi_\sigma(x),c_0\epsilon_n\delta_1)\subset\bigcup_{\sigma\in\Omega(\epsilon_n)}\phi_\sigma(A)$$

where δ_1 is such that $F^{\delta_1}\subset A$, then

$$J(\epsilon_n)\mathrm{m_d}(B(\phi_\sigma(x),c_0\epsilon_n\delta_1))\le\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon_n)}\phi_\sigma(A)),\tag{10}$$

$$J(\epsilon_n)\le\mathrm{const}.\epsilon_n^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon_n)}\phi_\sigma(A))\le\mathrm{const}.h\epsilon_n^{-\alpha_0 s}.$$

Therefore we have

$$\sum_{\sigma\in\Omega^*(\epsilon_n)}|B(\phi_\sigma(x),5\epsilon_n|A|)|^{\alpha_0 s}=J(\epsilon_n)(10|A|\epsilon_n)^{\alpha_0 s}\le\mathrm{const}.h,$$

which indicates $\mathcal{H}^{\alpha_0 s}(F)=0$. As a result we get $\mathcal{H}^{\alpha_0 s}(F)>0$ implies $\underline{\lim}_{\epsilon\to 0}\frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon)}\phi_\sigma(A))}{\sum_{\sigma\in\Omega(\epsilon)}c_\sigma^{s(1-\alpha_0)}}>0$. **QED**

Conjecture: If $\underline{\lim}_{\epsilon\to 0}\frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon)}\phi_\sigma(A))}{\sum_{\sigma\in\Omega(\epsilon)}c_\sigma^{s(1-\alpha_0)}}>0$ then $\mathcal{H}^{\alpha_0 s}(F)>0$.

Corollary 2.4 *If ϕ_i's satisfy the open set condition then* $\dim_H F=\dim_B F=\dim_P F=s$.

Proof Let bounded nonempty open set O make ϕ_i's satisfy the open set condition. Taking $A=O^1$ thus

$$\mathrm{const.}\ge\frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon)}\phi_\sigma(O^1))}{\mathrm{Card}\Omega(\epsilon)}\ge\frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon)}\phi_\sigma(\overline{O}))}{\mathrm{Card}\Omega(\epsilon)}\ge\mathrm{const.}>0,$$

which means $\alpha_0=1$. Therefore $\dim_H F=\dim_B F=\dim_P F=s$ by Theorem 2.3. **QED**

Remark 2.5 If the above **Conjecture** holds then it is easy to get

(a) F is a $\alpha_0 s$-set iff $0<\underline{\lim}_{\epsilon\to 0}\frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon)}\phi_\sigma(A))}{\sum_{\sigma\in\Omega(\epsilon)}c_\sigma^{s(1-\alpha_0)}}<\infty$;

(b) ϕ_i's satisfy the open set condition iff $\underline{\lim}_{\epsilon\to 0}\frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon)}\phi_\sigma(A))}{\mathrm{Card}\Omega(\epsilon)}>0$;

(c) $\mathcal{H}^s(F)=0$ iff $\underline{\lim}_{\epsilon\to 0}\frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega(\epsilon)}\phi_\sigma(A))}{\mathrm{Card}\Omega(\epsilon)}=0$.

3 Generalization to MW-construction and generalized recurrent set

Let $A_{n\times n} = (a_{ij})_{n\times n}$ be an irreducible $0-1$ matrix. $\{\phi_{ij} : a_{ij} = 1\}$ is a family of similar maps in $\mathbf{R}^d$ with the ratio c_{ij} for ϕ_{ij}. Let s be such that the spectral radius of $(a_{ij}c_{ij}^s)_{n\times n}$ is 1 where we take $a_{ij}c_{ij}^s = 0$ when $a_{ij} = 0$.Write

$$\Omega_A = \{\sigma \in \prod_1^\infty \{1, 2, \cdots, n\} : \sigma = (\sigma(1), \sigma(2), \cdots), a_{\sigma(l),\sigma(l+1)} = 1, l \in \mathbf{N}\},$$

$$\Omega_A^* = \{\sigma \in \bigcup_{i=2}^\infty \{1, 2, \cdots, n\}^i : \sigma = (\sigma(1), \cdots, \sigma(k)), a_{\sigma(l),\sigma(l+1)} = 1, 1 \le l \le k-1\}.$$

