HW # 2 Due on May 3rd 2016.

1. (★) Let X be a normed space which is not complete and let Y be a Banach space. Let $f\colon X \to Y$ be a bounded linear map.Use X' to denote the completion of X. We can extend f uniquely to a bounded linear map

$$f'\colon X' \longrightarrow Y$$

such that $f'|_X = f$ and $\|f'\|_{X'\to Y} = \|f\|_{X\to Y}$. Give such details on how to define X', how to extend f to f', and why $f'|_X = f$ and $\|f'\|_{X'\to Y} = \|f\|_{X\to Y}$.

2. (★) Let $c_0(\mathbb{N})$ be defined as in assignment #1, equipped with the supremum norm. Let $l^1(\mathbb{N})$ be the one defined in class, which is a Banach space equipped with the l^1 norm. Let $l^\infty(\mathbb{N})$ be the Banach space covered in class, equipped with the supremum norm. Prove that both $c_0(\mathbb{N})$ and $l^1(\mathbb{N})$ are separable. Then prove that $l^\infty(\mathbb{N})$ is *not* separable.

Note: Let μ be the Lebesgue measure on $[0,1]$, we know (one of the homework problems last term) that $L^\infty([0,1])$ is also *not* separable.

3. (★★) Let X be a Banach space and let M be a closed linear subspace of X. Define the quotient space X/M to be

$$X/M = \{x + M \colon x \in X\},$$

where $x + M = y + M$ if $x - y \in M$. For X/M, define the norm as

$$\|x + M\| = \inf_{m\in M} \|x + m\|.$$

Prove that X/M is also a Banach space under this above defined norm. This space is called a quotient space.

3.1 (★) In the Banach space $l^\infty(\mathbb{N})$, consider $c_0(\mathbb{N})$ as defined in assignment # 1. Define a subspace D in $l^\infty(\mathbb{N})$ to be the algebraic span of $\{e_i\}_{i=1}^\infty$, where e_i is the element in l^∞ whose i-th co-ordinate is

1 and j-th co-ordinate is 0 for all $j \neq i$. Prove that the closure of D is exactly c_0.

3.2 **($\star$)** Construct two bounded linear operator P and Q from the Banach space $l^\infty(\mathbb{N})$ to $\mathbb{C}$, such that $P \neq Q$ and P agrees with Q when restricted to the algebraic span of $\{e_i\}_{i=1}^{\infty}$, where e_i is defined as above.

4. **($\star$)** For a complex function $f\colon \Omega \to \mathbb{C}$, where Ω is a region, and for $z_0 \in \Omega$, if

$$\left.\frac{\partial f}{\partial \bar{z}}\right|_{z=z_0} = 0,$$

does it follow that f is differentiable at $z = z_0$? If so, prove it. If not, give a counter example.

Note: $\dfrac{\partial f}{\partial \bar{z}}$ is defined to be $\dfrac{1}{2}\dfrac{\partial f}{\partial x} - \dfrac{1}{2i}\dfrac{\partial f}{\partial y}$.

Hint: For $g\colon \mathbb{R}^2 \to \mathbb{R}$, what is the difference between the following two statements:

i) g is differentiable at (a, b).

ii) $\dfrac{\partial g}{\partial x}$ and $\dfrac{\partial g}{\partial y}$ both exists at (a, b).

4.1 **($\bigstar$)** For a complex function $f\colon \Omega \to \mathbb{C}$, where Ω is a region, and for $z_0 \in \Omega$, define

$$\left.\frac{\partial f}{\partial z}\right|_{z=z_0} = \left.\frac{\partial f}{\partial x}\right|_{z=z_0} + \left.\frac{1}{2i}\frac{\partial f}{\partial y}\right|_{z=z_0}.$$

If $\left.\dfrac{\partial f}{\partial z}\right|_{z=z_0}$ exists, does it follow that $f'(z_0)$ exits? That is, does it follow that f is differentiable at z_0 (that is, $f'(z_0)$ exists)?

On the other hand, if f is differentiable at z_0, prove that the derivative $f'(z_0)$ equals $\left.\dfrac{\partial f}{\partial z}\right|_{z=z_0}$.

5. **($\bigstar$)** For a complex function $f\colon \Omega \to \mathbb{C}$, where Ω is a region. Prove that the following are equivalent:

i) f is holomorphic in Ω.

ii) f is C^1 in Ω and $\left.\dfrac{\partial f}{\partial \bar{z}}\right|_{z=z_0} = 0$ for all $z_0 \in \Omega$.

6. (★★) In $\mathbb{C}$, consider the region Ω_1 and Ω_2 defined as

$$\Omega_1 = \{x + iy\colon x, y \in \mathbb{R}, x^2 + y^2 < 1\} \text{ and } \Omega_2 = \{x + iy\colon x, y \in \mathbb{R}, x^2 + (2y)^2 < 1\}.$$

It is clear that both Ω_1, Ω_2 and their closures are all inside

$$E = \{x + iy\colon \max\{|x|, |y|\} < 100\}.$$

Prove that we cannot find a holomophic map f from the open set E to itself (does not need to be one-to-one or onto) such that $f(\partial\Omega_1) = \partial\Omega_2$ and $f(0) = 0$.

Note: If we consider Ω_1 and Ω_2, according to the Riemann mapping theorem, there does exits conformal/bi-holomorphic mapping between Ω_1 and Ω_2.

6.5 $(\star)$ In $\mathbb{C}$, let Ω_1 and Ω_2 be defined the same as above. Prove that there exists a biholomorphic/conformal map

$$f\colon \Omega_1 \longrightarrow \Omega_2$$

such that $f(0) = 0$. You can use the Riemann mapping theorem direclty, which states that any simply connected **proper** open subset of $\mathbb{C}$ is biholomorphic/conformal equivalent to the open unit disk of $\mathbb{C}$.

