Key to partial exercises for *Mathematical Analysis*

[15-8] Under what conditions does the inverse of the function

$$y = \frac{ax+b}{cx+d}$$

equal to itself?

Solution. At first let's notice that a function has an inverse equal to itself if and only if its graph is symmetric with $y = x$. Suppose that $ad - bc \neq 0$ to avoid the degenerate case that $y =$constant when the inverse function doesn't exist. In the following we divide our problem into two cases.

Case I: $c = 0$. Then we have

$$y = \frac{ax+b}{d} = \frac{a}{d}x + \frac{b}{d}. \tag{1}$$

Thus we require that $\left(\frac{a}{d}x + \frac{b}{d}, x\right)$ satisfies (1) for all x in the domain of the original function, i.e.,

$$x \equiv \frac{a}{d}\left(\frac{a}{d}x + \frac{b}{d}\right) + \frac{b}{d} = \left(\frac{a}{d}\right)^2 x + \frac{b}{d}\left(\frac{a}{d} + 1\right).$$

Thus

$$x\left(1 - \frac{a}{d}\right)\left(1 + \frac{a}{d}\right) \equiv \frac{b}{d}\left(\frac{a}{d} + 1\right).$$

It gives that $a = -d$ or $a = d$ with $b = 0$.

Case II: $c \neq 0$. Then we have

$$y = \frac{ax+b}{cx+d} = \frac{a}{c} + \frac{\frac{bc-ad}{c^2}}{x - \left(-\frac{d}{c}\right)},$$

i.e.,

$$y - \frac{a}{c} = \frac{\frac{bc-ad}{c^2}}{x - \left(-\frac{d}{c}\right)}.$$

At this moment, the graph is symmetric with $y = x$ if and only if $a = -d$.

Summarizing the above two cases we get the conditions for our problem as follows.

$$\begin{cases} ad - bc \neq 0 \\ a = -d \end{cases} \quad \text{or} \quad \begin{cases} ad - bc \neq 0 \\ c = b = 0 \\ a = d \end{cases}$$

[20-7] Let f, g be bounded functions defined on D such that

$$f(x) \leq g(x), \quad x \in D.$$

Show that (1) $\sup_{x\in D} f(x) \leq \sup_{x\in D} g(x)$; (2) $\inf_{x\in D} f(x) \leq \inf_{x\in D} g(x)$.

Proof. For any $x \in D$ we have

$$f(x) \leq g(x) \leq \sup_{x\in D} g(x),$$

leading the conclusion (1). The conclusion (2) can be deduced in the same way.

[20-8] Let f be a bounded function defined on D. Show that

$$(1) \sup_{x\in D}\{-f(x)\} = -\inf_{x\in D} f(x); \quad (2) \inf_{x\in D}\{-f(x)\} = -\sup_{x\in D} f(x).$$

Proof. For any $x \in D$ we have

$$-f(x) \leq \sup_{x\in D}\{-f(x)\},$$

i.e.

$$f(x) \geq -\sup_{x\in D}\{-f(x)\}.$$

Thus $\inf_{x\in D} f(x) \geq -\sup_{x\in D}\{-f(x)\}$, i.e., $-\inf_{x\in D} f(x) \leq \sup_{x\in D}\{-f(x)\}$. On the other hand, for any $x \in D$ we have

$$f(x) \geq \inf_{x\in D} f(x),$$

i.e.

$$-f(x) \leq -\inf_{x\in D} f(x).$$

So $\sup_{x\in D}\{-f(x)\} \leq -\inf_{x\in D} f(x)$ and (1) is true.

As to (2), it can be directly deduced from (1). In fact from (1) it follows that

$$\inf_{x\in D}\{-f(x)\} = -\sup_{x\in D}\{-(-f(x))\} = -\sup_{x\in D} f(x).$$

[22-12] Let f, g be bounded functions defined on D. Show that

$$(1)\ \inf_{x\in D}\{f(x)+g(x)\} \le \inf_{x\in D} f(x) + \sup_{x\in D} g(x);$$
$$(2)\ \sup_{x\in D} f(x) + \inf_{x\in D} g(x) \le \sup_{x\in D}\{f(x)+g(x)\}.$$

Proof. For any $x \in D$ we have

$$f(x)+g(x) \le f(x) + \sup_{x\in D} g(x).$$

Then we get

$$\inf_{x\in D}\{f(x)+g(x)\} \le \inf_{x\in D}\{f(x)+\sup_{x\in D} g(x)\} = \inf_{x\in D} f(x) + \sup_{x\in D} g(x),$$

where the first inequality is obtained by the conclusion (2) in [20-7], and the second equality is clear and left to the readers for proof.

The conclusion (2) can be done by the same argument. We leave its proof to the readers.

[22-13] Let f, g be nonnegative bounded functions defined on D. Show that

$$(1)\ \inf_{x\in D} f(x)\cdot\inf_{x\in D} g(x) \le \inf_{x\in D}\{f(x)g(x)\}; \quad (2)\ \sup_{x\in D}\{f(x)g(x)\} \le \sup_{x\in D} f(x)\cdot\sup_{x\in D} g(x).$$

Proof. Note that for any $x \in D$

$$0 \le \inf_{x\in D} f(x) \le f(x) \le \sup_{x\in D} f(x),$$

and

$$0 \le \inf_{x\in D} g(x) \le g(x) \le \sup_{x\in D} g(x).$$

Therefore for any $x \in D$

$$\inf_{x\in D} f(x)\cdot\inf_{x\in D} g(x) \le f(x)g(x) \le \sup_{x\in D} f(x)\cdot\sup_{x\in D} g(x),$$

which gives both (1) and (2).