How smooth is a devil’s staircase?

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Abstract

Let the Cantor set $C$ in $\mathbb{R}$ be defined by $C = \bigcup_{j=0}^{r} h_j(C)$ with a disjoint union, where the $h_j$’s are similitude mappings with ratios $0 < a_j < 1$. Let $\mu$ on $C$ be the self-similar probability measure corresponding to the probability vector $(a_0^\xi, a_1^\xi, \cdots, a_r^\xi)$, where $\xi = \text{dim}_H C$ is the Hausdorff dimension of $C$. Let $S$ be the set of points at which the probability distribution function $F(x)$ of $\mu$ has no derivative, finite or infinite. We prove that $\text{dim}_H S = (\text{dim}_H C)^2$ and $\text{dim}_P S = \text{dim}_B S = \text{dim}_H C$.

Key Words: Hausdorff dimension; Packing dimension; Non-differentiability; Cantor function; Cantor measure.

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1 Introduction

Let $h_i(x) = ax + i(1 - a), i = 0, 1$ with $x \in [0, 1]$ and $0 < a < \frac{1}{2}$. Then there exists a unique non-empty compact set $C \subset [0, 1]$ such that

$$C = h_0(C) \cup h_1(C).$$

It is well-known that the Hausdorff dimension of $C$ equals $\dim_H C = -\frac{\log 2}{\log a}$. Let $\mu$ be the uniform probability measure on $C$. Consider the distribution function which is often referred to as the Devil’s staircase (for $a = \frac{1}{3}$):

$$F(x) = \mu([0, x]), \quad x \in [0, 1].$$

It is easy to check that the derivative of $F(x)$ is zero for all $x \in [0, 1] \setminus C$ and the upper derivative of $F(x)$ is infinite on $C$. Let $S$ be the set of points at which $F(x)$ is not differentiable, i.e., the set of points in $C$ at which the lower derivative of $F(x)$ is finite. $S$ can be decomposed into

$$S = N^+ \cup N^- \cup \{t : t \text{ is an endpoint of } C\}, \tag{1}$$

where $N^+ (N^-)$ is the set of non-end points of $C$ at which the lower right (left) derivative of $F(x)$ is finite. Each $t \in C$ can be encoded by a unique $0 - 1$ sequence, denoted by $\tilde{t} = (t(1), t(2), \cdots)$, which satisfies $\{t\} = \cap_{n=1}^{\infty} h_{t(1)} \circ \cdots \circ h_{t(n)}([0, 1])$. Now let $z(t, n)$ denote the position of the $n$-th zero in $\tilde{t}$. The set $N^+$ (symmetrically for $N^-$) is characterized by R. Darst [1] as follows:

[a] if $t \in N^+$, then $\limsup_{n \to \infty} \frac{z(t, n+1)}{z(t, n)} \geq -\frac{\log a}{\log 2}$;

[b] if $\limsup_{n \to \infty} \frac{z(t, n+1)}{z(t, n)} > -\frac{\log a}{\log 2}$, then $t \in N^+$.

By means of the above [a] and [b] R. Darst [1] proves that

$$\dim_H S = \dim_H N^+ = \left[\frac{\log 2}{\log a}\right]^2 = (\dim_H C)^2.$$

It is not difficult to show (see [1]) that $\dim_H S = \dim_H N^+ = \left[\frac{\log(r+1)}{\log a}\right]^2 = (\dim_H C)^2$ still holds for a little bit more general Cantor set $C$ with $C = \bigcup_{j=0}^{r} h_j(C)$, where $h_j(x) = a x + (1 - a)\frac{j}{r}$, $j = 0, 1, \cdots, r$ and $0 < a < \frac{1}{r+1}$. 

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The Cantor sets $C$ described above are all homogeneous in the sense that all similitude mappings $h_j(x)$ have the same scaling factor $a$ and the gaps between the images $h_j([0,1])$, $j = 0, 1, \cdots, r$, have the same length. In the following, let $r$ be a positive integer and let the Cantor set $C$ in $\mathbb{R}$ be defined by

$$C = \bigcup_{j=0}^{r} h_j(C),$$

where $h_j(x) = a_j x + b_j$, $j = 0, 1, \cdots, r$, with $0 < a_j < 1$. Without loss of generality we shall assume that $b_0 = 0$ and $a_r + b_r = 1$. We furthermore assume that the images $h_j([0,1])$, $j = 0, 1, \cdots, r$ are pairwise disjoint and are lined up from left to right. In this paper, we will determine the Hausdorff, box and packing dimensions of the set of non-differentiability points of the distribution function $F(x) = \mu([0, x])$ of the self-similar probability measure $\mu$ associated to the mappings $(h_j)_{j=0}^{r}$.

