HW # 3 Due by June 23 2016.

1. **($\star$)** As in the class, we use (f, D) to denote a holomorphic function f that is holomorphic on the open disk D (radius of D does not need to be 1). For two pairs of such (f, D) and (f', D'), we say $(f, D) \sim (f', D')$ if $D \cap D' \neq \emptyset$ and $f\,|_{D\cap D'} = f'\,|_{D\cap D'}$.

Construct (f_0, D_0), (f_1, D_1) and (f_2, D_2), such that $D_0 \cap D_1 \neq \emptyset$, $D_0 \cap D_2 \neq \emptyset$, $D_1 \cap D_2 \neq \emptyset$, $(f_0, D_0) \sim (f_1, D_1)$, $(f_0, D_0) \sim (f_2, D_2)$ but $(f_1, D_1) \nsim (f_2, D_2)$.

Note: To find such pairs, you need to make sure that $D_1 \cap D_2 \cap D_3 = \emptyset$. Else, according to what is covered in class, such $\sim$ relation is guaranteed to be transitive in this case.

2. **($\bigstar$)** In class, we demonstrated in sketch that there exists a continuous function $f\colon \overline{U} \longrightarrow \mathbb{C}$ such that $f(\overline{U}) = \partial U$, where U is the open unit disk, and $\overline{U}$ is the closure of it.

Do the following.

1) Give detailed construction of a continuous function $f\colon \overline{U} \to \mathbb{C}$ such that $f(\overline{U}) = \partial U$.

~~2) For any continuous function f over $\overline{U}$ such that $f(\overline{U}) = \partial U$, prove that $f(\partial U) = \partial U$.~~

Note: Thanks to one of the students for pointing out the error here. One quick example for $f(\overline{U}) = \partial U$ but $f(\partial U) \subsetneq \partial U$ can be done like this: Consider $\overline{U}$ as the closed unit disk in x-y plane of $\mathbb{R}^3$. We can find a homeomorphism between this $\overline{U}$ and $C = \{t \cdot (\cos\theta, \sin\theta, 0) + (1-t)\cdot(0,0,1) \colon t \in [0,1], \theta \in [0, 2\pi)\}$, where C can be regarded as the surface of a cone, and the base of the cone is exactly $\overline{U}$. Do the projection of C to the z-axis, we get a segment $[0,1]$ in z-axis. Connect the two ends of this segment to get a circle, which is homeomorphic to ∂U.

2) Find a continuous function f over $\overline{U}$, such that $f(\overline{U}) = \overline{U}$ but $f(\partial U)$ is a single point (thus surely we do not have $f(\partial U) = \partial U$).

Hint: Just read the note above, and recall things like the Peano curve.

3) For any continuous function $g\colon \overline{U} \to \overline{U}$ with $g(\partial U) = \partial U$, this g is not "path connected" to the identity map

$$\mathrm{id}\colon \overline{U} \longrightarrow \overline{U}, \quad z \mapsto z.$$

By "g is path connected to the identity map id", we mean there exits a continuous map

$$H\colon \overline{U} \times [0,1] \longrightarrow \overline{U}$$

such that $H(*,0) = \text{id}$, $H(*,1) = g$ and $H(*,\lambda)(\partial U) = \partial U$ for all $\lambda \in [0,1]$.

Remark: In 3), if we remove the requirement that "$H(\lambda,*)(\partial U) = \partial U$ for all $\lambda \in [0,1]$", then such a H always exits. Why?

2.1 (★) In $\mathbb{R}^2$, consider

$$A = \{(0,0)\} \cup \{(x,y) \in \mathbb{R}^2\colon x^2 + y^2 = 1\} \text{ and } B = \{(2,0)\} \cup \{(x,y) \in \mathbb{R}^2\colon x^2 + y^2 = 1\}.$$

Let the topology on $\mathbb{R}^2$ be the usual one, and let the topology π_A and π_B be the restricted topology from $\mathbb{R}^2$ to A and B correspondingly.

1) Prove that (A,π) and (B,π_B) are homeomorphic to each other. That is, there is a map $f\colon A \to B$, such that f is injective and onto, and both f and f^{-1} are continuous.

