HW # 1 Due on April 8th 2016.

1. ($\star$) Let H be an infinite dimensional Hilbert space and let $\{x_i\}_{i\in\mathcal{A}}$ be a Hamel basis of H. Prove that the index set $\mathcal{A}$ must be an uncountable set. In other words, for any infinite dimensional Hilbert space, each Hamel basis of it must be uncountable.

Hint: You might want to apply the Gram-Schmidt process first, and you might eventually need the Axiom of Independent Choice. Recall that we already proved in class that for any orthonormal basis in the infinite dimensional Hilbert space, it can *not* be a Hamel basis (why?).

2. ($\star$) Let X and Y be complex Banach spaces. Define

$$B(X,Y) = \{f\colon X \to Y \mid f \text{ is a bounded linear operator}\}.$$

i) Prove that if for any $\lambda \in \mathbb{C}$, and any $f, g \in B(X,Y)$, it follows that $\lambda\colon f$ and $f+g$ are also in $B(X,Y)$.

ii) Prove that $B(X,Y)$ is a normed space, where the norm of any f in $B(X,Y)$ is just the operator norm.

iii) Prove that $B(X,Y)$ is a complex Banach space.

iv) In the statement, if we just ask X to be a complex normed space, and still ask Y to be a complex Banach space, does it follow that $B(X,Y)$ is still a complex Banach space? Why or why not?

3. ($\bigstar$) Let X be a complex Banach space. We use X^* to denote $B(X,\mathbb{C})$, where $B(X,\mathbb{C})$ is defined as in the previous problem. X^* is called the dual space of X. We use $C[0,1]$ to denote the Banach space of all the continuous functions on $[0,1]$, where the norm is defined as

$$\|f\| = \sup_{x\in[0,1]} |f(x)|, \ \forall f \in C[0,1].$$

We say that a Banach space X is reflexive if $X \cong X^{**}$ via the natural embedding (in other words, the natural embedding is an isomorphism in the category of Banach spaces), where the double dual X^{**} is defined to be $(X^*)^*$.

Prove that $C[0,1]$ is *not* reflexive.

4. ($\star$) Let X be a real linear space and let $p\colon X \to \mathbb{R}_{\geq 0}$ be positively homogeneous and subadditive. Prove that

$$\{x \in X \colon p(x) \leq 1\}$$

is a convex subset and 0 is an interior point of it.

5. ($\star$) (Hahn-Banach Theorem) Let X be a real linear space and let $p\colon X \to \mathbb{R}_{\geq 0}$ be positively homogeneous and subadditive. Let M be a linear subspace of X and let $f_0\colon M \to \mathbb{R}$ be a linear functional on M satisfying

$$f_0(x) \leq p(x), \ \forall\, x \in M.$$

Prove that there exists a linear functional $f\colon X \to \mathbb{R}$ such that

$$f\,|_M = f_0 \ \text{ and } \ f(x) \leq p(x)\ \forall\, x \in X.$$

6. ($\star$) For metric spaces X and Y, with d_X and d_Y be the corresponding metrics on them, consider a map/function $f\colon X \to Y$. Choose $a \in X$. Prove that the following two statements are equivalent:

i) For any sequence of points $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \to a$, we have $f(x_n) \to f(a)$. In other words, for any sequence of points $\{x_n\}_{n\in\mathbb{N}}$ satisfying $d_X(x_n, a) \to 0$ as $n \to \infty$, it follows that $d_Y(f(x_n), f(a)) \to 0$ as $n \to \infty$.

ii) For any $\epsilon > 0$, there exists $\delta > 0$, such that for every $x \in X$ with $d_X(x, a) < \delta$, we have $d_Y(f(x), f(a)) < \epsilon$.

Hint: The hard direction is to show that i) implies ii). Given i), to show ii) holds, one approach is to assume the opposite of ii), and show how that will violate i).

6.5 ($\star$) For metric spaces X and Y, with d_X and d_Y be the corresponding metrics on them, consider a map/function $f\colon X \to Y$. Choose $a \in X$. Consider the following to statements:

i) For any open set E in Y such that $f(a) \in E$, $f^{-1}(E)$ is open.

ii) For any $\epsilon > 0$, there exists $\delta > 0$, such that for every $x \in X$ with $d_X(x, a) < \delta$, we have $d_Y(f(x), f(a)) < \epsilon$.

