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POINTS OF INFINITE DERIVATIVE OF CANTOR FUNCTIONS

Abstract

We consider self-similar Borel probability measures μ on a self-similar set E with strong separation property. We prove that for any such measure μ the derivative of its distribution function $F(x)$ is infinite for μ-a.e. $x \in E$, and so the set of points at which $F(x)$ has no derivative, finite or infinite is of μ-zero.

1 Introduction.

Let $E \subset \mathbb{R}$ be a Borel set, let μ be a finite, atomless Borel measure on E. For $0 < c < \infty$, set

$$Q_c^u = \left\{ x \in E : \limsup_{r \to 0+} \frac{\mu([x-r, x+r])}{r} \le c \right\},$$

and

$$Q_c^l = \left\{ x \in E : \liminf_{r \to 0+} \frac{\mu([x-r, x+r])}{r} \le c \right\}.$$

Then a classical result (ref. proposition 2.2 (a) and (c) in [4]) shows that $\mu(Q_c^u) \le c\mathcal{H}^1(Q_c^u)$ and $\mu(Q_c^l) \le c\mathcal{P}^1(Q_c^l)$, where $\mathcal{H}^1(\cdot)$ and $\mathcal{P}^1(\cdot)$ are, respectively, the one-dimensional Hausdorff and packing measures. Therefore, if both $\dim_H E$ and $\dim_P E$ are less than 1, then for μ-a.e. $x \in E$,

$$\limsup_{r \to 0+} \mu([x-r, x+r])/r = +\infty \text{ and } \liminf_{r \to 0+} \mu([x-r, x+r])/r = +\infty. \quad (1)$$

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The first equality in (1) implies that for μ-a.e. $x \in E$,

$$\max\left\{\limsup_{r\to 0+}\frac{\mu[x-r,x]}{r},\ \limsup_{r\to 0+}\frac{\mu[x,x+r]}{r}\right\} = +\infty.$$

It shows that the distribution function of μ has infinite upper derivatives μ almost everywhere. However, the second equality in (1) provides less information about its lower derivatives which for $x \in \mathbb{R}$ equal

$$\min\left\{\liminf_{r\to 0+}\frac{\mu[x-r,x]}{r},\ \liminf_{r\to 0+}\frac{\mu[x,x+r]}{r}\right\}.$$

In the following, we consider E as a class of self-similar sets, and μ as the self-similar measures on E. In the present paper, we show that their distribution functions have infinite derivatives for μ-a.e. $x \in E$.

A *self-similar set* E in $\mathbb{R}$ is defined as the unique nonempty compact set invariant under h_j's:

$$E = \bigcup_{j=0}^{r} h_j(E), \tag{2}$$

where $h_j(x) = a_j x + b_j$, $j = 0, 1, \ldots, r$, with $0 < a_j < 1$ and $r \geq 1$ being a positive integer. Without loss of generality, we shall assume that $b_0 = 0$ and $a_r + b_r = 1$. We furthermore assume that the images $h_j([0,1])$, $j = 0, 1, \ldots, r$ are pairwise disjoint (i.e., E satisfies the strong separation property) and are ordered from left to right. We remark that this assumption implies that the h_j's satisfy the open set condition with the open set $(0,1)$, which is less general than the usual one defined by [6]. It is well-known that $\dim_H E = \dim_B E = \dim_P E = \xi \in (0,1)$ and $0 < \mathcal{H}^{\xi}(E) < \mathcal{P}^{\xi}(E) < +\infty$ where ξ is given by $\sum_{j=0}^{r} a_j^{\xi} = 1$ (ref. [6]).