6.6 $(\star)$ In the region $\Omega = C \setminus \{0\}$, consider the function

$$f\colon \Omega \longrightarrow \mathbb{C}, \quad z \mapsto z + \frac{1}{z}.$$

It is easy to see that f is holomorphic on Ω.

Let $S = \{z \in \Omega\colon |z| = 2\}$. Prove that $f(S)$ is an ellipse.

Hint: It should be easy.

Remark: In the setup of Problem 6, the domain for holomorphic functions are regions without any holes inside. For this Problem 6.6, there is a hole $z = 0$ in the domain. That one single hole makes all the differences.

7. (★) Let D be a non-empty open set in $\mathbb{C}$. If D is connected, prove that D is path connected.

Hint: For any $a \in D$, use E_a to denote $\{x \in D\colon \text{There exists a path in } D \text{ connecting } a \text{ and } x\}$. First show that each E_a is open. Then for any $a, b \in D$, show that we do not have much choice on the relation between the two sets E_a and E_b ...

Remark: Based on this result, any region in $\mathbb{C}$, which can typically be used as the domain for holomorphic functions, must be path-connected.

7.5 (⋆) Let X be a topological space. We say that X is locally path-connected if there exists a topological basis $\{A_i\}_{i \in \mathcal{A}}$ of this topological space X such that each A_i is path-connected.

Prove the following: If the topological space X is connected and locally path-connected, then X is path-connected. That is, for topological spaces that are locally path-connected, "connectedness" and "path-connectedness" are just equivalent.

8. (★⋆) Let M be a m-dimentional real differential manifold. We say that the differential manifold M is orientable, or equivalently, M allows an orientation, if there exists a degree m exterior differential form Ω on M, such that Ω avoids zero everywhere in M. By Ω is a degree m exterior form on M, we mean Ω is a continuous section of $\wedge_{i=1}^m T^*M$, where T^*M is the cotangent bundle of M and $\wedge$ is the outer derivative contatenation.

As M is a differential manifold, we can write $M = \bigcup_{i \in \mathcal{A}} D_i$, where each D_i is diffeomorphic to an open subset in $\mathbb{R}^m$. Use φ_i to denote the diffeomorphism from D_i to the corresponding open set in $\mathbb{R}^n$. Note that φD_i is an open subset in $\mathbb{R}^n$. Thus $\varphi(D_i)$ is orientable (why?).

Prove that M is orientable if and only if there exists a class of m-forms $\{\omega_i\}_{i \in \mathcal{A}}$, such that each ω_i is defined locally on $\varphi(D_i)$, ω_i avoids 0 everywhere in $\varphi(D_i)$, and for any $i, j \in \mathcal{A}$ with $D_i \bigcap D_j \neq \emptyset$, the two forms ω_i and ω_j are compatible. By "two forms ω_i and ω_j are compatible", we mean the following: Restricted on $D_i \bigcap D_j$, if we write $\omega_i = f \cdot dx_1 \wedge \cdots \wedge dx_n$ and $\omega_j = g \cdot dy_1 \wedge \cdots \wedge y_n$, then

$$\frac{f}{g} \cdot \det \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \end{pmatrix} > 0 \ \text{ everywhere in } \ D_i \bigcap D_j.$$

Hint: Partition of the constant function 1, as mentioned in class. This technique is useful for both "gluing up" local properties into properties on the whole manifold and "passing" properties on the whole manifold to local open neighborhoods.

8.1 ($\star$) Let M be any Riemann surface. That is, M is a one-dimensional complex manifold. It can be regarded as a two dimensional real manifold in the natural way. Besides, as holomorphic maps are automatically C^∞ maps, it follows that we can regard M as a *smooth* two dimensional real manifold. With the terminology of orientation defined as above, prove that the Riemann surface M, when viewed as a two dimensional real manifold, does allow an orientation.

9. ($\bigstar$) Let Ω be a region in $\mathbb{C}$, and let E be a Banach space. Let

$$f : \Omega \longrightarrow E$$

be a map. We say that this map f is weakly holomorphic, if for any $L \in E^*$, the map

$$L \circ f : \Omega \longrightarrow \mathbb{C}$$

is a holomorphic function in the usual sense. We say that the map f is strongly holomorphic, if for any $a \in \Omega$, f is differentiable at a. By "f is differentiable at a", we mean there exists $S_a \in E$, such that

$$f(z) - f(a) = S_a \cdot (z - a) + o(z - a) \quad \text{as} \quad z \to a.$$

That is,

$$\lim_{z \to a} \frac{\|f(z) - f(a) - S_a \cdot (z - a)\|}{|z - a|} = 0.$$

In this case, S_a is called the derivative of f at a, which is denoted as $f'(a) = S_a$.

Do the following:

1) Show that if such S_a exists, then it is THE derivative of f at a. That is, the derivative of f at a, if exists, must be unique.

2) If $f : \Omega \to E$ is strongly holomorphic on Ω, prove that f is also weakly holomorphic. *Note: This should be kind of straightforward. Show the details.*

9.5 (★★⋆) With the same setup as above, prove that if $f \colon \Omega \to E$ is weakly holomorphic on Ω, then f is also strongly holomorphic on Ω.

Hint: Just recall how we prove in class that if a function $g \colon \Omega \to \mathbb{C}$ is holomorphic, then it is analytic. The main thing in the proof is to write g as certain proper form of the complex integration of a fraction with the numerator being a L^1 "function" and the denominator being things like $\xi - z$, where ξ is the phony/nominal variable used for the complex integration.

Keywords of some math you might need on the way: Vector-valued Riemann integrations, Hahn-Banach Theorem, power series with coefficients taken from a Banach space, etc.