Prove that i) implies ii). Also, give an example demonstrating that ii) might not imply i).

7. ($\star$) For topological spaces X and Y, and for a map $f\colon X \to Y$, we say that f is continuous at $x = a$ for some $a \in X$, if for any neighborhood D of $f(a)$, its preimage $f^{-1}(D)$ is still a neighborhood of a. We call this the "topological definition" of f being continuous at a point.

For metric spaces (X, d_X) and (Y, d_Y), and for a map $f\colon X \to Y$, we say that f is continuous at $x = a$ for some $a \in X$, we say that f is continuous at $x = a$ for certain $a \in X$, if for any $\epsilon > 0$, there exists $\delta > 0$, such that for all x' satisfying $d_X(x', a) < \delta$, we have $d_Y(f(x'), f(a)) < \epsilon$. We call this the "$\epsilon$-$\delta$ definition" of f being continuous at a point.

Let $f\colon X \to Y$ be a map between two metric spaces and let $a \in X$. Prove that f is continuous at $x = a$ in the sense of the "topological definition" above if and only if f is continuous at $x = a$ in the sense of the "ϵ-δ" definition above.

8. ($\bigstar$) Use c_0 to denote the following linear space

$$\{(x_1, x_2, \cdots)\colon x_i \in \mathbb{C} \text{ for all } i, \text{ and } \lim_{i \to \infty} x_i = 0\}.$$

For any $x = (x_1, x_2, \cdots) \in c_0$, define $\|x\|_{c_0}$ to be

$$\|x\|_{c_0} = \sup_{i \in \mathbb{N}_{\geq 1}} |x_i|.$$

One can check that this $\|\cdot\|_{c_0}$ is really a norm. It is also not hard to check that C_0 equipped with such a norm is a Banach space (why?).

We use l^1 to denote the following linear space

$$\{(x_1, x_2, \cdots)\colon x_i \in \mathbb{C} \text{ for all } i, \text{ and } \sum_{i=1}^{\infty} |x_i| < \infty\}.$$

For any $x = (x_1, x_2, \cdots) \in l^1$, define $\|x\|_{l^1}$ to be

$$\|x\|_{l^1} = \sum_{i=1}^{\infty} |x_i|.$$

One can check that this $\|\cdot\|_{l^1}$ is really a norm. It is also not hard to check that l^1 equipped with such a norm is a Banach space (why?).

Prove that $(c_0)^* \cong l^1$. That is, there exists a map $\varphi\colon (c_0)^* \longrightarrow l^1$, such that φ is an isomorphism between two linear spaces $(c_0)^*$ and l^1, and $\|\varphi^*(f)\|_{l^1} = \|f\|$ for all $f \in (c_0)^*$, where $\|f\|$ is just the operator norm.

Hint: You might want to construct a linear isomorphism first. Then check/hope that the isomorphism you constructed preserves the norm.

9. ($\bigstar$) Construct a topological space X which is *not* compact, and for each *sequence* in X, we can find a convergent subsequence of it.

Hint: It is clear that such X is not metrizable.

10. ($\star$) Let X be a set and use $\mathcal{P}(X)$ to denote the power set of X. For any non-empty $\mathcal{G} \subset \mathcal{P}(X)$, prove that the following are equivalent:

i) There exists a filter $\mathcal{F}$ on X, such that $\mathcal{G} \subset \mathcal{F}$.

ii) For any finite many elements $g_1, \cdots, g_n$ in $\mathcal{G}$, their intersection $g_1 \cap \cdots \cap g_n$ is not the empty set.

10.5 ($\star$) Let X be a set and use $\mathcal{P}(X)$ to denote the power set of X. For any collection of filters $\{\mathcal{F}_i\}_{i \in A}$ on X, prove that their intersection, while regarding each $\mathcal{F}_i$ as a subset of $\mathcal{P}(X)$, is still a filter. That is, you need to prove that $\bigcap_{i \in A} \mathcal{F}_i$ is still a filter.