In order to encode the elements of $C$, we introduce some notations. Let $\Omega = \{0, 1, \cdots, r\}$. We will write

(i) $\Omega^{\omega} = \{\sigma = (\sigma(1), \sigma(2), \cdots) : \sigma(j) \in \Omega\}$;

(ii) $\Omega^k = \{\sigma = (\sigma(1), \sigma(2), \cdots, \sigma(k)) : \sigma(j) \in \Omega\}$ for $k \in \mathbb{N}$ and $\Omega^* = \bigcup_{k=1}^{\infty} \Omega^k$;

(iii) $| \cdot |$ is used to denote the length of word. For any $\sigma, \tau \in \Omega^*$ write $\sigma * \tau = (\sigma(1), \cdots, \sigma(|\sigma|), \tau(1), \cdots, \tau(|\tau|))$, and write $\tau * \sigma = (\tau(1), \cdots, \tau(|\tau|), \sigma(1), \sigma(2), \cdots)$ for any $\tau \in \Omega^*$, $\sigma \in \Omega^\omega$;

(iv) $\sigma|k = (\sigma(1), \sigma(2), \cdots, \sigma(k))$ for $\sigma \in \Omega^\omega$ and $k \in \mathbb{N}$.

Denote $h_\sigma(x) = h_{\sigma(1)} \circ \cdots \circ h_{\sigma(k)}(x)$ for $\sigma \in \Omega^k$ and $x \in \mathbb{R}$. Then for $\sigma \in \Omega^k$, the intervals $h_{\sigma*0}([0,1]), h_{\sigma*1}([0,1]), \cdots, h_{\sigma*r}([0,1])$ are contained in $h_\sigma([0,1])$ in this order where the left endpoint of $h_{\sigma*0}([0,1])$ coincides with the left endpoint of $h_\sigma([0,1])$, and the right endpoint of $h_{\sigma*r}([0,1])$ coincides with the right endpoint of $h_\sigma([0,1])$. Moreover the length of interval $h_\sigma([0,1])$ equals $\lambda(h_\sigma([0,1])) = \prod_{j=1}^{k} a_{\sigma(j)} =: a_\sigma$ for $\sigma \in \Omega^k$. For $j = 1, 2, \cdots$, we define

$$C_j = \bigcup_{\sigma \in \Omega^j} h_\sigma([0,1]).$$

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Then $C_j \downarrow C$ as $j \to \infty$ and $x \in C$ can be encoded by a unique $\sigma \in \Omega^\omega$ satisfying $
{\{x\} = \bigcap_{k=1}^\infty h_{\sigma^k}([0,1])}$. We usually denote this unique code of $x$ by $\tilde{x}$ and use $x(k)$ to denote the $k$-th component of $\tilde{x}$.

It is well-known that $\dim_H C = \dim_B C = \dim_P C = \xi$ where $\xi$ is given by

$$
\sum_{j=0}^r a_{\xi_j} = 1. \tag{2}
$$

Let $\mu$ be the self-similar probability measure on $C$ corresponding to the probability vector $(a_{\xi_0}, a_{\xi_1}, \ldots, a_{\xi_r})$, i.e., the measure satisfying

$$
\mu(h_{\sigma}([0,1])) = \prod_{j=1}^k a_{\sigma(j)}^\xi = a_{\sigma}^\xi, \quad \text{for any } \sigma \in \Omega^k, \ k \in \mathbb{N}.
$$

Consider the distribution function of the probability measure $\mu$, also called Cantor function:

$$
F(x) = \mu([0,x]), \quad x \in [0,1].
$$

Figure 1: *The graph of $F(x)$ for the case $a_0 = 0.5, b_0 = 0, a_1 = 0.2$ and $b_1 = 0.8$.*

It is easy to check that the derivative of $F(x)$ is zero for all $x \in [0,1] \setminus C$. We will show that the upper derivative of $F(x)$ is infinite on $C$. Let $S$ be the set of points at which $F(x)$ is not differentiable, i.e., the set of points in $C$ at which the lower derivative of $F(x)$ is finite. The set $S$ can be decomposed in the same way as in (1). The endpoints of
For any $e$ of $C$ lies in $C$ and except for a finite number of terms, its coding $\tilde{e}$ consists of either only the symbol 0 if $e$ is the left endpoint of some $h_\sigma([0,1])$, or only the symbol $r$ if $e$ is the right endpoint of some $h_\sigma([0,1])$.

