HW # 6 Due on Jan. 6th 2016.

1. Let $(X, \mathcal{M}, \mu)$ be a measure space. For $p \in [1, +\infty]$, assume $\{f_n\}$ is a sequence of functions in $L^p(X, \mathcal{M}, \mu)$ and f is a measurable function such that $f_n \xrightarrow{\mu} f$. This is the set-up shared by all the questions below.

i) In case $p = +\infty$, does it follow that f is in $L^\infty(X)$? If so, prove it. If not, give a counter example.

ii) Assume in extra that $\mu(X) < \infty$, do i).

iii) For a given $p \in [1, +\infty)$, does it follow that f is in $L^p(X)$? Does your answer depend on this $p \in [1, +\infty)$? Why? A proof or counter example is needed.

iv) Assume in extra that $\mu(X) < \infty$, do iii).

v) With no such requirements as $\mu(X) < \infty$, for $p \in [1, +\infty)$, if there exists $g \in L^p(X)$, such that $|f_n| \le |g|$ for all $n \in \mathbb{N}_{\ge 1}$, does it follow that $f \in L^p(X)$? If so, prove it. If not, give a counter example.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $p \in (1, +\infty)$. If there are functions $\{f_n\}_{n \in \mathbb{N}}$ and f in $L^p(X)$, such that $\|f_n - f\|_p \to 0$ as $n \to \infty$, prove that $f_n \xrightarrow{\mu} f$ as $n \to \infty$.

2.5 Let $[0, 1]$ be equipped with the Lebesgue measure μ. Let $p \in (1, +\infty)$. Construct a sequence of measurable functions f_n, such that $\|f_n - 0\|_p \to 0$, but we do not have $f_n \to 0$ pointwise a.e. on $[0, 1]$. *Note: Recall the stuff covered in class, this result also implies that there is a subsequence of $\{f_n\}$ that converges pointwise a.e. to a measurable function.*

2.6 Let $[0, 1]$ be equipped with the Lebesgue measure μ. Fix $p \in (1, +\infty)$. Construct a measurable function f and a sequence of measurable functions f_n, such that $f_n \to f$ pointwise, but we do not have $\|f_n - f\|_p \to 0$ as $n \to \infty$.

2.7 Let $[0, 1]$ be equipped with the Lebesgue measure μ. Fix $p \in (1, +\infty)$. Construct a measurable function f and a sequence of measurable functions f_n, such that $f_n \xrightarrow{\mu} f$, but we do not have $\|f_n - f\|_p \to 0$ as $n \to \infty$.

3. Let H be a Hilbert space. Prove/check/verify/whatever the parallelogram law. That is, for any

$x, y \in H$, we have

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

4. Let H be a *complex* Hilbert space. Prove the polarization identify. That is, for any $x, y \in H$, we have

$$\langle x, y \rangle = \frac{1}{4} \sum_{j=1}^{4} (\sqrt{-1})^j \left\| x + \sqrt{-1}^j y \right\|^2.$$

5. Let M be a normed space. Use D to denote $\{x \in M \colon \|x\| \leq 1\}$. Prove that D is a convex set. *Note: Combined with the "unit disk" we described for L^p-spaces in class, this explains why $L^p(X)$ might not be a normed space when $p \in (0,1)$.*

6. In the first assignment, we proved that $\mathbb{R}^2$ cannot be covered by countably many straight lines. Now, use the Baire Category Theorem to prove this result again.

Remark 1: In the first assignment/homework, you are supposed to give a rudimentary proof of the fact that "$\mathbb{R}^2$ cannot be covered by countably many straightly lines". Mathematically speaking, that rudimentary proof is based on two facts: Fact 1, every real quadratic function has at most two real solutions. Fact 2, a circle in $\mathbb{R}^2$ contains uncountably many points. None of these two facts needs the Axiom of Choice. Thus that rudimentary proof does not rely on the Axiom of Choice. The Baire Category Theorem we covered in class (for complete metric spaces), however, does assume the Axiom of Choice (or certain weaker form: the Axiom of Dependent Choice) in the proof. So a proof to this problem 6 using that Baire Category Theorem is different from the above mentioned rudimentary proof in the sense that the proof here (the one using the Baire Category Theorem covered in class) need to assume certain version of choice axioms, while the rudimentary proof you have done in the first assgiment does not.