2) Let f be one of such homeomorphisms as above. Prove that we cannot find a continuous map

$$H\colon A \times [0,1] \to \mathbb{R}^2$$

such that for all $\lambda \in [0,1]$, $H(*,\lambda)$ is a homeomorphism from A to $H(*,\lambda)(A)$, $H(*,0) = \text{id}_A$ and $H(*,1) = f$, where id_A is the identity map on A. In short, we cannot find "a continuous path of homeomorphisms" that connects id_A and a homeomorphism from A to B.

3) In $\mathbb{R}^3$, let

$$A' = \{(x,y,0) \in \mathbb{R}^3\colon (x,y) \in A\} \text{ and } B' = \{(x,y,0) \in \mathbb{R}^3\colon (x,y) \in B\}.$$

Let the topology of A' be the topology restrected from $\mathbb{R}^3$ to A', and let the topology of B' be the topology restrected from $\mathbb{R}^3$ to B', where the topology on $\mathbb{R}^3$ is the usual one.

Similar to 1), we can show that, under the above mentioned topologies, A' is homeomorphic to B'.

Find a continuous map

$$H\colon A'\times[0,1]\to\mathbb{R}^3$$

such that for all $\lambda\in[0,1]$, $H(*,\lambda)$ is a homeomorphism between A' and $H(*,\lambda)(A')$, $H(*,0)=\mathrm{id}_{A'}$ and $H(*,1)(A')=B'$, where $\mathrm{id}_{A'}$ is the identity map on A'. In short, find "a continuous path of homeomorphisms" that connects $\mathrm{id}_{A'}$ and a homeomorphism from A' to B'.

3. **($\star$)** Give an example of a holomorphic function f defined on a simply connected open set Ω such that $f(\Omega)$ is not simply connected.

3.1 **($\bigstar$)** Let A be a convex set in $\mathbb{C}$. From the definition of convexity, for any $a,b\in A$, the segment connecting a and b also lies in A. Then a convex set in $\mathbb{C}$ is always path-connected, thus connected.

1) Let Ω be a non-empty bounded convex set in $\mathbb{C}$. Prove that Ω is simply connected.

2) Give an example of a <u>bi</u>-holomorphic function f defined on a convex open set Ω, such that $f(\Omega)$ is not convex.

3) Give an example of a holomorphic function f defined on a convex open set Ω, such that $f(\Omega)$ is not simply connected.

4) For the example you found in 3), is it bi-holomorphic? Can you find a bi-holomorphic map that maps a simply connected open set to a simply connected open set? Why or why not?

3.5 ($\star$) Let X and Y be two Riemann surfaces, and assume that the Riemann surface X is compact. Let $f\colon X\to Y$ be a holomorphic/analytic function such that $f(X)=Y$ (i.e, f is surjective). Prove that for any open set D in X, its image $f(D)$ is also open in Y.

4. ($\star$) Assume that Ω is a region in $\mathbb{C}$. In class, we mentioned that for an $f\colon\Omega\to\mathbb{C}$ such that both the real part $u(x,y)$ and the imaginary part $v(x,y)$ are C^1 functions, "at every $(x_0,y_0)\in\Omega$, if $\det\left.\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}\right|_{(x_0,y_0)}\neq 0$, then f map an open neighborhood of (x_0,y_0) to an open set" is not enough to ensure that this f is holomorphic. A typical example is $f(z)=\bar{z}$. For this f, even the metric structure of $\mathbb{C}$ is preserved. However, it is clear that this f is not holomorphic as $\dfrac{\partial f}{\partial \bar{z}} = 1 \neq 0$. In this case, note that f does not preserve the angle. In fact, this f reverses the angle.

In contrast, as also mentioned in class, "f preserves the angle" kind of ensures that f is holomorphic. Do the following:

Let Ω be a region in $\mathbb{C}$. Let $f\colon \Omega \to \mathbb{C}$ be a complex funciton. We write it as

$$f(z) = f(x+iy) = u(x,y) + iv(x,y).$$

That is, we use u and v to denote the real part of imaginary part of f.

Assume that $f \in C^1(\Omega)$. That is, both u and v are in $C^1(\Omega)$. For simplicity, we also assume that for any $(x_0, y_0) \in \Omega$ (we abuse the notation a bit and just pretend that Ω is in $\mathbb{R}^2$), we have

$$\det \left. \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \right|_{(x_0,y_0)} \neq 0.$$

If "f preserves the angle" everywhere in Ω, prove that f is holomorphic in Ω.