In this paper, we will prove that $\dim_H S = \dim_H N^+ = \dim_H N^- = (\dim_H C)^2 = \xi^2$ and $\dim_B S = \dim_P S = \dim_H C = \xi$.

2 Codes of non-differentiability points

In this section, we characterize the set $S$ of non-differentiability points by means of the behavior of their codings. We focus on $N^+$. Results on $N^-$ can be obtained symmetrically.

Proposition 2.1 The upper derivative of $F(x)$ is infinite for all $x \in C$.

Proof For any $t$ in $C$, $t$ not a right endpoint, let the code be $\tilde{t} = (t(1), t(2), \cdots)$. Then $\tilde{t}$ has infinitely many entries lying in $\Omega \setminus \{r\}$. Suppose $\tilde{t}$ has a entry from $\Omega \setminus \{r\}$ in position $j$. Then $t$ lies in the interval $h_{\tilde{t}(j-1)}([0,1])$ but is not equal to the right endpoint $u$ of $h_{\tilde{t}(j-1)}([0,1])$, where

$$
\tilde{u} = (t(1), \cdots, t(j-1), r, r, \cdots).
$$

Note that $u$ is also the right endpoint of $h_{\tilde{u}(j)}([0,1])$ and that $t \notin h_{\tilde{u}(j)}([0,1])$. Thus we have that $t, u \in h_{\tilde{u}(j-1)}([0,1])$ and $(t, u) \supseteq h_{\tilde{u}(j)}([0,1])$. Consider the slope of the line segment from the point $P = (t, F(t))$ on the graph of $F$ to the point $Q = (u, F(u))$. We have

$$
\frac{F(u) - F(t)}{u - t} = \frac{\mu((t, u])}{u - t} \geq \frac{\mu(h_{\tilde{u}(j)}([0,1]))}{|h_{\tilde{u}(j-1)}([0,1])|} = \frac{a_{\tilde{u}(j-1)}^\xi a_r^\xi}{a_{\tilde{u}(j-1)}} = \frac{a_r^\xi (a_{\tilde{u}(j-1)})^{\xi-1}}{a_{\tilde{u}(j-1)}} \to \infty \text{ as } j \to \infty.
$$

Symmetrically the upper left derivative of $t$ at a non-left-end point of $C$ is infinite. QED

Proposition 2.2 Let $\Gamma = \{0, 1, \cdots, r - 1\}$. Let $a = \min_{j \in \Gamma} a_j$ and $\bar{a} = \max_{j \in \Gamma} a_j$. Let $t \in C$ be not an endpoint of $C$ and let $z(t, n)$ denote the position of the $n$-th occurrence of elements of $\Gamma$ in $\tilde{t}$, then
Therefore, defining \( i = \frac{n}{z(t, n)}(1 - \xi^{-1}) \left( \frac{\log a}{\log ar} - 1 \right) + \frac{z(t, n + 1)}{z(t, n)} - \xi^{-1} \geq 0; \)

(II) if \( \lim \sup_{n \to \infty} \left[ \frac{n}{z(t, n)}(1 - \xi^{-1}) \left( \frac{\log a}{\log ar} - 1 \right) + \frac{z(t, n + 1)}{z(t, n)} - \xi^{-1} \right] > 0, \) then \( t \in N^+ . \)

**Proof** We first prove the statement (I), i.e., the lower-right derivative of \( F(x) \) is infinite at a non-endpoint \( t \) of \( C \) when \( \lim \sup_{n \to \infty} \left[ \frac{n}{z(t, n)}(1 - \xi^{-1}) \left( \frac{\log a}{\log ar} - 1 \right) + \frac{z(t, n + 1)}{z(t, n)} - \xi^{-1} \right] < 0. \)