Remark 2: If you really want to prove the fact in problem 6 using the Baire Category Theorem but not assuming any choice axioms (that is, your proof is based on ZF model only), that is doable. The fact is, the Baire Category Theorem for $\mathbb{R}$ (not for general complete metric spaces now) can be proved without using any choice axioms. The main thing here is that $\mathbb{R}$ is separable. The proof is not so hard if you have some understanding of choice axioms, but it is way beyond this course and we will stop here.

7. As is mentioned in class, while considering $\mathbb{R}$ with the usual topology, $\mathbb{Q}$ is not a G_δ set. Prove this fact. Also, show that $\mathbb{R}-\mathbb{Q}$ is a G_δ set (*this should be much easier*).

Note: If you worked out this problem, you had also shown that $\mathbb{R}-\mathbb{Q}$ *is not a* F_σ *subset in* $\mathbb{R}$. *Why?*

Hint: The Baire Category Theorem and its consequences.

7.5 This is related to one of the homework problems in Rudin's book. The original problem is like this: Does there exists a sequence of continuous function $\{f_n\}$ on $\mathbb{R}$ (equipped with the usual topology), such that $f_n(x) \to +\infty$ if and only if $x \in \mathbb{Q}$? Does there exists a sequence of continuous function $\{f_n\}$ on $\mathbb{R}$ (equipped with the usual topology), such that $f_n(x) \to +\infty$ if and only if $x \in \mathbb{R}-\mathbb{Q}$?

We start with relatively simple cases. We first assume that $\{f_n\}$ *is an* ***increasing*** *sequence of continuous functions on* $\mathbb{R}$.

1) Prove the following: Given a sequence of continuous functions $\{g_n\}$ on $\mathbb{R}$, and given $x \in \mathbb{R}$, "$g_n(x) \to +\infty$" does not happen if and only if there exists certain $N \in \mathbb{N}$, and certain subsequence $\{n_k \colon k \in \mathbb{N}\}$ such that

$$x \in \bigcap_{k=1}^{\infty} \{t \in \mathbb{R} \colon g_{n_k}(t) \le N\}.$$

2) Based on the result in 1), prove the following: Given an **increasing** sequence of continuous functions $\{f_n\}$ on $\mathbb{R}$, and given $x \in \mathbb{R}$, "$f_n(x) \to +\infty$" does not happen if and only if there exists certain $N \in \mathbb{N}$ such that

$$x \in \bigcap_{n=1}^{\infty} \{t \in \mathbb{R} \colon f_n(t) \le N\}.$$

3) Assume $\{f_n\}$ is an **increasing** sequence of continuous functions on $\mathbb{R}$. Prove that the subset formed by all those x satisfying $\lim_{n\to\infty} f_n(x) = +\infty$ is a G_δ set.

4) Prove that we can not find an **increasing** sequence of continuous functions $\{f_n\}$ on $\mathbb{R}$, such that $f_n(x) \to +\infty$ if and only if $x \in \mathbb{Q}$.

We consider the case that $\{f_n\}$ *is just a sequence of continuous functions on* $\mathbb{R}$, *not surely an increasing sequence. We assume that there exists function* f *(not necessarily continuous) on* $\mathbb{R}$, *such* *that* $\lim_{n\to} f_n(x) = f(x)$ *for all* $x \in \mathbb{R}$.