There exist unqiue compact sets $F_1, F_2, \cdots, F_n$ which sometimes is called MW-construction such that

$$F_i = \bigcup_{\{j:a_{ij}=1\}} \phi_{ij}(F_j), \quad 1 \le i \le n. \tag{11}$$

It is well-known that when $\{\phi_{ij} : a_{ij} = 1\}$ satisfy the open condition, i.e., there is nonempty bounded open sets $O_1, O_2, \cdots, O_n$ such that

$$O_i \supset \bigcup_{\{j:a_{ij}=1\}} \phi_{ij}(O_j), \quad 1 \le i \le n,$$

with the right hand being disjoint union,we have

$$\dim_H F_i = \dim_B F_i = \dim_P F_i = s, \quad 1 \le i \le n,$$

and F_i are all s-set.

Furthermore in Li[6] we prove that

Proposition 3.1 $\{\phi_{ij} : a_{ij} = 1\}$ *satisfies the open set condition iff* F_i *is an* s*-set for some* $1 \le i \le n$ *where* s *is given above.* **QED**

Now for $1 \le i \le n$ let

$$\alpha_i = \sup\{\alpha : \underline{\lim}_{\epsilon\to 0} \frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega_i(\epsilon)} \phi_\sigma(A_{\sigma(|\sigma|)}))}{\sum_{\sigma\in\Omega_i(\epsilon)} c_\sigma^{s(1-\alpha)}} = \infty\}, \tag{12}$$

$$\beta_i = \sup\{\beta : \overline{\lim}_{\epsilon\to 0} \frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega_i(\epsilon)} \phi_\sigma(A_{\sigma(|\sigma|)}))}{\sum_{\sigma\in\Omega_i(\epsilon)} c_\sigma^{s(1-\beta)}} = \infty\},$$

where $A_i \supset F_i$ are bounded open sets; $|\sigma|$ denotes the lengh of σ; $\Omega_i(\epsilon) = \{\sigma \in \Omega_A^* : \sigma(1) = i,\ c_\sigma \le \epsilon$ and $c_{\sigma|(|\sigma|-1)} > \epsilon\}$; $c_\sigma = c_{\sigma(1),\sigma(2)}c_{\sigma(2),\sigma(3)} \cdots c_{\sigma(|\sigma|-1),\sigma(|\sigma|)}$; $\phi_\sigma = \phi_{\sigma(1),\sigma(2)} \circ \phi_{\sigma(2),\sigma(3)} \circ \cdots \circ \phi_{\sigma(|\sigma|-1),\sigma(|\sigma|)}$. Write $c_0 = \min_{a_{ij}=1} c_{ij}$.

In usual, we always take some bounded open set A with $A \supset \bigcup_i F_i$ instead of A_i's in (12). Similarly it is easy to get

Proposition 3.2 **(1)** $0 \le \alpha_i \le \beta_i \le 1$ *for* $1 \le i \le n$; **(2)** *When* $\{\phi_{ij} : a_{ij} = 1\}$ *satisfies the open set condition ,we have* $\alpha_i = \beta_i = 1$ *for all* $1 \le i \le n$. **QED**

Similar to Theorem 2.3 we have

Theorem 3.3 **(I)** *All* α_i *and* β_i *are equal, denoting by* α_0 *the common value. And* $\dim_H F_i = \dim_B F_i = \dim_P F_i = \alpha_0 s$ *for* $1 \le i \le n$.

(II) $\mathcal{H}^{\alpha_0 s}(F_i) < \infty$ *for some* $1 \le i \le n$ *iff* $\underline{\lim}_{\epsilon\to 0} \frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega_i(\epsilon)}\phi_\sigma(A_{\sigma(|\sigma|)})}{\sum_{\sigma\in\Omega_i(\epsilon)} c_\sigma^{s(1-\alpha_0)}} < \infty$ *for some* $1 \le i \le n$. *And if* $\mathcal{H}^{\alpha_0 s}(F_i) > 0$ *for some* $1 \le i \le n$ *then* $\underline{\lim}_{\epsilon\to 0} \frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega_i(\epsilon)}\phi_\sigma(A_{\sigma(|\sigma|)})}{\sum_{\sigma\in\Omega_i(\epsilon)} c_\sigma^{s(1-\alpha_0)}} > 0$ *for all* $1 \le i \le n$.