Consider such a point \( t \) with \( \tilde{t} = (t(1), t(2), \cdots) \). Let \( k \) be a positive integer such that

\[
\frac{n}{z(t, n)}(1 - \xi^{-1}) \left( \frac{\log a}{\log ar} - 1 \right) + \frac{z(t, n + 1)}{z(t, n)} - \xi^{-1} < q, \tag{3}
\]

for some negative real number \( q \) whenever \( n \geq k \). Let \( u \) be a positive number such that \( u \) is smaller than the distance between \( t \) and \( [0, 1] \setminus C_{\tilde{t}, l} \) with \( l = z(t, k) \). Let \( x \) be a point in the segment \( (t, t + u) \). Then \( t, x \in h_{\tilde{t}, l}([0, 1]) \). We will see that \( (F(x) - F(t))/(x - t) \) is large relative to \( k \), so \( t \) is not in \( N^+ \). Let \( i \) denote the level at which \( x \notin h_{\tilde{t}, l}([0, 1]) \) but \( x \in h_{\tilde{t}, l(i-1)}([0, 1]) \). Note that also \( t \in h_{\tilde{t}, l(i-1)}([0, 1]) \). Thus \( x - t \leq |h_{\tilde{t}, l(i-1)}([0, 1])| = a_{\tilde{t}, l(i-1)} \); also \( i = z(t, n) \) for some \( n > k \). Put \( j = z(t, n + 1) - 1 \), and by \( v \) we denote the right endpoint of \( h_{\tilde{t}, l(j)}([0, 1]) \), which implies that \( \tilde{v} = (t(1), \cdots, t(j), r, r, \cdots) \) and \( (t, v] \supseteq h_{\tilde{t}, l(j+1)}([0, 1]) \). Then we have \( t < v < x \) and \( F(v) - F(t) = \mu((t, v]) \geq \mu(h_{\tilde{t}, l(j+1)}([0, 1])) = (a_{1,j})^\xi a_r^\xi \).

Therefore, defining

\[
\Pi_n = \prod_{i = 1, t(i) \neq r}^{z(t, n) - 1} a_{t(i)},
\]

we have

\[
\frac{F(x) - F(t)}{x - t} \geq \frac{(a_{1,j})^\xi a_r^\xi}{a_{\tilde{t}, l(i-1)}} = \frac{a_r^\xi}{\prod_{i = 1}^{z(t, n) - 1} a_{t(i)}} \frac{\prod_{i = 1}^{z(t, n) - 1} a_{t(i)}}{a_r^\xi} \left[ a_r^{z(t, n) - 1 - n} a_{t(z(t, n)) \Pi_n} \right]^\xi
\]

\[
= a_r^\xi \frac{\prod_{i = 1}^{z(t, n)} a_{t(i)}}{a_r^{z(t, n) - 1 - n} \Pi_n} \left[ a_r^{z(t, n) - 1 - n} a_{t(z(t, n))} \right] \left[ a_r^{z(t, n) - 1 - n} a_{t(z(t, n))} \Pi_n \right]^\xi
\]

\[
= a_r^\xi \left[ a_r^{(z(t, n) - 1) - 1 + \frac{z(t, n) - 1}{z(t, n)} - \frac{n}{z(t, n)}} a_{t(z(t, n))} \left[ a_r^{\frac{z(t, n) - 1}{z(t, n)}} z(t, n) \right] \right]
\]

\[
= a_r^\xi \left[ a_r^{\frac{n(z(t, n) - 1)}{z(t, n)}} z(t, n) \right] ^\xi
\]

Let

\[
Q = \left( \frac{a_r}{a_r} \right)^{\frac{n(z(t, n) - 1)}{z(t, n)} z(t, n) \xi - 1}
\]

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Taking logs and by (3) we have
\[
\log Q = \xi \log a_r \left[ \frac{n}{z(t, n)} (1 - \xi^{-1}) \left( \frac{\log a}{\log a_r} - 1 \right) + \frac{z(t, n + 1)}{z(t, n)} - \xi^{-1} \right] 
\geq \xi q \log a_r > 0. 
\tag{5}
\]
Since \( t \) is a non-end point, \( z(t, n) \to \infty \) and the lower-right derivative of \( F(x) \) is infinite at \( t \) by (4) and (5).