5) Let $\{f_n\}$ be a sequence of continuous functions on $\mathbb{R}$. Assume that there exists $f: \mathbb{R} \to [-\infty, +\infty]$ such that $\lim_{n\to} f_n(x) = f(x)$ for all $x \in \mathbb{R}$. In other words, for each $x \in \mathbb{R}$, $\lim_{n\to\infty} f_n(x)$ always exists. Prove that for any $r \in \mathbb{R}$, we have

$$\{x\colon f(x) < r\} = \bigcup_{K=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N+1}^{\infty} \left\{x\colon f_n(x) \leq r - \frac{1}{K}\right\}.$$

Remark: Following the proof of 5), for a sequence of continuous functions $\{f_n\}$ on $\mathbb{R}$ such that $\lim_{n\to\infty} f_n(x)$ always exists for all $x \in \mathbb{R}$, we can show that $\{x\colon f(x) < +\infty\}$ is a F_σ set in $\mathbb{R}$. In fact, we just need to note that

$$\{x\colon f(x) < +\infty\} = \bigcup_{K=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N+1}^{\infty} \{x\colon f_n(x) \leq K\}.$$

As we already know that $\mathbb{R} - \mathbb{Q}$ is not F_σ, we can claim immediately:

" For a sequence of continuous functions $\{f_n\}$ on $\mathbb{R}$ such that $\lim_{n\to\infty} f_n(x)$ always exists for all $x \in \mathbb{R}$, it is impossible that $f_n(x) \to +\infty$ if and only if x is a rational number. "

Now, we consider the original case in Rudin book. That is, $\{f_n\}$ *is just a sequence of continuous functions on* $\mathbb{R}$*, not surely an increasing sequence or a decreasing sequence of functions, and we do not know anything about the existence of* $\lim_{n\to\infty} f_n(x)$ *for* $x \in \mathbb{R}$.

6) Prove the following: Let $\{f_n\}$ be a sequence of continuous functions on $\mathbb{R}$. Then

$$\{x\colon f_n(x) \to +\infty\} = \bigcap_{M=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N+1}^{\infty} \{t\colon f_n(t) \geq M\}.$$

Remark: As each f_n is continuous, $\bigcap_{n=N+1}^{\infty}\{t\colon f_n(t) \geq M\}$ is always closed. Thus $\{x\colon f_n(x) \to +\infty\}$ is a countable intersection of F_σ sets. Use $F_{\sigma\delta}$ to denote the set of all the sets that can be written as a countable intersection of F_σ sets. Under this notation, we can say that $\{x\colon f_n(x) \to +\infty\}$ is a $F_{\sigma\delta}$ set. Given a sequence of continuous function f_n, 6) says that $\bigcap_{n=N+1}^{\infty}\{t\colon f_n(t) \geq M\}$ is always $F_{\sigma\delta}$. A known result is: In $\mathbb{R}$ with the usual topology, if a subset D is $F_{\sigma\delta}$, then there exists a sequence of continuous functions f_n on $\mathbb{R}$, such that $\bigcap_{n=N+1}^{\infty}\{t\colon f_n(t) \geq M\} = D$. With this in mind, note that both $\mathbb{Q}$ and $\mathbb{R}-\mathbb{Q}$ are $F_{\sigma\delta}$, it follows immediately that there exists one sequence of continuous functions f_n and another sequence of continuous functions g_n, such that $f_n(x) \to +\infty$ if and only if x is rational, and $g_n(x) \to +\infty$ if and only if x is irrational.

8. In real analysis, the example of a function on $\mathbb{R}$ (or $[a,b]$) that is continuous but nowhere differentiable is quite classical. You can find concrete constructions in most of the textbooks. Now, we just consider the space of continuous functions on $[0,1]$. The standard fact is: "most" of the functions in $C[0,1]$ are nowhere differentiable. By "most", we mean a subset of second category.

In $C[0,1]$, define the metric/distance to be

$$d(f,g) = \sup_{x\in[0,1]} |f(x)-g(x)|.$$

It is a well-known fact that $(C[0,1],d)$ is a complete metric space (why?). Thus $(C[0,1],d)$ is of second category.

In this complete metric space, use F_n to denote all those functions $f \in C[a,b]$, such that there exists one point (say, x) in $[0,1]$, satisfying

$$\left|\frac{f(y)-f(x)}{y-x}\right| \leq n$$

for all $y \in [0,1]-\{x\}$ with $|y-x| < 1/n$.

Prove the following:

1) For a function $f \in C[0,1]$, if f is differentiable at at least one point, then $f \in \bigcup_{n=1}^{\infty} F_n$.

2) Each F_n is a closed subset in the metric space $(C[0,1],d)$.