Proof **(I)**Without loss of generality we suppose that $\alpha_1 = \min_{1\le i\le n}\alpha_i,\ \beta_1 = \min_{1\le i\le n}\beta_i,\ \beta_n = \max_{1\le i\le n}\beta_i$.

Fix some $j,\ 1 \le j \le n$. First step we prove $\dim_H F_j \ge \alpha_1 s$. Taking $x_i \in F_i$ and writing $\delta = \max_i |F_i|$. We choose the subfamily $\Omega_i^*(\epsilon)$ from $\Omega_i(\epsilon)$ such that

(1) for any $\sigma, \tau \in \Omega_i^*(\epsilon)$ and $\sigma \ne \tau$

$$\rho(\phi_\sigma(x_{\sigma(|\sigma|)}), \phi_\tau(x_{\tau(|\tau|)})) > 4\delta\epsilon;$$

(2) if $\sigma \in \Omega_i(\epsilon) \setminus \Omega_i^*(\epsilon)$ there exists $\tau \in \Omega_i^*(\epsilon)$ such that

$$\rho(\phi_\sigma(x_{\sigma(|\sigma|)}), \phi_\tau(x_{\tau(|\tau|)})) \le 4\delta\epsilon.$$

Let $J_i(\epsilon) = \mathrm{Card}\Omega_i^*(\epsilon)$. Thus

$$\bigcup_{\sigma\in\Omega_i^*(\epsilon)} B(\phi_\sigma(x_{\sigma(|\sigma|)}), 5\delta\epsilon) \supseteq \bigcup_{\sigma\in\Omega_i(\epsilon)} B(\phi_\sigma(x_{\sigma(|\sigma|)}), \delta\epsilon) \supseteq \bigcup_{\sigma\in\Omega_i(\epsilon)} \phi_\sigma(A_{\sigma(|\sigma|)}).$$

Therefore we have

$$J_i(\epsilon)\mathrm{m_d}B(\phi_\sigma(x_{\sigma(|\sigma|)}), 5\delta\epsilon) \ge \mathrm{m_d}(\bigcup_{\sigma\in\Omega_i(\epsilon)} \phi_\sigma(A_{\sigma(|\sigma|)})),$$

$$J_i(\epsilon)\epsilon^{\alpha s} \ge \frac{\epsilon^{-d}\mathrm{m_d}(\bigcup_{\sigma\in\Omega_i(\epsilon)}\phi_\sigma(A_{\sigma(|\sigma|)})}{\sum_{\sigma\in\Omega_i(\epsilon)} c_\sigma^{s(1-\alpha)}}(\sum_{\sigma\in\Omega_i(\epsilon)} c_\sigma^{s(1-\alpha)})\delta^{-d}\mathrm{const}.\epsilon^{\alpha s}.$$

Now let $(m_1, \cdots, m_n)$ be the strictly positive right eigenvector responding to the eigenvalue 1. Then

$$(c_{ij}^s a_{ij})_{n\times n} \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}.$$

Therefore

$$\left[\frac{\min m_i}{\max m_i}\right]^2 \le \sum_{\sigma\in\Omega_i(\epsilon)} c_\sigma^s \le \left[\frac{\max m_i}{\min m_i}\right]^2.$$

In addition

$$1 \le (\epsilon c_\sigma^{-1})^{\alpha s} \le c_0^{-\alpha s}.$$

Therefore

$$J_i(\epsilon)\epsilon^{\alpha s} \ge \frac{\epsilon^{-d} m_d(\bigcup_{\sigma\in\Omega_i(\epsilon)} \phi_\sigma(A_{\sigma(|\sigma|)})}{\sum_{\sigma\in\Omega_i(\epsilon)} c_\sigma^{s(1-\alpha)}} \delta^{-d}\text{const.} \tag{13}$$