Now we turn to the proof of statement (II). Let \( t \) be such that
\[
\limsup_{n \to \infty} \left[ \frac{n}{z(t, n)} (1 - \xi^{-1}) \left( \frac{\log a}{\log a_r} - 1 \right) + \frac{z(t, n + 1)}{z(t, n)} - \xi^{-1} \right] = c > 0. 
\]
Then there exists a sequence \( \{n_k\} \) of positive integers such that
\[
\frac{n_k}{z(t, n_k)} (1 - \xi^{-1}) \left( \frac{\log a}{\log a_r} - 1 \right) + \frac{z(t, n_k + 1)}{z(t, n_k)} - \xi^{-1} > c, 
\tag{6}
\]
for some positive constant \( c \). Let \( x_k \) be the left endpoint of \( h_{\tilde{t}(j_k)}(t(j_k+1)+1)([0,1]) \), where \( j_k = z(t, n_k) - 1 \). Thus we have \( \tilde{x}_k = (t(1), \ldots, t(j_k), t(j_k+1)+1, 0, \ldots, 0) \). Let \( u_k \) be the right endpoint of \( h_{\tilde{t}(j_k+1)}([0,1]) \). Then \( \tilde{u}_k = (t(1), \ldots, t(j_k), t(j_k+1), r, r, \ldots) \). Thus
\[
[u_k, x_k] \text{ is the gap on the right side of } h_{\tilde{t}(j_k+1)}([0,1]) \text{ and } \lambda([u_k, x_k]) = x_k - u_k = a_{\tilde{t}(j_k+1)} \beta_{(t(j_k+1))}
\]
where by \( \beta_j, j = 0, 1, \ldots, r - 1 \), we denote length of the gap between images \( h_j([0,1]) \) and \( h_{j+1}([0,1]) \). Note that \( [t, x_k] \supseteq [u_k, x_k] \) and \( \mu([(t, x_k)]) = \mu((t, u_k)) + \mu((u_k, x_k)) = \mu((t, u_k)) \leq \mu(h_{\tilde{t}(z(t, n_k+1)+1)}([0,1])) \) since \( \tilde{t}([z(t, n_k+1)+1]) = \tilde{u}_k([z(t, n_k+1)+1]) \). Therefore we have
\[
F(x_k) - F(t) = \mu((t, x_k)) \leq \mu(h_{\tilde{t}(z(t, n_k+1)+1)}([0,1])) = a_{\tilde{t}(z(t, n_k+1)+1)}^\xi, 
\]
and
\[
x_k - t \geq \lambda([u_k, x_k]) = a_{\tilde{t}(z(t, n_k)+1)} \beta_{t(z(t, n_k))}.
\]
Denote \( \beta = \min_{j \in \{0,1,\ldots, r - 1\}} \beta_j \). Then we obtain with a similar reasoning which led to (4)
\[
\frac{F(x_k) - F(t)}{x_k - t} \leq \frac{a_{\tilde{t}(z(t, n_k+1)+1)}}{a_{\tilde{t}(z(t, n_k)+1)} \beta_{t(z(t, n_k))}} \leq \frac{\left( \prod_{i=1}^{z(t, n_k+1)-1} a_{\tilde{t}(i)} \right)^\xi}{\beta \prod_{i=1}^{z(t, n_k)-1} a_{\tilde{t}(i)}} 
\leq \frac{a^\xi a^{-\xi}}{\tilde{a}^\xi \beta} \left[ \left( \frac{a}{a_r} \right) ^{\frac{n_k(-1)}{a_r}} \frac{z(t, n_k+1)}{z(t, n_k)} \xi^{-1} \right] \frac{z(t, n_k)}{z(t, n_k+1)}. 
\tag{7}
\]
Let

\[ Q = \left( \frac{a}{a_r} \right)^{\frac{z(t,n_k)}{z(t,n_k)}} \left( \frac{a}{a_r} \right)^{-\xi^{-1}} \]

Taking logs and using (6) we obtain

\[
\log Q = \xi \log a_r \left[ \frac{n_k}{z(t,n_k)} (1 - \xi^{-1}) \left( \frac{\log a}{\log a_r} - 1 \right) + \frac{z(t,n_k + 1)}{z(t,n_k)} - \xi^{-1} \right] \\
\leq \xi c \log a_r < 0.
\]

From (7) and (8) it follows that the right lower derivative at \( t \) is finite by letting \( k \to \infty \).

QED

### 3 Dimensions of the set of non-differentiability points

In this section, we determine the dimensions of \( S \). The following lemma is a special case of a result in [4].

**Lemma 3.1** Let \( \Gamma = \{0, 1, \cdots, r-1\} \) and \( z(t,n) \) denote the position of the \( n \)-th occurrence of elements of \( \Gamma \) in \( \tilde{t} \). For given \( 0 < p \leq 1 \), let

\[ C(p) = \left\{ t \in C \setminus \{ \text{right endpoints of } C \} : \limsup_{n \to \infty} \frac{z(t,n+1)}{z(t,n)} \geq p^{-1} \right\}. \]  

Let \( \eta = \eta(p) \) be such that

\[ p \log \sum_{j \in \Omega} a_j^p + (1 - p) \log a_r^p = 0. \]

Then we have \( \dim_H C(p) = \eta \) and \( \dim_P C(p) = \dim_B C(p) = \dim_H C = \xi \) where \( \xi \) is defined in (2).