Note: As for the meaning of "f preserves the angle" at a point, check your class notes weeks ago.

Hint: Just note that (why?) $\mathrm{d}f = \dfrac{\partial f}{\partial z} \cdot \mathrm{d}z + \dfrac{\partial f}{\partial \bar{z}} \cdot \mathrm{d}\bar{z}$, which implies that (why?)

$$f(z) - f(z_0) = \frac{\partial f}{\partial z}(z_0) \cdot (z - z_0) + \frac{\partial f}{\partial \bar{z}}(z_0) \cdot \overline{z - z_0} + o(z - z_0).$$

In case $\frac{\partial f}{\partial z}(z_0) \neq 0$, the map from $z - z_0$ to $\frac{\partial f}{\partial z}(z_0) \cdot (z - z_0)$ is a multiplication of a non-zero complex constant, which "preserves the angle". In case $\frac{\partial f}{\partial \bar{z}}(z_0) \neq 0$, the map from $z - z_0$ to $\frac{\partial f}{\partial \bar{z}}(z_0) \cdot \overline{z - z_0}$ is the composition of the complex conjugacy map and the "multiply by non-zeoro complex constant $\frac{\partial f}{\partial \bar{z}}(z_0)$" map. As the complex conjugacy map "reverses the angle" and the "multipliy by non-zero complex constant" map "preserves the angle", their composition just "reverses the angle". As for $o(z - z_0)$, it is the relatively small term which has no contribution to the final result of angles (why?). Then ...

5. ($\star$) Let U be the unit open disk of $\mathbb{C}$. Let $f\colon U \to \mathbb{C}$ be a holomorphic function such that $|f(z)| = |z|$ for all $z \in U$. Prove that $f(z) = e^{i\theta} z$ for all z in U, where θ is a constant value in $\mathbb{R}$.

5.5 (★⋆) Let U be the unit open disk of $\mathbb{C}$. Let $f\colon U \to \mathbb{C}$ be a function (not surely holomorphic) such that $|f(z_1) - f(z_2)| = |z_1 - z_2|$ and $f(0) = f(0)$ for all $z_1, z_2 \in U$ (that is, f preserves the metric structure and maps 0 to 0). Prove that either $f(z) = e^{i\theta}z$ for all $z \in U$ or $f(z) = e^{i\theta}\bar{z}$ for all $z \in U$, where θ is a constant value in $\mathbb{R}$.

6. (⋆) Let $f\colon [0,1] \to [0,1]$ be the Cantor funtion as defined in class. We already proved last term that f is continuous. As $[0,1]$ is a bounded closed subset of $\mathbb{R}$, it follows that this f is uniformly continuous. Prove that this f is *not* absolutely continuous.

Remark: Continuity is a local property, while uniform continuity is not. Besides, absolute conitnuity is not a local property either.

6.1 (★⋆) For the above defined f, prove that it is *not* weakly differentiable. That is, show that the weak derivative of the Cantor function does not exist.

Hint: According to the Lemma we proved in class, it is not hard to show that if the weak derivative of f exists (say, g), then $g = 0$ in $L^1(0,1)$. That is, we have

$$\int_{(0,1)} 0 \cdot h \,\mathrm{d}x = -\int_{(0,1)} f \cdot h' \,\mathrm{d}x \quad \forall\, h \in C_c^\infty(0,1).$$

In other words, we have

$$\int_{(0,1)} f \cdot h' \,\mathrm{d}x = 0 \quad \forall\, h \in C_c^\infty(0,1).$$

Observe that this is equivalent to (why?)

$$\int_{(0,1)} f \cdot h \,\mathrm{d}x = 0 \quad \forall\, h \in C_c^\infty(0,1) \text{ with } \int_{(0,1)} h \,\mathrm{d}x = 0.$$

Let $C = \int_{(0,1)} f \,\mathrm{d}x$ and let $\tilde{f} = f - C$. If we can show that $\tilde{f} = 0$, then we are done, as we know that the Cantor function f is not a constant function at all. In order to get $\tilde{f} = 0$, we just need to show that

$$\int_{(0,1)} \tilde{f} \cdot g \,\mathrm{d}x = 0 \quad \text{for all } g \in C_0(0,1).$$