As usual, the elements of E in (2) can be encoded by digits in $\Omega = \{0, 1, \ldots, r\}$ as follows. We write $\Omega^{\mathbb{N}} = \{\sigma = (\sigma(1), \sigma(2), \ldots) : \sigma(j) \in \Omega\}$ and $\Omega^* = \bigcup_{k=1}^{\infty} \Omega^k$ with $\Omega^k = \{\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(k)) : \sigma(j) \in \Omega\}$ for $k \in \mathbb{N}$. $|\sigma|$ is used to denote the length of the word $\sigma \in \Omega^*$. For any $\sigma, \tau \in \Omega^*$, write $\sigma * \tau = (\sigma(1), \ldots, \sigma(|\sigma|), \tau(1), \ldots, \tau(|\tau|))$, and write $\tau * \sigma = (\tau(1), \ldots, \tau(|\tau|), \sigma(1), \sigma(2), \ldots)$ for any $\tau \in \Omega^*$, $\sigma \in \Omega^{\mathbb{N}}$. $\sigma|k = (\sigma(1), \sigma(2), \ldots, \sigma(k))$ for $\sigma \in \Omega^{\mathbb{N}}$ and $k \in \mathbb{N}$. Let $h_\sigma(x) = h_{\sigma(1)} \circ \cdots \circ h_{\sigma(k)}(x)$ for $\sigma \in \Omega^k$ and $x \in \mathbb{R}$. Then for $\sigma \in \Omega^k$, the intervals $h_{\sigma*0}([0,1])$, $h_{\sigma*1}([0,1]), \ldots, h_{\sigma*r}([0,1])$ are contained in $h_\sigma([0,1])$ in this order where the left endpoint of $h_{\sigma*0}([0,1])$ coincides with the left endpoint of $h_\sigma([0,1])$, and the right endpoint of $h_{\sigma*r}([0,1])$ coincides with the right endpoint of $h_\sigma([0,1])$. Moreover, the length of the interval $h_\sigma([0,1])$ equals $\lambda(h_\sigma([0,1])) = \prod_{j=1}^{k} a_{\sigma(j)}$ $=: a_\sigma$ for $\sigma \in \Omega^k$, where $\lambda(\cdot)$ denotes the one-dimensional Lebesgue measure.

For $j = 1, 2, \ldots$, let $E_j = \cup_{\sigma \in \Omega^j} h_\sigma([0,1])$. Then $E_j \downarrow E$ as $j \to \infty$ and $x \in E$ can be encoded by a unique $\sigma \in \Omega^{\mathbb{N}}$ satisfying

$$\{x\} = \bigcap_{k=1}^{\infty} h_{\sigma|k}([0,1]).$$

Throughout this paper we sometimes denote this unique code of x by $\tilde{x}$ and use $x(k)$ to denote the k-th component of $\tilde{x}$; i.e., use $\tilde{x} = (x(1), x(2), \ldots)$ for the code of $x \in E$. In this way one can establish a continuous one-to-one correspondence between $\Omega^{\mathbb{N}}$ and E. The endpoints of $h_\sigma([0,1])$ for a $\sigma \in \Omega^*$ will be called the endpoints of E. So the set of endpoints of E is countable. Obviously, any endpoint e of E lies in E and except for a finite number of terms, its coding $\tilde{e}$ consists of either only the symbol 0 if e is the left endpoint of some $h_\sigma([0,1])$, or only the symbol r if e is the right endpoint of some $h_\sigma([0,1])$.

Let μ be a *self-similar Borel probability measure* on E corresponding to the probability vector $(p_0, p_1, \ldots, p_r)$, where each $p_i > 0$ and $\sum_{i=0}^r p_i = 1$; i.e., the measure satisfying

$$\mu(A) = \sum_{j=0}^{r} p_j \mu(h_j^{-1}(A)) \ \text{ for any Borel set } A,$$

and so

$$\mu(h_\sigma([0,1])) = \prod_{j=1}^{k} p_{\sigma(j)} =: p_\sigma, \ \text{ for any } \sigma \in \Omega^k, \ k \in \mathbb{N}. \tag{3}$$

Obviously, μ is atomless. Consider the distribution function of such a probability measure μ, also called *Cantor function* or a self-affine 'devil's staircase' function,

$$F(x) = \mu([0,x]), \ x \in [0,1]. \tag{4}$$

Then $F(x)$ is a non-decreasing continuous function with $F(0) < F(1)$; that is, constant off the support of μ. Obviously, the derivative of $F(x)$ is zero for each $x \in [0,1] \setminus E$. In particular, the set S of *points of non-differentiability* of $F(x)$; that is, those x where

$$\lim_{\delta \to 0} \frac{F(x+\delta) - F(x)}{\delta} = \lim_{\delta \to 0} \frac{\mu((x, x+\delta])}{\delta} \ \left(\text{or } \frac{\mu((x+\delta, x])}{-\delta} \text{ if } \delta < 0 \right)$$

does not exist either as a finite number or ∞, has Lebesgue measure 0. The Hausdorff dimension of S has been obtained (ref. [1, 2, 3, 5] for the case $p_i = a_i^\xi$, [8] for the case $p_i = a_i (\sum_{i=0}^r a_i)^{-1}$ and [7] for the case $p_i > a_i$). Let

$$E^* = E \setminus \{\text{endpoints of } E\},$$

and

$$T = \left\{ t \in E^* : \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log \frac{p_{t(i)}}{a_{t(i)}} = \sum_{i=0}^{r} p_i(\log p_i - \log a_i) \right\}. \tag{5}$$

Then $\mu(T) = 1$ by the law of large numbers. We decompose the set S into

$$S = N^+ \cup N^- \cup Z,$$

where $N^+(N^-)$ is the set of points in E^* at which the right (left) derivative of $F(x)$ doesn't exist, finite or infinite, Z is a subset of the set of endpoints of E, so at most countable. In the present paper, we prove the following theorem.