It is easy to verify that \( \eta(p) \) is strictly increasing and continuous in \( p \) and that \( \eta(0+) < \eta(p) \leq \eta(1) = \xi \). We also consider for \( 0 < p \leq 1 \) and with the same \( \Gamma \)

\[ C^*(p) = \left\{ t \in C \setminus \{ \text{right endpoints of } C \} : \limsup_{n \to \infty} \frac{z(t,n+1)}{z(t,n)} > p^{-1} \right\}. \]

Directly from Lemma 3.1 it follows that \( \dim_P C^*(p) = \dim_B C^*(p) = \dim_H C = \xi \). Moreover, \( \dim_H C^*(p) = \eta(p) \). To see that this follows from Lemma 3.1, approximate \( C^*(p) \) by a union of \( C(p_k) \)'s, where \( p_k \uparrow p \).
Lemma 3.2 Let $\Gamma = \{0, 1, \ldots, r-1\}$, $a = \min_{j \in \Gamma} a_j$, $\bar{a} = \max_{j \in \Gamma} a_j$, and let

$$p_1^{-1} = \max \left\{ \xi^{-1}, (\xi^{-1} - 1) \frac{\log a}{\log a_r} + 1 \right\}; \quad p_2^{-1} = \min \left\{ \xi^{-1}, (\xi^{-1} - 1) \frac{\log \bar{a}}{\log a_r} + 1 \right\}.$$ 

Then $C^\ast(p_1) \subseteq N^+ \subseteq C(p_2)$ and $\eta(p_1) \leq \dim_H(N^+) \leq \eta(p_2)$.

**Proof** By Lemma 3.1 it suffices to prove $C^\ast(p_1) \subseteq N^+ \subseteq C(p_2)$. Let $t \in N^+$. By Proposition 2.2 we have

$$\limsup_{n \to \infty} \left[ \frac{n}{z(t, n)} \left(1 - \xi^{-1} \right) \left( \frac{\log \bar{a}}{\log a_r} - 1 \right) + \frac{z(t, n + 1)}{z(t, n)} - \xi^{-1} \right] \geq 0.$$ 

Now if $\frac{\log \bar{a}}{\log a_r} > 1$, then $\limsup_{n \to \infty} \frac{z(t, n + 1)}{z(t, n)} \geq \xi^{-1}$. If not, we use that $\frac{n}{z(t, n)} \leq 1$ and so $\limsup_{n \to \infty} \frac{z(t, n + 1)}{z(t, n)} \geq \xi^{-1} - \left(1 - \xi^{-1} \right) \left( \frac{\log \bar{a}}{\log a_r} - 1 \right)$.

So we find that

$$\limsup_{n \to \infty} \frac{z(t, n + 1)}{z(t, n)} \geq p_2^{-1},$$

i.e., $t \in C(p_2)$. On the other hand, $t \in C^\ast(p_1)$ implies in a similar way that $t \in N^+$.

QED

Theorem 3.3 Let $C = \bigcup_{j=0}^r h_j(C)$ be the Cantor set determined by $\{h_j(x) = a_jx + b_j : 0 \leq j \leq r\}$, and let $\xi$ be its Hausdorff dimension. If $S$ is the set of non-differentiability points of the Cantor distribution determined by $a_0, \ldots, a_r$ on $C$, then $\dim_H S = \xi^2 = (\dim_H C)^2$ and $\dim_B S = \dim_P S = \dim_H C = \xi$.