According to the lectures in class, $C_c^\infty(0,1)$ is dense, with respect to the supreme norm, in $C_0(0,1)$. Thus we just need to show

$$\int_{(0,1)} \tilde{f} \cdot g \,\mathrm{d}x = 0 \text{ for all } g \in C_0(0,1).$$

Check the following:

i)

$$\int_{(0,1)} \tilde{f} \cdot h \,\mathrm{d}x = 0 \;\; \forall\, h \in C_c^\infty(0,1) \text{ with } \int_{(0,1)} h \,\mathrm{d}x = 0.$$

ii)

$$\int_{(0,1)} \tilde{f} \cdot 1 \,\mathrm{d}x = \int_{(0,1)} f \,\mathrm{d}x - C = 0.$$

The hint ends here with still some decent work left to be finished.

6.5 ($\star$) With the Cantor function f defined as above, define

$$g \colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} f(x), & x \in [0,1] \\ 0, & \text{else.} \end{cases}$$

It is easy to check that g is positive, measurable and $g \in L^1(\mathbb{R}, \mu)$, where μ is the Lebesgue measure on $\mathbb{R}$. Use v to denote the positive measure on $\mathbb{R}$ induced by g and μ as

$$v(E) = \int_{\mathbb{R}} \chi_E \cdot g \,\mathrm{d}\mu.$$

One can check that v is a bounded measure with $v(\mathbb{R}) = \|f\|_{L^1(\mathbb{R},\mu)}$. Do the following:

1) Find a Lebesgue measurable subset E of $\mathbb{R}$, such that $g(E)$ is also Lebesgue measurable, $\mu(E) = 0$, and $\mu(g(E)) > 0$.

2) Prove that this measure v is absolutely continuous with respect to μ. That is, prove that $v \ll \mu$ does hold.

Remark: Let $(\mathbb{R}, \mu)$ be the standard Lebesgue measure space and let $h \in L^1(\mathbb{R}, \mu)$. Then we can derive a bounded measure v satisfying $v(E) = \int \chi_E \cdot h \,\mathrm{d}\mu$ for any μ-measurablet subet E. This construction of v is also denoted as $\mathrm{d}v = h\mathrm{d}\mu$. This v is also called the integral measure over μ with respect to f. From the definition of v, it is straightforward to check that $\mu(E) = 0$ implies $v(E) = 0$.

Thus $v \ll \mu$. Under this setup, by Radon-Nykodym Theorem, the measure v is differentiable with respect to the orginal measure μ, and $\frac{\mathrm{d}v}{\mathrm{d}\mu} = h$ (this h is also called the Radon-Nykodym derivative). Besides, in case we have another measure v' on $\mathbb{R}$ such that every Lebesgue measurable subset is also v'-measurable and $v' \ll \mu$, it follows that the integral of the Radon-Nykodym derivative of v' (with respect to μ) is exactly μ. For details on differentiation of measures, see Rudin book.

Remark: Let X be the standard Cantor set in $[0, 1]$, which is constructed by consecutively taking off the "middle one-thirds". For any separable (i.e, second countable) compact metric space Y, it is a well-known result (the standard proof of this result is quite understandable, if you are interested) that there exists (not surely uniquely though, in fact, "almost" always not unique) a continuous map $f : X \to Y$ such that $f(X) = Y$. In the language of category theory, this result just says that the Cantor set is an **initial object** in the catogory of separable compact metric spaces with $\mathrm{Hom}(A, B)$ being all the continuous surjective maps from A to B, where both A and B are separable compact metric spaces. This is also called the universal property of the Cantor set.

Furthur Question: Based on the definition of absolute continuity for functions, and the definition of absolute continuity for measures, noting Problem 6 and 6.5, when restricted to $\mathbb{R}$, can you make a guess on the relation between the absolute continuity of functions and the aboslute continuity of measures. For example, a guess like the following.

"

Let μ be the Lebesgue meausre on $\mathbb{R}$, and let v be another positve measure on $\mathbb{R}$ such that every Lebesgue measurable set is also v measurable, $v(\mathbb{R}) < \infty$ and blablah...(you can fill in what you need or what you think is appropriate).

Define

$$g : \mathbb{R} \longrightarrow \mathbb{R}, \quad t \mapsto \int_{\mathbb{R}} \chi_{(-\infty,t)} \, \mathrm{d}v.$$

Then v is absolutely continuous with respect to the Lebesgue measure μ if and only if the function g is absolutely continuous on $\mathbb{R}$. "

Prove or disprove your guess.