Theorem 1.1. *Let $(p_0, p_1, \ldots, p_r)$ be an arbitrarily given probability vector. Let μ and $F(x)$ be determined by* (3) *and* (4) *respectively. Then $F'(x) = +\infty$ for μ-a.e. $x \in E$. So $\mu(S) = 0$.*

2 Proofs.

In this section, we first prove in the following Proposition 2.1 that $F(x)$ has infinite upper derivatives for μ-a.e. $x \in E$ (although it can be obtained directly from (1)) by showing that both of the upper right and the upper left derivatives of $F(x)$ are infinite for each $x \in T$. Then the set $T \cap N^+$ $(T \cap N^-)$ consists of those points of T at which $F(x)$ has finite lower right (left) derivatives by the definition of N^+ (N^-). We characterize $T \cap N^+$ $(T \cap N^-)$ by the coding property of its elements in Lemma 2.2. Theorem 1.1 then is proved by showing that $\mu(T \cap N^+) = 0$ $(\mu(T \cap N^-) = 0)$.

Proposition 2.1. *Both the upper right and the upper left derivatives of $F(x)$ are infinite for each $x \in T$.*

PROOF. Let $t \in T$ with code $\tilde{t} = (t(1), t(2), \ldots)$. Then $\tilde{t}$ has infinitely many entries lying in $\Omega \setminus \{r\}$. Suppose $\tilde{t}$ has an entry from $\Omega \setminus \{r\}$ in position j. Then t lies in the interval $h_{\tilde{t}|(j-1)}([0,1])$, but is not equal to the right endpoint u of $h_{\tilde{t}|(j-1)}([0,1])$, where $\tilde{u} = (t(1), \ldots, t(j-1), r, r, \ldots)$. Note that u is also the right endpoint of $h_{\tilde{u}|j}([0,1])$ and that $t \notin h_{\tilde{u}|j}([0,1])$. Thus we have that $t, u \in h_{\tilde{t}|(j-1)}([0,1])$ and $(t, u] \supseteq h_{\tilde{u}|j}([0,1])$. Consider the slope of the line segment from the point $P = (t, F(t))$ on the graph of $F(x)$ to the point $Q = (u, F(u))$. We have

$$\begin{aligned} \frac{F(u)-F(t)}{u-t} &= \frac{\mu((t,u])}{u-t} \geq \frac{\mu(h_{\tilde{u}|j}([0,1]))}{|h_{\tilde{t}|(j-1)}([0,1])|} = \frac{p_{\tilde{t}|(j-1)}p_r}{a_{\tilde{t}|(j-1)}} \\ &= p_r \exp\left((j-1)\frac{1}{j-1}\sum_{i=1}^{j-1}\log\frac{p_{t(i)}}{a_{t(i)}}\right). \end{aligned} \tag{6}$$

Note that by corollary 1.5 in [4],

$$\sum_{i=0}^{r} p_i(\log p_i - \log a_i) \geq -\log\sum_{j=0}^{r} a_j > 0.$$

Thus, the upper right derivative of $F(x)$ at t is infinite by (6) and (5) when $j \to +\infty$. Symmetrically, the upper left derivative of $F(x)$ at t of E is also infinite. □

Lemma 2.2. *Let $\Gamma = \{0, 1, \ldots, r-1\}$. Let $t \in E^*$ and let $z(t,n)$ denote the position of the n-th occurrence of elements of Γ in $\tilde{t}$. Then*

(I) $T \cap N^+ \subseteq T \cap \left\{t \in E^* : \limsup\limits_{n\to\infty} \frac{z(t,n+1)}{z(t,n)} \geq 1 - \frac{1}{\log p_r}\sum\limits_{i=0}^{r} p_i(\log p_i - \log a_i)\right\}$;

(II) $T \cap \left\{t \in E^* : \limsup_{n\to\infty} \frac{z(t,n+1)}{z(t,n)} > 1 - \frac{1}{\log p_r}\sum_{i=0}^{r} p_i(\log p_i - \log a_i)\right\} \subseteq T \cap N^+$.