If $\alpha_1 = 0$, it is trival. We assume $\alpha_1 > 0$ and take $0 < \alpha < \alpha_1$. Thus we have

$$\underline{\lim}_{\epsilon\to 0} J_i(\epsilon)\epsilon^{\alpha s} = \infty,$$

by (13) for $1 \le i \le n$. Take $\epsilon_1 > 0$ such that $J_i(\epsilon_1)\epsilon_1^{\alpha s} c_0^{\alpha s} \ge 2$ for all $1 \le i \le n$. Considering the arbitrary finite open $c_0\epsilon_1\delta$-covering $\{V_i\}$ of F_j, thus

(a) if there exists some V_i with $|V_i| \ge (c_0\epsilon_1)^2\delta$ then

$$\sum_i |V_i|^{\alpha s} \ge (c_0\epsilon_1)^{2\alpha s}\delta^{\alpha s}; \tag{14}$$

(b) otherwise we have

$$\sum_i |V_i|^{\alpha s} = \epsilon_1^{\alpha s} \sum_i |\epsilon_1^{-1} V_i|^{\alpha s}.$$

For each $\sigma \in \Omega_j^*(\epsilon_1)$, let $\mathcal{V}_\sigma = \{V_i :\ V_i \cap B(\phi_\sigma(x_{\sigma(|\sigma|)}), \epsilon_1\delta) \ne \emptyset\}$. Thus $\mathcal{V}_\sigma$ is a covering of $\phi_\sigma(F_{\sigma(|\sigma|)})$ and for any $\sigma, \tau \in \Omega_j^*(\epsilon_1)$, $\sigma \ne \tau$, we have $\mathcal{V}_\sigma \cap \mathcal{V}_\tau = \emptyset$. Take $\lambda_1 \in \Omega_j^*(\epsilon_1)$ such that

$$\sum_{V_i\in\mathcal{V}_{\lambda_1}} |V_i|^{\alpha s} = \min_{\sigma\in\Omega_j^*(\epsilon_1)} \sum_{V_i\in\mathcal{V}_\sigma} |V_i|^{\alpha s}.$$

Therefore

$$\begin{aligned}\sum_i |V_i|^{\alpha s} &\ge J(\epsilon_1) \sum_{V_i\in\mathcal{V}_{\lambda_1}} |V_i|^{\alpha s} \ge J(\epsilon_1)\epsilon_1^{\alpha s} \sum_{V_i\in\mathcal{V}_{\lambda_1}} |\epsilon_1^{-1} V_i|^{\alpha s} \ge 2c_0^{-\alpha s} \sum_{V_i\in\mathcal{V}_{\lambda_1}} |\epsilon_1^{-1} V_i|^{\alpha s} \\ &= 2(c_{\lambda_1} c_0^{-1}\epsilon_1^{-1})^{\alpha s} \sum_{V_i\in\mathcal{V}_{\lambda_1}} |\phi_{\lambda_1}^{-1} V_i|^{\alpha s} \ge 2 \sum_{V_i\in\mathcal{V}_{\lambda_1}} |\phi_{\lambda_1}^{-1} V_i|^{\alpha s}. \end{aligned}\tag{15}$$

Since $\mathcal{V}_{\lambda_1}$ is a covering of $\phi_{\lambda_1}(F_{\lambda_1(|\lambda_1|)})$, $\phi_{\lambda_1}^{-1}\mathcal{V}_{\lambda_1}$ is a finite open $c_0\epsilon_1\delta$-covering of $F_{\lambda_1(|\lambda_1|)}$. Denoting $\phi_{\lambda_1}^{-1}\mathcal{V}_{\lambda_1}$ by $\{u_i\}$ as above we have

(a') if there exists $u_i \in \phi_{\lambda_1}^{-1}\mathcal{V}_{\lambda_1}$ such that $|u_i| \ge (c_0\epsilon_1)^2\delta$ then (14) holds by (15).

(b') otherwise repeating the above step and considering Card$\{V_i\}$ finite, thus (14) holds after finite steps. Therefore $\dim_H F_j \ge \alpha s$ for any $0 < \alpha < \alpha_1$ which means $\dim_H F_j \ge \alpha_1 s$.