3) Each F_n has empty interior.

4) In $(C[0,1],d)$, the subset formed by those function g that is nowhere differentiable is of second category. *Note: In fact, as each g in $(\bigcup_{n=1}^{\infty} F_n)^c$ is nowhere differentiable, one just need to show that $(\bigcup_{n=1}^{\infty} F_n)^c$ is of second category.*

Note: The following problems (9, 9.5, 10, 10.5, 10.6 and 10.7) might help you to better understand some differences between the geometries on metric spaces and the geometries on norm spaces.

9. Let (X,d) be a metric space.

1) For any $r > 0$ and for any $a \in X$, prove that

$$\overline{\{x\colon d(x,a) < r\}} \subset \{x\colon d(x,a) \le r\}.$$

2) In general, with the same setup as above, we shall not expect to have

$$\overline{\{x\colon d(x,a) < r\}} = \{x\colon d(x,a) \le r\}.$$

Construct a metric space to show this phenomenon.

Hint: Consider the case the metric space is discrete.

9.5 Let $(M, \|\cdot\|)$ be a normed space. For any $a \in M$ and $r > 0$, prove that

$$\overline{\{x\colon \|x-a\| < r\}} = \{x\colon \|x-a\| \le r\}.$$

Note: You just need to prove for the case $a = 0$.

10. In a normed space $(M, \|\cdot\|)$, use $B_x(r)$ to denote the open ball $\{y \in X\colon d(y,x) < r\}$, where $d(x,y)$ is defined to be $\|x-y\|$ for $x, y \in M$. If there exist $x_1, x_2 \in M$ and $r_1, r_2 \in \mathbb{R}_{>0}$ such that

$$B_{x_1}(r_1) \subset B_{x_2}(r_2),$$

does it follow that $r_1 \le r_2$? If so, prove it. If not, find a counter example.

10.5 Construct a metric space (X, d), such that we can find $x_1, x_2 \in X$ and $r_1, r_2 \in \mathbb{R}_{>0}$, such that

$$B_{x_1}(r_1) \subset B_{x_2}(r_2)$$

but $r_1 > r_2$, where $B_{x_i}(r_i)$ is defined to be $\{y \in X\colon d(y,x_i) < r_i\}$ for $i = 1, 2$.

10.6 Construct a metric space (X, d), such that we can find $x_1, x_2 \in X$ and $r_1, r_2 \in \mathbb{R}_{>0}$, such that

$$B_{x_1}(r_1) \subsetneq B_{x_2}(r_2)$$

but $r_1 > r_2$, where $B_{x_i}(r_i)$ is defined to be $\{y \in X\colon d(y, x_i) < r_i\}$ for $i = 1, 2$.

Hint for 10.5 and 10.6: The statement in problem 10.7 and 10.8, which you are supposed to prove. It might help.

10.7 Let (X, d) be a metric space. For any $x_1, x_2 \in X$ and $r_1, r_2 \in \mathbb{R}_{>0}$, assume that

$$B_{x_1}(r_1) \subset B_{x_2}(r_2),$$

where $B_{x_i}(r_i)$ is defined to be $\{y \in X\colon d(y, x_i) < r_i\}$ for $i = 1, 2$. Prove that

$$B_{x_1}(r_1) \subset B_{x_1}(2r_2).$$

Remark: In a metric space (X, d), if $B_x(r_1) \subset B_x(r_2)$, in general, there is nothing we can say about the relationship between r_1 and r_2. For example, consider the case X is made up of one point x. Then for any $r_1 > 0$ and any $r_2 > 0$, we have $B_x(r_1) \subset B_x(r_2)$.

Note: This form of problem 10.8 owes credit to one of the students' suggestions during the office hour.