**Proof** Since $S = N^+ \cup N^- \bigcup \{\text{the endpoints of } C\}$ and because of the symmetry between $N^+$ and $N^-$, it suffices to determine the dimensions of $N^+$. By Lemma 3.2 and Lemma 3.1, $\dim_B N^+ = \dim_P N^+ = \dim_H C = \xi$ is trivial. We define $a = \min_{j \in \Omega} a_j$. Let $0 < \delta < a^2$, and let

$$\Omega_\delta = \{ \sigma \in \Omega^* : a_\sigma \leq \delta \text{ and } a_{|\sigma| - 1} > \delta \}.$$ 

Note that for each $\sigma \in \Omega_\delta$ we have $a\delta \leq a_\sigma \leq \delta$. Thus for any $\sigma, \tau \in \Omega_\delta$

$$\frac{\log \delta}{\log a + \log \delta} \leq \frac{\log a_{\sigma}}{\log a_r} \leq \frac{\log a + \log \delta}{\log \delta}.$$ (12)

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Here we would like to relate $\Omega_\delta$ to $\xi$ by showing that
\[
\lim_{\delta \to 0} \frac{\log \#\Omega_\delta}{-\log \delta} = \xi,
\]
where $\xi$ is defined in (2) and $\#\Omega_\delta$ denotes the number of elements of $\Omega_\delta$. By $N_\delta(C)$ we denote the smallest number of sets of diameter at most $\delta$ that cover $C$. By Lemma 9.2 in [2] there exists a positive constant $q$ independent of $\delta$ such that
\[
q \#\Omega_\delta \leq N_\delta(C) \leq \#\Omega_\delta.
\]
Hence (13) holds by (14) and the fact $\lim_{\delta \to 0} \frac{\log N_\delta(C)}{-\log \delta} = \xi$. Now let $H_\delta = \{h_\sigma : \sigma \in \Omega_\delta\}$. Note that $h_\sigma$ is a similitude mapping with ratio $0 < a_\sigma < 1$ for each $\sigma \in H_\delta$, that the family $H_\delta$ of similitude mappings still satisfies the open set condition, and that the unique self-similar set determined by $H_\delta$ equals $C$:
\[
C = \bigcup_{\sigma \in H_\delta} h_\sigma(C).
\]
We also have that $\xi$ defined in (2) satisfies $\sum_{\sigma \in \Omega_\delta} a_\sigma^\xi = 1$. If we denote $\mu_\delta$ the self-similar probability measure on $C$ corresponding to the probability vector $(a_\sigma^\xi : \sigma \in \Omega_\delta)$, then $\mu_\delta = \mu$ since $\mu_\delta(h_\tau([0,1])) = \mu(h_\tau([0,1]))$ for any $\tau \in \Omega_\delta^k$ and $k \in \mathbb{N}$. Hence for the corresponding non-differentiability points $N_\delta^+$ we have $N_\delta^+ = N^+$ for all $\delta > 0$. There exists a unique $\sigma \in \Omega_\delta$ with $\sigma(j) = r$ for $j = 1, 2, \cdots, |\sigma|$. We denote this special element by $\sigma_\delta$. Note that $\sigma_\delta$ plays the same role in $\Omega_\delta$ as $r$ in $\Omega$, in the sense that $h_{\sigma_\delta}([0,1])$ is the right most interval in $[0,1]$ of the intervals $(h_\sigma([0,1]))_{\sigma \in \Omega_\delta}$. Let
\[
\Gamma_\delta := \Omega_\delta \setminus \{\sigma_\delta\} \quad \text{and} \quad a_\delta := a_{\sigma_\delta} = a_{\sigma_\delta}^{[\sigma_\delta]}.
\]
We will use the notations $a_\delta = \min_{\sigma \in \Gamma_\delta} a_\sigma$, $a_\delta = \max_{\sigma \in \Gamma_\delta} a_\sigma$,
\[
p_1(\delta) = \left( \max \left\{ \xi^{-1}, (\xi^{-1} - 1) \frac{\log a_\delta}{\log a_\delta} + 1 \right\} \right)^{-1}
\]
and
\[
p_2(\delta) = \left( \min \left\{ \xi^{-1}, (\xi^{-1} - 1) \frac{\log a_\delta}{\log a_\delta} + 1 \right\} \right)^{-1}.
\]
Then by Lemma 3.1 and Lemma 3.2 with $\Omega$ replaced by $\Omega_\delta$, $\Gamma$ replaced by $\Gamma_\delta$, $a_r$ replaced by $a_\delta$ and $p_1, p_2$ by $p_1(\delta)$ respectively $p_2(\delta)$, we have

$$\eta(p_1(\delta)) \leq \dim_H N^+_\delta \leq \eta(p_2(\delta)),$$

(16)

where $\eta(p_1(\delta))$ and $\eta(p_2(\delta))$ are defined by formula (10):

$$p_1(\delta) \log \sum_{\sigma \in \Omega_\delta} a_\delta^{\eta(p_1(\delta))} + (1 - p_1(\delta)) \log a_\delta^{\eta(p_1(\delta))} = 0,$$