Similar to the proof of Theorem 2.3 we also get $\alpha_1 s \le \dim_H F_j \le \underline{\dim}_B F_j \le \alpha_1 s$ and $\beta_1 s \le \dim_H F_j \le \overline{\dim}_B F_j \le \beta_1 s$ and $\overline{\dim}_B F_j = \beta_n s$. Thus we complete the proof. In addition it is easy to find that all α_i's and β_i's are equal and independent of the choice of A_i' s.

(II) Finally using the same method as those in proof of Theorem 2.3 (III) and (IV) we can complete the proof of (II). **QED**

Corollary 3.4 *When* $\{\phi_{ij} : a_{ij} = 1\}$ *satisfies the open set condition, we have for every* $0 \le i \le n$, $\dim_H F_i = \dim_B F_i = \dim_P F_i = s$. **QED**

Conjecture: if

$$\underline{\lim}_{\epsilon\to 0} \frac{\epsilon^{-d} \mathrm{m}_d(\bigcup_{\sigma\in\Omega_i(\epsilon)} \phi_\sigma(A_{\sigma(|\sigma|)}))}{\sum_{\sigma\in\Omega_i(\epsilon)} c_\sigma^{s(1-\alpha)}} > 0$$

for some $1 \le i \le n$, then $\mathcal{H}^{\alpha s}(F_i) > 0$ for all $1 \le i \le n$.

Remark 3.5 **(1)** Since the recurrent set (Dekking [2]) and the generalized recurrent set (Li [8]) are all the special cases of MW-construction (Bedford [1] & Li [7]) the Theorem 3.3 also works there. Thus our Theomem 3.3 actually improves the main results of [11] [12] which discussed the lower bound of Hausdorff dimension of recurrent sets and self-similar sets.

(2) If the above conjecture is ture, it is easy to get

(a) F_i is an αs-set for some $1 \le i \le n$ iff for some $1 \le i \le n$

$$0 < \underline{\lim}_{\epsilon\to 0} \frac{\epsilon^{-d} \mathrm{m}_d(\bigcup_{\sigma\in\Omega_i(\epsilon)} \phi_\sigma(A_{\sigma(|\sigma|)}))}{\sum_{\sigma\in\Omega_i(\epsilon)} c_\sigma^{s(1-\alpha)}} < \infty;$$

(b) F_i's satisfy the open set condition iff for some $1 \le i \le n$

$$\underline{\lim}_{\epsilon\to 0} \frac{\epsilon^{-d} \mathrm{m}_d(\bigcup_{\sigma\in\Omega_i(\epsilon)} \phi_\sigma(A_{\sigma(|\sigma|)}))}{\mathrm{Card}\Omega_i(\epsilon)} > 0.$$

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References

[1] Bedford , T., *Dimension and dynamics of fractal recurrent set.* J. London Math.Soc. ,**33** (1986), 89–100.

[2] Dekking,F.M., *Recurrent sets.* Advances in Math., **44** (1982), 78–104.

[3] Falconer,K., *Fractal geometry – Mathematical foundation and applications.* Chichester: John Wiley & Sopens Ltd., 1990.

[4] Falconer,K., *Dimensions and measures of quasi self-similar sets.* Proc. Amer. Math. Soc.,**106**(1989), 543–554.

[5] Hutchinson, J.E., *Fractals and self-similarity*, Indiana Univ. Math. J., **30**(1981), 713–747.

[6] Li, W.X., *Separation properties for MW-fractals.* to appear in Acta Mathematica Sinica, **41**(1998).

[7] Li, W.X., *Researches on a class of partially-self-similar sets.* Chinese Ann. of Math., **15A**(1994), 681–688.

[8] Li, W.X., *Generalized recurrent set*. Acta Math. Sinica. **39**(1996) 78–88.

[9] Mauldin, R.D.& William, S.C., *Hausdorff dimension in graph directed constructions*. Trans. Amer. Math. Soc. **309**(1988) 811–829.

[10] Schief, A., *Separation properties for self-similar sets* . Proc. Amer. Math. Soc. **122**(1994), 111–115.

[11] Wen, Z.Y., Wu, L.M.,Zhong, H.L., *A note on recurrent sets*. Chin. Ann. of Math. **15B**(1994), 321–326.

[12] Wu, M., *The Hausdorff dimensions of several kinds of set*. Doctoral dissertation, Wuhan University. China, 1993.