10.8 Let (X, d) be a metric space. For any $x_1, x_2 \in X$ and $r_1, r_2 \in \mathbb{R}_{>0}$, assume that

$$B_{x_1}(r_1) \subsetneq B_{x_2}(r_2),$$

where $B_{x_i}(r_i)$ is defined to be $\{y \in X\colon d(y, x_i) < r_i\}$ for $i = 1, 2$. Prove that

$$r_1 \leq 2r_2.$$

Hint: If we just have $B_{x_1}(r_1) \subset B_{x_2}(r_2)$ instead of $B_{x_1}(r_1) \subsetneq B_{x_2}(r_2)$, according to the Remark above, we shall not always expect to get $r_1 \leq 2r_2$ (in fact, there is almost nothing we can say about the relation between r_1 and r_2 in general). So, it should not be so surprising that your proof should start with something like "Choose $y \in B_{x_2}(r_2) - B_{x_1}(r_1)$".

11. A metric space (X, d) is separable if it has a countable dense subset.

1) For the metric space (X, d), and for $x \in X$ and $r > 0$, use $B_x(r)$ to denote the open ball

$\{y \in X \colon d(y,x) < r\}$. Assume that we can find an uncountable subset $\{x_i\}_{i\in\mathcal{A}}$ in X, and $\{r_i\}_{i\in\mathcal{A}} \in \mathbb{R}_{>0}$, such that for any pair of **distinct** $i, j \in \mathcal{A}$, the open balls $B_{x_i}(r_i)$ and $B_{x_j}(r_j)$ are disjoint. Prove that (X,d) is not separable.

2) For the metric space (X,d), if we can find an uncountable subset $\{x_i\}_{i\in\mathcal{A}}$ in X, such that for any index $i, j \in \mathcal{A}$,

$$d(x_i, x_j) = \begin{cases} 1 & i \neq j \\ 0 & i = j \end{cases},$$

prove that (X,d) is not separable.

3) Let $\mathbb{N}$ be equipped with the counting measure μ. Prove that $L^\infty(\mathbb{N},\mu)$ is not separable.

4) Let $[0,1]$ be equipped with the Lebesgue measure μ. Prove that $L^\infty([0,1],\mu)$ is not separable.

12. This is about G_δ and F_σ sets, about sets of first category and sets of second category, and how the Baire Category Theorem can come into play.

1) Consider $\mathbb{R}$ with the usual topology. First prove that $[0,1)$ is a G_δ set. Then prove that $[0,1)$ is also a F_σ set. *Note: This is a non-trivial (not $\emptyset$, not $\mathbb{R}$) example of a set which is both G_δ and F_σ.*

Note: For $\mathbb{R}$ with the usual topology, do you have examples of a set that is G_δ but not F_σ? F_σ but not G_δ? I think it is already covered in class or homework assignments.

2) Consider $\mathbb{R}$ with the usual topology. Surely there are sets that is neither G_δ nor F_σ. For example, as G_δ sets and F_σ sets are all Borel sets, and all Borel sets are Lebesgue measurable, it immediately follows that all the non-measurable sets are neither G_δ nor F_σ. According to problem 10 of homework 3, we have sets that is Lebesgue measurable but not Borel. That is, we have those sets that is Lebesgue measurable but is neither G_δ nor F_σ (because they are not Borel). Now, we keep pushing forward. Consider $\mathbb{R}$ with the usual topology, can you construct a Borel subset in $\mathbb{R}$, such that it is neither G_δ nor F_σ? (This is an optional problem. It is totally OK if you cannot figure it out.)

Hint: As for $\mathbb{R}$ with the usual topology, you already know examples of a set A which is G_δ but not F_σ, and you also know examples of a set B that is F_σ but not G_δ. Can you "combine"/"mix" these two sets A and B together to get a set which is a Borel set, but neither G_δ nor F_σ?

3) Let (X,d) be a metric space (not necessarily complete). Let D be a F_σ subset of X. Prove/show

that D is either of first category or has non-empty interior (that is, there exists $x \in X$ and $r > 0$, such that $\{y \in X : d(y,x) < r\} \subset D$).

4) Let X be a topological space. Note that subsets of first category and subsets of second category can still be defined on this *topological* space X. Just that we can not apply the Baire Categor Theorem and claim that X is of second category. Let $E \subset X$ be a subset of first category. Prove that every subset of E is also of first category. Let $F \subset X$ be a subset of X, prove that every superset of F (that is, every set containing F as a subset) is also of second category.