(17)

and

$$p_2(\delta) \log \sum_{\sigma \in \Omega_\delta} a_\delta^{\eta(p_2(\delta))} + (1 - p_2(\delta)) \log a_\delta^{\eta(p_2(\delta))} = 0.$$

Since $N^+_\delta = N^+$ for all $\delta > 0$ it follows from (16) that $\dim_H N^+ = \xi^2$ holds if for any $\varepsilon > 0$ there exists a $\delta^* > 0$ such that when $0 < \delta < \delta^*$ we have $|\eta(p_1(\delta)) - \xi^2| < \varepsilon$ and $|\eta(p_2(\delta)) - \xi^2| < \varepsilon$. Verification of this claim will be given in the following only for $\eta(p_1(\delta))$, since the same argument can be employed for $\eta(p_2(\delta))$. To alleviate the notation we will write $p_\delta := p_1(\delta)$ and $\eta_\delta := \eta(p_1(\delta))$. For $x \in (0, \xi]$ let

$$T_\delta(x) = p_\delta \log \frac{\sum_{\sigma \in \Omega_\delta} a_\delta^x}{x \log a_\delta} + 1 - p_\delta$$

Note that

$$a_\delta^x \delta^x \# \Omega_\delta \leq \sum_{\sigma \in \Omega_\delta} a_\delta^x \leq \delta^x \# \Omega_\delta,$$

which implies

$$\frac{x \log \delta + \log \# \Omega_\delta}{x \log a_\delta} \leq \log \sum_{\sigma \in \Omega_\delta} a_\delta^x \leq \frac{x \log a + x \log \delta + \log \# \Omega_\delta}{x \log a_\delta}.$$

(18)

Therefore by (18) we have

$$0 \leq T_\delta(x) - \left[ 1 + p_\delta \left( \frac{\log \delta}{\log a_\delta} - 1 \right) + \frac{1}{x} \frac{p_\delta \log \# \Omega_\delta}{\log a_\delta} \right] \leq p_\delta \log a \log a_\delta.$$

(19)

Note that by (12) and (15) we have

$$\lim_{\delta \downarrow 0} p_\delta = \xi.$$

(20)

Since $a \delta < a_\delta \leq \delta$, and by (13) and (20) we have

$$\lim_{\delta \downarrow 0} \left( \frac{\log \delta}{\log a_\delta} - 1 \right) = \lim_{\delta \downarrow 0} \frac{p_\delta \log a}{\log a_\delta} = 0 \quad \text{and} \quad \lim_{\delta \downarrow 0} \frac{p_\delta \log \# \Omega_\delta}{\log a_\delta} = -\xi^2.$$
Thus by (19) we have for all \( x \in (0, \xi] \)

\[
\lim_{\delta \downarrow 0} T_\delta(x) = 1 - \frac{\xi^2}{x} =: T_0(x). \tag{21}
\]

Now for any given \( \varepsilon > 0 \) satisfying \( 0 < \xi^2 - \varepsilon < \xi^2 + \varepsilon < \xi < 1 \), we see that

\[
T_0(\xi^2 - \varepsilon) = \frac{-\varepsilon}{\xi^2 - \varepsilon} < -\varepsilon \quad \text{and} \quad T_0(\xi^2 + \varepsilon) = \frac{\varepsilon}{\xi^2 + \varepsilon} > \varepsilon.
\]

By (21) we can take \( \delta^* > 0 \) so that for \( 0 < \delta < \delta^* \)

\[
|T_\delta(\xi^2 - \varepsilon) - T_0(\xi^2 - \varepsilon)| < \frac{\varepsilon}{2} \quad \text{and} \quad |T_\delta(\xi^2 + \varepsilon) - T_0(\xi^2 + \varepsilon)| < \frac{\varepsilon}{2}.
\]

Then necessarily for these \( \delta \)

\[
T_\delta(\xi^2 - \varepsilon) < -\frac{\varepsilon}{2} \quad \text{and} \quad T_\delta(\xi^2 + \varepsilon) > \frac{\varepsilon}{2}.
\]

Then for \( 0 < \delta < \delta^* \)

\[
\xi^2 - \varepsilon < \eta_\delta < \xi^2 + \varepsilon,
\]

since \( T_\delta(x) \) is strictly increasing in \( x \) and \( T_\delta(\eta_\delta) = 0 \). \textbf{QED}
References


