Take-home midterm

On Stokes' Theorem.

Stokes' Theorem is about the relation between the integration of a degree $m-1$ exterior differential form ω over the boundary ∂M of a m-dimensional differential manifold M and the integration of the exterior derivative of this $m-1$ form, i.e, $\mathrm{d}\omega$ over the manifold M itself, where the manifold M is orientable and the orientation of ∂M is the one derived from the orientation of M. Note that we do need the orientations on M and ∂M in order to define integrations of m forms and $m-1$ forms over them correspondingly.

To simplify the work without losing the essentials of math, we just assume this differential manifold M is compact.

As M can be regarded as certain well-behaved subsets of $\mathbb{R}^m$ patched up together, the first thing needed in the proof of the Stokes Theorem is to prove the special case under which M resides in $\mathbb{R}^m$. Then we need to "glue up" the results above in certain compatible way to get the final version of Stokes's Theorem on general orientable differential manifold M, during which the fact that M is orientable (thus so is ∂M) will be needed. Also, during the "glue up" process, a classical and standard tool is the partition of the unit 1, which is a constant function, into continuous functions that are supported on local neighborhoods such that each of those neighborhoods resides in one atlas only.

Definition (Topological manifolds with boundary)

A topological manifold M with boundary (denoted as ∂M) is a Hausdorff topological space that is locally homeomorphic to $\mathbb{R}^n_+$, where H^n is a subset of $\mathbb{R}^n$, with the induced topology, and is defined as

$$R^n_+ = \{(x_1, \cdots, x_n) \colon x_i \geq 0, x_i \in \mathbb{R} \text{ for all } i \in 2, \cdots, n\}.$$

By "M is locally homeomorphic to $\mathbb{R}^n_+$", we mean "for any point in M, there is an open neighborhood of that point which is homeomorphic to an open set in $\mathbb{R}^n_+$".

Definition (Differentiable manifolds with boundary)

A differentiable manifold M with boundary (denoted as ∂M) is a topological manifold with boundary such that the transformation maps among different atlases are diffeomorphisms (instead of being merely homeomorphisms).

Hint on how to get the orientation on ∂M from the orientation on M

Let M be a differentiable manifold with boundary, and assume that M is orientable. How to derive an orientation of ∂M? In case $M = \mathbb{R}^n_+$, just consider the following natural approach: If the orientation of $\mathbb{R}^n_+$ is defined to be $dx_1 \wedge \cdots \wedge dx_n$, then the orientation on $\partial\mathbb{R}^n_+ = \{(0, x_2, \cdots, x_n) \colon x_i \in \mathbb{R} \text{ for } i \in 2, \cdots, n\}$ is just defined to be $dx_2 \wedge \cdots \wedge dx_n$. With this in mind, you should figure out how to derive the orientation of ∂M from the orientation of M.

Now, your turn.

1. Assume the compact m-dimensional differential manifold M just resides in $\mathbb{R}^n$. Prove the Stokes' Theorem.

Remark: If $m = 1$, then what you are asked to prove can be derived from the Newton-Leibniz formula.

2. Let M be a compact m-dimensional differential manifold and let ∂M be its boundary with the derived orientation. Prove that for any $m-1$ form ω on ∂M, we have

$$\int_{\partial M} \omega = \int_M \mathrm{d}\omega.$$