5) For a topological space (X, π), we say that it is completely metrizable if there exists a metric space (Y, d), such that the metric space (Y, d) is complete and the *topological space* (X, π) is homeomorphic to the *topological space* (Y, π_d), where π_d is the topology on Y induced by the metric d. As you have already seen in homework 5, $(0, 1)$ with usual topology is completely metrizable because $(0, 1)$ is homeomorphic to $\mathbb{R}$ with the topology derived from the usual distance/metric structure d in $\mathbb{R}$, and it is a well known fact that the metric space $(\mathbb{R}, d)$ is complete. Given those background above, consider $\mathbb{Q}$ with the usual topology $\pi_{\mathbb{Q}}$, which is restricted from the usual topology on $\mathbb{R}$. If we consider the usual metric/distance structure d_0 on $\mathbb{Q}$ defined as $d_0(x, y) = |x - y|$ for all $x, y \in \mathbb{Q}$, it is clear that the topology derived from the metric d_0 is exactly $\pi_{\mathbb{Q}}$ and the metric space $(\mathbb{Q}, d_0)$ is not complete. Prove that there is no metric structure d on $\mathbb{Q}$ such that the topology derived from this metric d is exactly $\pi_{\mathbb{Q}}$ and the metric space $(\mathbb{Q}, d)$ is complete. In other words, prove that the topological space $(\mathbb{Q}, \pi_{\mathbb{Q}})$ is not completely metrizable.

Hint: Assume the opposite, and apply the Baire Category Theorem.

***13.** The Riesz Representation Theorem as an exercise. **This is an optional long exercise**, not difficult to figure out the outline of the proof (assuming you have a good understanding of how the Caratheodory Extension Theorem works, especially about the part on how to derive a measure from an outer measure). It does require some work, but the outline is clear and just follows that of the Caratheodory Extension Theorem. Some sketches and remarks will be given. If interested, you can follow those sketches/remarks, connect the dots, and get the complete picture of the proof for the Riesz Representation Theorem, which, from certain point of view, is about deriving measure structures from certain topological structures (via a bounded positive linear functional).

Remark: The purpose of this problem/reading stuff is to show that the Riesz Representation Theorem you have seen in the Rudin book is nothing but a form of the standard Caratheodory Measure Extension approach. Although the approach to the Riesz Representation Theorem in Rudin book might look different from the Caratheodory Extention approach at first glance, they are just equivalent mathematically.

Theorem (Riesz Representation Theorem) Let X be a locally compact Hausdorff space. Let $L\colon C_c(X) \to \mathbb{R}$ be a positive linear functional on $C_c(X)$ which is bounded on all the subspaces $C_c(X)_K$, where K is a compact subset of X and $C_c(X)_K$ is defined as $\{h\colon h \in C(X),\ \operatorname{supp}(h) \subset K\}$. By "$L$ is positive", we mean $L(f) \geq 0$ for any positive function $f \in C_c(X)$. Then there exists a unique measure μ on X, such that μ is complete, both inner regular (when restricted to compact subsets) and outer regular for all measurable sets (see Rudin book for detailed definitions), and for any $g \in C_c(X)$, we have $L(g) = \int_X g \,\mathrm{d}\mu$.

We will just focus on the main thing of the Riesz Representation Theorem, that is, how to derive such a measure. The hints/comments we give here does not closely follow the approach in Rudin book. Instead, it follows the line of the lectures in our class, that is, the process related to Caratheodory Extension Theorem.

<u>Step 0:</u>

The requirement that "L is bounded on all the subspaces $C_c(X)_K$" is redundant. In fact, we will show that any positive linear functional on $C_c(X)$ is automatically bounded on those $C_c(X)_K$, where K is a compact subset.