Note: This is not required as for homework. If you really want to crack this question, read the stuff related to differentiation of measures in Rudin book, which should give you enough information to solve this question.

7. (★) This is about some basic properties/facts of the test function $C_c^\infty(\mathbb{R})$ that you have seen in class. During the class, test functions are used to define weak derivatives, and they also serve as mollifying functions (a.k.a. mollifiers), etc.

1) Prove that $C_c^\infty(\mathbb{R}) = C^\infty(\mathbb{R}) \bigcap C_c(\mathbb{R})$.

2) Prove that $C_c^\infty(\mathbb{R})$ is an algebra. That is, for any $f, g \in C_c^\infty(\mathbb{R})$, and for any $\lambda \in \mathbb{R}$, $f + g$, $\lambda \cdot f$ and $f \cdot g$ are all in $C_c^\infty(\mathbb{R})$.

3) In $C_c^\infty(\mathbb{R})$, define a the pointwise convergence as $f_n \to f$ if and only if $f_n(x) \to f(x)$ as $n \to \infty$ for any $x \in \mathbb{R}$. Given this convergence, we can define clousure of a subset in $C_c^\infty(\mathbb{R})$. For $A \subset C_c^\infty(\mathbb{R})$, we say that A is closed if the closure of A under the above defined pointwise covergence is A itself. Then we say that $E \subset C_c^\infty(\mathbb{R})$ is open if and only if E^c is closed as in the sense above. Now, we can get a topology on $C_c^\infty(\mathbb{R})$ if we can verify that i) $\varnothing$ is an open set; ii) $C_c^\infty(\mathbb{R})$ is an open set; and iii) the unions (might be uncountably many) of open sets is still an open set.

Your job: Verify i), ii) and iii).

Remark: With the above defined topology, one can immedaitely check that $C_c^\infty(\mathbb{R})$ is a topological vector space. That is, under this topology, additions (of two functions) and scalar multiplications are all continuous.

4) In $C_c^\infty(\mathbb{R})$, construct $\{f_n\}_{n=1}^\infty$ and f, such that for each $x \in \mathbb{R}$, we have $f_n(x) \to f(x)$ while $n \to \infty$ but

$$\bigcup_{n=1}^\infty \operatorname{supp}(f_n) = \mathbb{R}.$$

Recall that for a continuous function $g\colon X \to \mathbb{R}$, $\operatorname{supp}(g)$ is defined to be the closure of $\{x \in X : g(x) \neq 0\}$.

5) Find $f \in C^\infty(\mathbb{R})$ such that f is not the constant function 0 and $f(0) = f'(0) = f^{(2)}(0) = f^{(3)}(0) = \cdots = 0$. Roughly speaking, find a function that is smooth everywhere but not analytic at 0. By a function f is analytic at a, we mean there exists $\delta > 0$, such that, when restricted to $(a - \delta, a + \delta)$, we have

$$f(x) = f(a) + f'(a)x + \frac{f''(a)}{2!}x^2 + \frac{f^{(3)}(a)}{3!}x^3 + \cdots .$$

Remark: If $f\colon \mathbb{C} \to \mathbb{C}$ is holomorphic and $f(0) = f'(0) = f^{(2)}(0) = f^{(3)}(0) = \cdots = 0$, using knowledge of complex analysis we already covered in class (to be more precise, being holomorphic is equivalent to being analytic), it is easy to show that $f \equiv 0$. This demonstarte how different real anlaysis and complex analysis are.

6) For any $f \in C^{\infty}(\mathbb{R})$ and for any $[a,b] \subset \mathbb{R}$, prove that we can find $g \in C_c^{\infty}(\mathbb{R})$ such that $f\big|_{[a,b]} = g\big|_{[a,b]}$.

7) For any $f \in C^{\infty}[a,b]$ and for any $[c,d] \subset [a,b]$ with $a < c$ and $d < b$, prove that we can find $g \in C_c^{\infty}((a,b))$ such that $\|g\|_{C^{\infty}[a,b]} \leq \|f\|_{C^{\infty}[a,b]}$ and $f\big|_{[a,b]} = g\big|_{[a,b]}$.