Symmetrically, if we replace Γ by $\{1, 2, \ldots, r\}$, then

(I') $T \cap N^- \subseteq T \cap \left\{t \in E^* : \limsup\limits_{n\to\infty} \frac{z(t,n+1)}{z(t,n)} \geq 1 - \frac{1}{\log p_0}\sum\limits_{i=0}^{r} p_i(\log p_i - \log a_i)\right\}$;

(II') $T \cap \left\{t \in E^* : \limsup_{n\to\infty} \frac{z(t,n+1)}{z(t,n)} > 1 - \frac{1}{\log p_0}\sum_{i=0}^{r} p_i(\log p_i - \log a_i)\right\} \subseteq T \cap N^-$.

PROOF. We first prove statement (I); i.e., the lower-right derivative of $F(x)$ is infinite at $t \in T$ when

$$\limsup_{n\to\infty} \frac{z(t,n+1)}{z(t,n)} < 1 - \frac{1}{\log p_r}\sum_{i=0}^{r} p_i(\log p_i - \log a_i). \tag{7}$$

Consider such a point t with $\tilde{t} = (t(1), t(2), \ldots)$. By (7) and (5) let k be a positive integer such that for $n \geq k$

$$\frac{z(t,n+1)}{z(t,n)} < 1 - \frac{1}{\log p_r}\sum_{i=0}^{r} p_i(\log p_i - \log a_i) + 2q, \tag{8}$$

and

$$\frac{1}{z(t,n)}\sum_{i=1}^{z(t,n)}\log\frac{p_{t(i)}}{a_{t(i)}} > \sum_{i=0}^{r} p_i(\log p_i - \log a_i) - q\log p_r, \tag{9}$$

for some negative real number q. Let u be a positive number such that u is smaller than the distance between t and $[0,1]\setminus E_{\tilde{t}|l}$ with $l = z(t,k)$. Let x be a point in the segment $(t, t+u)$. Then $t, x \in h_{\tilde{t}|l}([0,1])$. We will see that $(F(x)-F(t))/(x-t)$ is large relative to k, so t is not in N^+. Let i denote the level at which $x \notin h_{\tilde{t}|i}([0,1])$ but $x \in h_{\tilde{t}|(i-1)}([0,1])$. Note also that $t \in h_{\tilde{t}|(i-1)}([0,1])$. Thus $x - t \le |h_{\tilde{t}|(i-1)}([0,1])| = a_{\tilde{t}|(i-1)}$; also $i = z(t,n)$ for some $n > k$. Put $j = z(t,n+1)-1$, and by v we denote the right endpoint of $h_{\tilde{t}|j}([0,1])$, which implies that $\tilde{v} = (t(1),\ldots,t(j),r,r,\ldots)$ and $(t,v] \supseteq h_{\tilde{v}|(j+1)}([0,1])$. Then we have $t < v < x$ and $F(v)-F(t) = \mu((t,v]) \ge \mu(h_{\tilde{v}|(j+1)}([0,1])) = p_{\tilde{t}|j}p_r$. Therefore, we have

$$\begin{aligned}\frac{F(x)-F(t)}{x-t} &\ge \frac{p_{\tilde{t}|j}p_r}{a_{\tilde{t}|(i-1)}} = \frac{p_r\prod_{m=1}^{z(t,n+1)-1}p_{t(m)}}{\prod_{m=1}^{z(t,n)-1}a_{t(m)}} \\ &= a_{t(z(t,n))}p_r^{z(t,n+1)-z(t,n)}\prod_{m=1}^{z(t,n)}\frac{p_{t(m)}}{a_{t(m)}} \\ &\ge \left(\min_{0\le m\le r} a_m\right)\left[p_r^{\frac{z(t,n+1)}{z(t,n)}-1}\left(\prod_{m=1}^{z(t,n)}\frac{p_{t(m)}}{a_{t(m)}}\right)^{\frac{1}{z(t,n)}}\right]^{z(t,n)}.\end{aligned} \tag{10}$$

Let

$$Q = p_r^{\frac{z(t,n+1)}{z(t,n)}-1}\left(\prod_{m=1}^{z(t,n)}\frac{p_{t(m)}}{a_{t(m)}}\right)^{\frac{1}{z(t,n)}}.$$

Taking logs, and by (8) and (9), we have

$$\log Q = \left(\frac{z(t,n+1)}{z(t,n)} - 1\right)\log p_r + \frac{1}{z(t,n)}\sum_{m=1}^{z(t,n)}\log\frac{p_{t(m)}}{a_{t(m)}} > q\log p_r. \tag{11}$$

Since t is a non-end point, $z(t,n)\to\infty$ and the lower-right derivative of $F(x)$ is infinite at t by (10) and (11).