Remark: Based on the results of 10 and 10.5, it follows that given any subset $\mathcal{G}$ of $\mathcal{P}(X)$ satisfying the condition that all those finite intersections of elements inside $\mathcal{G}$ are non-empty, we can find a smallest filter that contains this $\mathcal{G}$. This filter is called the filter generated by $\mathcal{G}$.

11. ($\bigstar$) Let X be a set and let $\mathcal{F}$ be a filter on it. Prove that the $\mathcal{F}$ is an ultrafilter if and only if for any filter $\mathcal{H}$ satisfying $\mathcal{F} \subset \mathcal{H}$, it follows that $\mathcal{F} = \mathcal{H}$ (that is, the filter $\mathcal{F}$ is maximal).

12. ($\star$) Consider the set $X = \mathbb{Z}$. Define

$$\mathcal{S} = \{E \subset X \colon E^c \text{ is a finite set}\}.$$

i) Prove that $\mathcal{S}$ is a filter. In fact, it is called the cofinite filter.

ii) Show that $\mathcal{S}$ is not a ultrafilter.

iii) For any subset D of $\mathbb{Z}$, prove (*it is not hard*) that $\mathcal{S} \neq \mathcal{P}(X)_{\geq D}$, where $\mathcal{P}(X)_{\geq D}$ is defined to be

$$\{G \in \mathcal{P}(X) \colon G \supset D\}.$$

iv) Start from the filter $\mathcal{S}$, by extending this filter to a larger filter whenever possible and applying the Zorn's Lemma, and we can prove that there exists (not surely unique though) an ultrafilter $\mathcal{U}$ such that $\mathcal{S} \subset \mathcal{U}$. For any such ultrafilter $\mathcal{U}$, and for any $a \in X = \mathbb{Z}$, prove that

$$\mathcal{U} \neq \mathcal{P}(X)_{\geq \{a\}}.$$

13. ($\star$) In the class, for a topological space X, we covered "the limits of nets" in X, and we also covered "limits of filters" in X. Starting from "limits of nets", for any subset/subspace D of X, we defined the "closure of D" in the sense of nets, and proved that a subset D is a closed set in the topological space X if and only if the closure of D in the sense of nets is D itself.

Similar fact holds in the sense of "closure in sense of filters". But we need to define "closure in sense of filters" first, and this problem is about such details.

Question: Let X be a topological space and let D be a subset of X. For a point $x \in X$, how to describe such things as "an ultra filter in D converges to x"? As D might not be X, we can not do a word by word copy/paste of the original definition of ultrafilter convergence in X (even in case $x \in D$). The main thing here is that a filter in D might not be a filter in X at all.

i) Given a non-empty topological space X, let D be a non-empty subset of X. For a filter $\mathcal{F}$ on D, it might not be a filter on X. Give such an example.

In spite of the obvious trouble as indicated above, we can still manage to define the converge of a filter on a subset/subspace like this:

Given a non-empty topological space X, let D be a non-empty subset of X. For a filter $\mathcal{F}$ on D, regarding it as a subset of $\mathcal{P}(X)$, as for any finitely many $E_1, \cdots, E_n$ in $\mathcal{F}$, we always have $\bigcap_{i=1}^{n} E_i \neq \varnothing$, such $\mathcal{F}$ will generate a filter on X, which is denoted by $\mathcal{F}'$. For this filter $\mathcal{F}$ on D and for a point $x \in X$, we say that the filter $\mathcal{F}$ on D converges to x, denoted as $\mathcal{F} \to x$, if the filter $\mathcal{F}'$ on X converges to x, which is already defined in class. That is, we define $\mathcal{F} \to x$ as $\mathcal{F}' \to x$.

With this definition in mind, do ii).

ii) In a topological space X, let D be a non-empty subset of X, and let $x \in X$. Prove that x is a limit point of D if and only if there exists a filter $\mathcal{F}$ on D, such that $\mathcal{F} \to x$. Note that we say x is a limit point of D, if for any neighborhood N of x, we have $N \cap D \neq \varnothing$.

Remark: From the result in ii), for any subset D of a topological space X, its topological closure agrees with its closure in the sense of convergence filters in D, which then implies that D is closed if and only if the closure of D in sense of filters is D itself.