8) Give an example of $f \in C_c^{\infty}(\mathbb{R})$ with $\sup\{\|f^{(n)}\|_{\infty}\}_{n=0}^{\infty} = \infty$, where $f^{(0)}$ is just f itself and $\|\cdot\|_{\infty}$ is the supremum norm for continuous functions.

9) For a set of continuous functions $\{f_i \in C(\mathbb{R})\}_{i\in\mathcal{A}}$, we say that they are equi-continuous, if for any $\epsilon > 0$, there exists $\delta > 0$, such that for any $x, y \in \mathbb{R}$ with $|x-y| < \delta$ and for any $i \in \mathcal{A}$, we have $|f_i(x) - f_i(y)| < \epsilon$. Find a sequence of functions $g_n \in C_c^{\infty}(\mathbb{R})$ such that $g_n \to 0$ pointwise for all $x \in \mathbb{R}$, where the 0 denotes a constant function whose value is always 0, but $\{g_n\}$ is not equi-continuous.

10) In $C^{\infty}(0,1)$, let

$$BC^{\infty}(0,1) = \{f \in C^{\infty}(0,1)\colon \sup_{i\in\mathbb{N}_{\geq 0}} \|f^{(i)}\|_{\infty} < \infty\},$$

where $\|\cdot\|_{\infty}$ is the supremum norm of continuous functions on $(0,1)$.

For any $f \in BC^{\infty}(0,1)$, it is easy to check that f' is also in $BC^{\infty}(0,1)$. Besides, it is also easy to check that the map (derivative operator)

$$Der\colon BC^{\infty}(0,1) \longrightarrow BC^{\infty}(0,1),\ f \mapsto f'$$

is linear.

For any $f \in BC^{\infty}(0,1)$, we define

$$\|f\|_{BC^{\infty}(0,1)} = \sup_{i\in\mathbb{N}_{\geq 0}} \|f^{(i)}\|_{\infty}.$$

Recall that $C_c^{\infty}(0,1)$ is a set of all the functions g such that $g \in C^{\infty}(0,1)$ and $\mathrm{supp}(g)$ is a compact subset of $(0,1)$. According to 8) of this problem, for $g \in C_c^\infty(0,1)$, $\sup_{i\in\mathbb{N}_{\geq 0}} \left\|g^{(i)}\right\|_\infty$ might be ∞. Let

$$BC_c^\infty(0,1) = \{g \in C_c^\infty(0,1)\colon \sup_{i\in\mathbb{N}_{\geq 0}} \left\|g^{(i)}\right\|_\infty < \infty\},$$

For any $g \in BC_c^\infty(0,1)$, we define, noting that $BC_c^\infty(0,1) \subset BC^\infty(0,1)$, $\|g\|_{BC_c^\infty(0,1)}$ to be $\|g\|_{BC^\infty(0,1)}$.

Define the integration operator

$$Int\colon BC_c^\infty(0,1) \longrightarrow BC^\infty(0,1), \quad g \mapsto h, \text{ where } h(t) = \int_0^t g(x)\, \mathrm{d}x.$$

Easy to check that Int is well-defined. In other words, for any $g \in BC_c^\infty(0,1)$, the integral function h above is in $BC^\infty(0,1)$.

i) Check that this above defined $\|\cdot\|_{BC^\infty(0,1)}$ is really a norm on $BC^\infty(0,1)$.

ii) Prove that this linear operator Der (derivative operator) from $(BC^\infty(0,1), \|\cdot\|_{BC^\infty(0,1)})$ to itself is bounded.

iii) Regarding the linear operator Int (integral operator) as

$$Int\colon (BC_c^\infty(0,1), \|\cdot\|_{BC_c^\infty(0,1)}) \longrightarrow (BC^\infty(0,1), \|\cdot\|_{BC^\infty(0,1)}),$$

prove that this operator is bounded.

8. ($\star$) Let μ be the Lebesgue measure on $\mathbb{R}^n$, and let $p \in [1, +\infty)$. Let $f \in L^p(\mathbb{R}^n, \mu)$. Let $g \in C_c^\infty(\mathbb{R}^n)$ such that $g \geq 0$ and $\int_{\mathbb{R}^n} g\, \mathrm{d}\mu = M < \infty$. Prove that

$$\|f * g\|_{L^p(\mathbb{R}^n,\mu)} \leq M \cdot \|f\|_{L^p(\mathbb{R}^n,\mu)}$$

Hint: Check that $f * g = (M \cdot f) * \frac{g}{M}$. Then apply the result we proved in class, which is about the case $M = 1$.