Now we turn to the proof of statement (II). Let $t \in T$ be such that

$$\limsup_{n\to\infty}\frac{z(t,n+1)}{z(t,n)} > 1 - \frac{1}{\log p_r}\sum_{i=0}^{r}p_i(\log p_i - \log a_i).$$

Then there exists a sequence $\{n_k\}$ of positive integers such that for some positive constant c,

$$\frac{z(t,n_k+1)}{z(t,n_k)} > 1 - \frac{1}{\log p_r} \sum_{i=0}^{r} p_i(\log p_i - \log a_i) + 2c, \tag{12}$$

and in addition by (5),

$$\frac{1}{z(t,n_k)} \sum_{i=1}^{z(t,n_k)} \log \frac{p_{t(i)}}{a_{t(i)}} < \sum_{i=0}^{r} p_i(\log p_i - \log a_i) - c \log p_r. \tag{13}$$

Let x_k be the left endpoint of $h_{(\tilde{t}|j_k)*(t(j_k+1)+1)}([0,1])$, where $j_k = z(t,n_k) - 1$. Thus we have $\tilde{x}_k = (t(1),\dots,t(j_k),t(j_k+1)+1,0,\dots,0,\dots)$. Let u_k be the right endpoint of $h_{\tilde{t}|(j_k+1)}([0,1])$. Then $\tilde{u}_k = (t(1),\dots,t(j_k),t(j_k+1),r,r,r,\dots)$. Thus, (u_k,x_k) is the gap on the right side of $h_{\tilde{t}|(j_k+1)}([0,1])$ and $\lambda([u_k,x_k]) = x_k - u_k = a_{\tilde{t}|j_k}\beta_{t(j_k+1)}$ where by β_j, $j = 0,1,\dots,r-1$, we denote length of the gap between images $h_j([0,1])$ and $h_{j+1}([0,1])$. Note that $[t,x_k] \supseteq [u_k,x_k]$ and $\mu((t,x_k]) = \mu((t,u_k]) + \mu((u_k,x_k]) = \mu((t,u_k]) \le \mu(h_{\tilde{t}|(z(t,n_k+1)-1)}([0,1]))$ since $\tilde{t}|(z(t,n_k+1)-1) = \tilde{u}_k|(z(t,n_k+1)-1)$. Therefore we have

$$F(x_k) - F(t) = \mu((t,x_k]) \le \mu(h_{\tilde{t}|(z(t,n_k+1)-1)}([0,1])) = p_{\tilde{t}|(z(t,n_k+1)-1)},$$

and

$$x_k - t \ge \lambda([u_k,x_k]) = a_{\tilde{t}|(z(t,n_k)-1)}\beta_{t(z(t,n_k))}.$$

Let $\beta_* = \min_{j\in\{0,1,\dots,r-1\}} \beta_j$ and $a^* = \max_{j\in\{0,1,\dots,r\}} a_j$. Then we obtain, by a similar reasoning which led to (10),

$$\begin{aligned}
\frac{F(x_k)-F(t)}{x_k - t} &\le \frac{p_{\tilde{t}|(z(t,n_k+1)-1)}}{a_{\tilde{t}|(z(t,n_k)-1)}\beta_{\tilde{t}(z(t,n_k))}} \\
&= \frac{a_{z(t,n_k)}}{\beta_{\tilde{t}(z(t,n_k))}p_r} p_r^{z(t,n_k+1)-z(t,n_k)} \prod_{i=1}^{z(t,n_k)} \frac{p_{t(i)}}{a_{t(i)}} \\
&\le \frac{a^*}{\beta_* p_r} \left(p_r^{\frac{z(t,n_k+1)}{z(t,n_k)}-1} \left(\prod_{i=1}^{z(t,n_k)} \frac{p_{t(i)}}{a_{t(i)}} \right)^{\frac{1}{z(t,n_k)}} \right)^{z(t,n_k)}.
\end{aligned} \tag{14}$$

