HW # 4 Lecture related problems.

1. Classify all the biholomorphic functions from D to itself, where D is the open unit disk.

2. Prove that $C_c^\infty(\mathbb{R})$ is dense in $C_0(\mathbb{R})$, where the norm on $C_0(\mathbb{R})$ is the supreme norm.

3. For $1 < p < q < \infty$, show that $L^p(\mathbb{R})$ is not a subset of $L^q(\mathbb{R})$ and also show that $L^q(\mathbb{R})$ is not a subset of $L^p(\mathbb{R})$.

3.5 For any $p > 1$, according to problem 3, we shall not expect to have $L^1(\mathbb{R}) \supset L^p(\mathbb{R})$, nor shall we expect $L^1(\mathbb{R}) \subset L^p(\mathbb{R})$. Prove that $L^1_{loc}(\mathbb{R}) \supset L^p_{loc}(\mathbb{R})$. **Note:** This fact kind of explains why we define weak derivatives on $L^1_{loc}(\mathbb{R})$.

4. [The interpolation property] For $0 < p < s < q$, and for a measurable function f over a measure space (X, μ), if $f \in L^p(X, \mu)$ and $f \in L^q(X, \mu)$, prove that f is also in $L^s(X, \mu)$.

5. Construct a function $f \in C_c^\infty(\mathbb{R})$, such that $\chi_{[-1,1]} \le f \le \chi_{[-2,2]}$.

6. Let X be a set and let $\mathcal{M}$ be a semi-ring of $\mathcal{P}(X)$. Let Y be a set and let $\mathcal{N}$ be a semi-ring of $\mathcal{P}(Y)$. Let $\mathcal{M} \times \mathcal{N} = \{A \times B \colon A \in \mathcal{M} \text{ and } B \in \mathcal{N}\}$. Prove that $\mathcal{M} \times \mathcal{N}$ is a semi-ring of $\mathcal{P}(X \times Y)$.

7. Given $f \colon \{x + iy \colon y \ge 0\} \to \mathbb{C}$ such that i) $f(z) \in \mathbb{R}$ for all $z \in \mathbb{R}$; ii) f is continuous on $\{x + iy \colon y \ge 0\}$; iii) f is holomorphic in $\{x + iy \colon y > 0\}$. Define $g \colon \mathbb{C} \to C$ as $g(z) = f(z)$ for all $z \in \{x + iy \colon y \ge 0\}$ and $g(z) = \overline{f(\overline{z})}$ for all z with $\mathrm{Im}(z) < 0$. Prove that f is holomorphic.

8. Let X and Y be two topological spaces. Let $f \colon X \to Y$ be a continuous function. Assume f is a bijection, and also assume that X is compact. Prove that f^{-1} is also continuous. That is, prove that f is a homeomorphism.

9. In the previous problem, if we remove the assumption that X is compact. Show that f^{-1} might not be continuous.

10. For $\{f_n\} \subset C_c^\infty(\mathbb{R})$, if $\|f_n\|_\infty \to 0$, does it follow that $\|f_n'\|_\infty \to 0$? Why or why not?

11. Assume $g \in C_c^\infty(\mathbb{R})$, g is non-negative and $\int_{\mathbb{R}} g \, \mathrm{d}x = 1$. For any $f \in C_0(\mathbb{R})$, prove that $f * g \in C_0^\infty(\mathbb{R})$.

12. For $f(x) = \mathrm{sgn}(x) = \begin{cases} 1 & x \ge 0 \\ 0 & x < 0 \end{cases}$. Prove that the weak derivative of f does not exist. **Note:** The following facts, which we proved in class, can be used directly: Fact 1: Let $g \in L^1_{loc}(\mathbb{R})$. The weak derivative of g, if exists, must be unique. Fact 2: Let $g \in L^1_{loc}(\mathbb{R})$ such that the weak derivative of g exists, which is denoted by $D_w(g)$. If there exists $(a,b) \subset \mathbb{R}$ such that $g\,\big|_{(a,b)} = 0$ a.e, then $D_w(g)\,\big|_{(a,b)} = 0$ a.e.

13. Find *all* the possible bihomomorphic maps between the upper half open plane in $\mathbb{C}$ and the open unit disk in $\mathbb{C}$, which maps the point $2i$ in upper half plane to 0 in the unit disk.

14. Prove that $(l^\infty)^*$ is not isomorphic to l^1.

15. For $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$, consider the map

$$\varphi\colon \{z \in \mathbb{C}\colon cz + d \neq 0\} \to \mathbb{C},\ z \mapsto \frac{az+b}{cz+d}.$$

Use CIRCLE to denote straight lines in $\mathbb{C}$ and ordinary circles in $\mathbb{C}$. Prove that φ maps CIRCLES to CIRCLES.

16. Does holomorphic functions always map convex sets to convex sets? Why or why not?

17. Let $\{X_i\}$ be a sequence of sets and let $\mathcal{M}_\rangle$ be a semi-ring of $\mathcal{P}(X_i)$ for each i. Let $\prod_{i=1}^\infty \mathcal{M}_i = \{A_1 \times A_2 \times \cdots : A_i \in \mathcal{M}_i \text{ for all } i\}$. Prove that $\prod_{i=1}^\infty \mathcal{M}_i$ is *not* a semi-ring of $\mathcal{P}(\prod_{i=1}^\infty X_i)$.

18. Let B be a Banach space and let $x \in B \setminus \{0\}$. Prove that there exists $\rho \in B^*$ such that $\|\rho\| = 1$ and $|\rho(x)| = \|x\|$. Based on this, prove that for any $y \in B$, $y = 0$ if and only if $\varphi(y) = 0$ for all $\varphi \in B^*$. **Note:** You can feel free to use the Hahn-Banach theorem directly.

19. For the Cantor function $f\colon [0,1] \to [0,1]$, prove that f is continuous and uniform continuous. Then prove that f is not absolutely continuous.

20. Write down the definition of "net" first. Then write down the definition of "net limits" for a topological space X. Besides, in case X is Hausdorff, show "net limit", if exits, must be unique.

21. Let D be a non-empty open subset of $\mathbb{R}^n$, where the topology on $\mathbb{R}^n$ is the "usual" one. Under this setup, prove that D is connected if and only if D is path-connected.

22. Let φ be a function in $C_c^\infty(\mathbb{R})$ such that $\varphi \geq 0$ and $\int_{\mathbb{R}} \varphi \, d\mu = 1$, where μ is the Lebesgue measure on $\mathbb{R}$. For any $(a,b) \subset \mathbb{R}$, it is clear that $\chi_{(a,b)} \in L^\infty(\mathbb{R}, \mu)$. According to what is covered in class, we know that that $\chi_{(a,b)} * \varphi$ is in $C^\infty(\mathbb{R})$. Assuming $\operatorname{supp}(\varphi) \subset [-r, r]$ for some $r > 0$ and $b - a \geq 2r$, show that $\left\|\chi_{(a,b)} * \varphi - \chi_{(a,b)}\right\|_{L^\infty(\mathbb{R},\mu)} \geq 1/2$.