9. ($\star$) Let φ be a function in $C_c^\infty(\mathbb{R})$ such that $\varphi \geq 0$ and $\int_{\mathbb{R}} \varphi\, \mathrm{d}\mu = 1$, where μ is the Lebesgue measure on $\mathbb{R}$. For any $(a,b) \subset \mathbb{R}$, it is clear that $\chi_{(a,b)} \in L^\infty(\mathbb{R}, \mu)$. According to what is covered in class, we know that that $\chi_{(a,b)} * \varphi$ is in $C^\infty(\mathbb{R})$.

1) Show that $\chi_{(a,b)} * \varphi \in C_c^\infty(\mathbb{R})$.

2) Assuming $\operatorname{supp}(\varphi) \subset [-r, r]$ for some $r > 0$ and $b - a \geq 2r$, show that

$$\left\|\chi_{(a,b)} * \varphi - \chi_{(a,b)}\right\|_{L^\infty(\mathbb{R},\mu)} \geq 1/2.$$

10. ($\star$) Let μ be the Lebesgue measure on $\mathbb{R}$. We can construct a product measure on $\mathbb{R} \times \mathbb{R}$ via μ. Use μ^2 to denote this product measure on $\mathbb{R}^2$. According to lectures in class, we start with a semiring $\mathcal{R}$ which is made up of all such elements as $X \times Y$, where both X and Y are μ-measurable, and assume $\mu^2(X \times Y) = \mu(X) \times \mu(Y)$. Then we applied the Caratheodory Extension Theorem to get the product measure. For ease of use, we complete the measure in the end. In other words, we assume that μ^2 is a complete measure.

For a subset $E \subset \mathbb{R}^2$ such that E is μ^2-measurable, it is easy (following the construction of μ^2 and using the translation invariance property of the Lebesgue measure μ) to check that for any $(a, b) \in \mathbb{R}^2$, $E + (a, b)$ is also μ^2-measurable and its measure is the same as $\mu^2(E)$. Here $E + (a, b)$ denotes $\{(x, y) + (a, b) \colon (x, y) \in E\}$.

Your job: For any two by two real matrix A with $\det(A) \neq 0$, and for any μ^2-measurable subset E in $\mathbb{R}^2$, show that $A \cdot E$ is also μ^2-measurable and

$$\mu^2(A \cdot E) = |\det(A)| \cdot \mu^2(E),$$

where $A \cdot E$ denotes $\{A \cdot e \colon e \in E\}$.

Hint: As μ^2 is a σ-finite measure (why?), without loss of generality, we can assume that $\mu^2(E) < \infty$. As μ^2 is outer regular (follows from Caratheodory Extension Theorem, why?), we can find a Borel set E_b such that $E \subset E_b$ and $\mu^2(E) = \mu^2(E_b)$. As we already assumed $\mu^2(E) < \infty$, here we can, starting from $\mu^2(E) = \mu^2(E_b)$, get $\mu^2(E_b - E) = 0$. Then, you just need to show things like the following: i) In $\mathbb{R}^2$, multiplying by A maps Borel sets to Borel sets. ii) For any Borel set B in $\mathbb{R}^2$ with $\mu^2(B) < \infty$, $\mu^2(A \cdot B) = |\det(A)| \cdot \mu^2(B)$. iii) Multiplying by A maps measurable zero sets to measure zero sets. In fact, as each measure zero set can be covered by a Borel set with measure arbitrarily small, and using the result achieved in ii), and note that $|\det(A)| < \infty$, the result needed for iii) follows easily. The hint ends here to ensure that you still have something left for discover or adventure.

Remark: Assuming that we have already proved the measurability of $A \cdot E$, the following proof of $\mu^2(A \cdot E) = |\det(A)| \cdot \mu^2(E)$ is *not* acceptable.

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As μ^2 is a σ-finite measure, without loss of generality, we can assume that $\mu^2(E) < \infty$. That is $\chi_E \in L^1(\mathbb{R}^2, \mu^2)$.