Let

$$Q = p_r^{\frac{z(t,n_k+1)}{z(t,n_k)}-1} \left(\prod_{i=1}^{z(t,n_k)} \frac{p_{t(i)}}{a_{t(i)}} \right)^{\frac{1}{z(t,n_k)}}.$$

Taking logs and using (12) and (13), we obtain

$$\log Q = \left(\frac{z(t,n_k+1)}{z(t,n_k)} - 1\right)\log p_r + \frac{1}{z(t,n_k)}\sum_{i=1}^{z(t,n_k)} \log \frac{p_{t(i)}}{a_{t(i)}} < c\log p_r < 0. \quad (15)$$

From (14) and (15) it follows that the lower-right derivative of $F(x)$ at t is finite by letting $k \longrightarrow \infty$. Finally, (I') and (II') can be proved similarly. □

PROOF OF THEOREM 1.1. Since μ is atomless, we only need to prove that $\mu(N^+ \bigcap T) = \mu(N^- \bigcap T) = 0$. Below we prove $\mu(N^+ \bigcap T) = 0$; $\mu(N^- \bigcap T) = 0$ can be obtained in the same way. By lemma 2.2 (I), we have $N^+ \bigcap T \subseteq M$ where

$$M = \left\{ t \in T : \limsup_{n\to\infty} \frac{z(t,n+1)}{z(t,n)} \geq 1 - \frac{1}{\log p_r}\sum_{i=0}^{r} p_i(\log p_i - \log a_i)\right\}.$$

Now fix a positive real number

$$\alpha < -\frac{1}{\log p_r}\sum_{i=0}^{r} p_i(\log p_i - \log a_i). \quad (16)$$

Choose n^* large enough to assure that when $k \geq n^*$

$$-\frac{2\log k}{k\log p_r} < \frac{\alpha}{2} \text{ and } \frac{1}{k} < \frac{\alpha}{8}. \quad (17)$$

Now for each $k \geq n^*$, we can choose $u_k > k$ such that

$$\frac{u_k}{k} > 1 - \frac{1}{\log p_r}\sum_{i=0}^{r} p_i(\log p_i - \log a_i) - \frac{\alpha}{2}, \quad (18)$$

and

$$\frac{u_k - 1}{k} \leq 1 - \frac{1}{\log p_r}\sum_{i=0}^{r} p_i(\log p_i - \log a_i) - \frac{\alpha}{2}. \quad (19)$$

Then we have

$$\begin{aligned} 1 - \frac{1}{\log p_r}\sum_{i=0}^{r} p_i(\log p_i - \log a_i) - \frac{\alpha}{2} &< \frac{u_k}{k} \\ &< 1 - \frac{1}{\log p_r}\sum_{i=0}^{r} p_i(\log p_i - \log a_i) - \frac{\alpha}{4}, \end{aligned} \quad (20)$$

by (18), (19) and the second inequality in (17). Let

$$J_k = \{x \in E : x(i) = r \text{ for } k < i \leq u_k\}, \; k \geq n^*,$$

and

$$J^\infty = \limsup_{k \to \infty} J_k = \bigcap_{m=n^*}^{\infty} \bigcup_{k \geq m} J_k.$$

Now for each point $t \in M$, there exists a strictly increasing sequence $\{n_i, i \in \mathbb{N}\}$ of positive integers such that $z(t, n_1) \geq n^*$ and

$$\frac{z(t, n_i + 1)}{z(t, n_i)} > 1 - \frac{1}{\log p_r} \sum_{i=0}^{r} p_i(\log p_i - \log a_i) - \frac{\alpha}{4}. \tag{21}$$

Taking $k_i = z(t, n_i)$ and using (21) as well as the second inequality in (20), we have $z(t, n_i + 1) > u_{k_i}$, which implies that $t \in J_{k_i}$. Thus we have $M \subseteq J^\infty$. Note that for $k \geq n^*$ and by the first inequality in (17), (18) and (16),

$$\frac{u_k}{k} - 1 \geq -\frac{2 \log k}{k \log p_r}; \; i.e., \; p_r^{u_k - k} \leq k^{-2}. \tag{22}$$

Therefore for any $m \geq n^*$,

$$\mu(N^+ \cap T) \leq \mu(M) \leq \mu(\bigcup_{k \geq m} J_k) \leq \sum_{k \geq m} p_r^{u_k - k} \leq \sum_{k \geq m} k^{-2},$$

by (22). Finally, we obtain $\mu(N^+ \cap T) = 0$ by letting $m \to \infty$. □

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