Assume $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. For $x, y \in \mathbb{R}$, we define x', y' as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

That is, $\begin{pmatrix} x' \\ y' \end{pmatrix} = A \cdot \begin{pmatrix} x \\ y \end{pmatrix}$. Then, noting that $(x', y') \in A \cdot E$ if and only if $(x, y) \in E$, we have

$$\begin{aligned}
\mu^2(A \cdot E) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi_{A \cdot E}((x', y'))\, \mathrm{d}x' \mathrm{d}y' && \text{[Fubini's Theorem]} \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi_E(x, y)\, \mathrm{d}x' \mathrm{d}y' \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi_E(x, y) \cdot |\det(A)|\, \mathrm{d}x \mathrm{d}y \\
&= |\det(A)| \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi_E(x, y)\, \mathrm{d}x \mathrm{d}y \\
&= |\det(A)| \cdot \mu^2(E).
\end{aligned}$$

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The reason is simple: the proof above relies heavily on “$\mathrm{d}x' \mathrm{d}y' = |\det(A)|\, \mathrm{d}x \mathrm{d}y$”, which is not yet proved for Lebesgue integration of the Lebesgue measures. Indeed, this Problem 10 is exactly about the proof of this fact (why?).

10.5 ($\star$) Based on the result in Problem 10, do the following:

Let μ be the Lebesgue measure on $\mathbb{R}^2$ and let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function in $L^1(\mathbb{R}^2, \mu)$. For any $A \in M_2(\mathbb{R})$ with $\det(A) \neq 0$, we can regard A as a linear map from $\mathbb{R}^2$ to itself. Use $f \circ A$ to denote the function from $\mathbb{R}^2$ to $\mathbb{R}$ with $(f \circ A)(x) = f(Ax)$ for all $x \in \mathbb{R}^2$.

1) Prove that $f \circ A$ is also a measurable function.

2) Prove that $f \circ A \in L^1(\mathbb{R}^2, \mu)$ and

$$\|f \circ A\|_{L^1} = \frac{\|f\|_{L^1}}{|\det(A)|}.$$

Remark: I just give the statement for $\mathbb{R}^2$, surely similar things hold for $\mathbb{R}^n$ with any $n \in \mathbb{N}_{\geq 1}$.

Hint: Approximate the function f above by simple functions.

10.6 ($\star$) Let μ be the Lebesgue measure on $\mathbb{R}^2$ and let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a positive measurable function on $(\mathbb{R}^2, \mu)$. Note that $\int_{\mathbb{R}^2} f \, d\mu$ always exists, which is either a finite non-negative value or $+\infty$. For any $A \in M_2(\mathbb{R})$ with $\det(A) \neq 0$, we can regard A as a linear map from $\mathbb{R}^2$ to itself. Use $f \circ A$ to denote the function from $\mathbb{R}^2$ to $\mathbb{R}$ with $(f \circ A)(x) = f(Ax)$ for all $x \in \mathbb{R}^2$. Similar to 1) fo the previous problem, we can show that $f \circ A$ is also measurable. Thus we have $f \circ A$ is positive and measurable. It then follows that $\int_{\mathbb{R}^2} f \circ A \, d\mu$ always exits, which is a either a finite non-negative value or $+\infty$.

Prove that we always have

$$\int_{\mathbb{R}^2} f \circ A \, d\mu = \frac{\int_{\mathbb{R}^2} f \, d\mu}{|\det(A)|}.$$

Note that both sides above might be $+\infty$.

Hint: In case $\int_{\mathbb{R}^2} f \, d\mu \in [0, \infty)$, it is already covered by the result of the previous problem. So what really remains to be proved is just the following: "If $\int_{\mathbb{R}^2} f \, d\mu = \infty$, then $\int_{\mathbb{R}^2} f \circ A \, d\mu = \infty$", which is not so hard to prove if you check against definitions of Lebesgue integrations.

11. ($\star$) Let $f_1, \cdots, f_n$ be measurable functions in (X, μ) such that $f_i \in L^{p_i}(X, \mu)$ for $1 \leq i \leq n$, where each $p_i > 0$. Let

$$\frac{1}{r} = \sum_{i=1}^{n} \frac{1}{p_i}.$$

Prove that

$$\left\| \prod_{i=1}^{n} f_i \right\|_{L^r} \leq \prod_{i=1}^{n} \|f_i\|_{L^{p_i}}$$

Hint: In case $n = 2$, the result is already proved in class (or assigned as some pleasant exercise), which was then called the generalized Hölder inequality. The gap between the case of $n > 2$ and $n = 2$ is nothing but some induction work.