To achieve this, as X is locally compact and Hausdorff, noting that for any compact set K of X, according to the result in problem 13 of homework 5, which is also a key result used to show the Urysohn's Lemma, we can find an open set U containing K, such that its closure $\overline{U}$ is also compact. In that case, we have (note that $\overline{U}$ is also compact)

$$K \subset U \subset \overline{U} \subset X.$$

With this in mind, by Urysohn's Lemma, we can construct a continuous **positive** function f such that

$f|_{\overline{U}} = 1$. For any $h \in C_c(X)_K$, we then have $\sup_{x\in K}|h(x)| < +\infty$. It is obvious that $h^+ \le \sup_{x\in K}|h(x)|$ and $h^- \le \sup_{x\in K}|h(x)|$. As $K \subset U \subset \overline{U}$, one can check that

$$0 \le h^+ \le \sup_{x\in K}|h(x)| \cdot f \quad \text{and} \quad 0 \le h^- \le \sup_{x\in K}|h(x)| \cdot f.$$

Note that the functional L is positive. Then we have (why?)

$$0 \le L(h^+) \le \sup_{x\in K}|h(x)| \cdot L(f) \quad \text{and} \quad 0 \le L(h^-) \le \sup_{x\in K}|h(x)| \cdot L(f).$$

As $L(f) \in \mathbb{R}$, it must be finite. So far, we have proved that

$$|L(h)| = |L(h^+ - h^-)| = |L(h^+) - L(h^-)| \le |L(h^+)| + |L(h^-)| \le 2 \cdot \sup_{x\in K}|h(x)| \cdot L(f)$$

for all $h \in C_c(X)_K$.

Step 1: We start with constructing a pre-measure μ_0 on certain "simple" subsets of X, just like defining the pre-measure of $[a, b)$ to be $b-a$ while deriving the Lebesgue measure using the Caratheodory Extension Theorem. In our case, we try to define μ_0 on all the open sets of X. To be precise, for any open set D, we define $\mu_0(D)$ to be

$$\mu_0(D) = \sup\{L(f)\colon f\colon X \to [0,1], \exists \text{ certain compact subset } K \text{ in } D, \text{such that } f|_{X-K} = 0\ \}.$$

For this μ_0, easy to check that $\mu_0(\emptyset) = 0$ and $\mu_0(A) \le \mu_0(B)$ if $A \subset B$.

Step 2: Just like how the outer measure on $\mathbb{R}$ is defined in the constructure of the Lebesgue measure on $\mathbb{R}$, for any subset E of X, we define the outer meaure λ of E to be (different from the one in Rudin book, but can be proved later that these two definitions are the same)

$$\lambda(E) = \inf\left\{\sum_{i=1}^{\infty}\mu_0(D_i)\colon \text{each } D_i \text{ is open and } E \subset \bigcup_{i=1}^{\infty} D_i\right\}.$$

As is already covered in class, this λ is automatically an outer measure. So far, we are not yet sure that $\lambda = \mu_0$ when restricted to the set of open subsets in X, and that is where the Caratheodory Extension Theorem will come into play.

Step 2.1

If you check the approach in Rudin book, you will realize a slight "difference" in the definition of the outer measure λ. In Rudin book, for any subset E of X, the outer measure, denoted here as λ', is defined as:

$$\lambda'(E) = \inf \left\{ \mu_0(D) \colon D \text{ is open and } E \subset D \right\}.$$

These two definitions of outer measures are equivalent here. That is, for any subset E, we have $\lambda(E) = \lambda'(E)$. To show this, we just need to proof the following claim.

Claim: With the setup as above, for any open subset D and a sequence of open subsets D_i, such that $D \subset \bigcup_{i=1}^{\infty} D_i$, we have

$$\mu_0(D) \leq \sum_{i=1}^{\infty} \mu_0(D_i).$$

Sketchy proof: To prove this claim, we just need to following the definitions. To achieve $\mu_0(D)$, just consider a continuous functions $f \colon X \to [0,1]$ such that $\mathrm{supp}(f)$ is a subset of certain comapct set K, with $K \subset D$ and with $L(f)$ "close" to $\mu_0(f)$. As f is supported on a compact subset K, $K \subset D \subset \bigcup_{i=1}^{\infty} D_i$, we can find a finite subcovering of K, say $K \subset \bigcup_{i=K}^{\infty} D_i$. As X is locally compact and Hausdorff, we can (check Rudin book for details) write f as the sum of f_i for $1 \leq i \leq K$, where each f_i is continuous and is supported inside D_i. With this in mind, you should be able to finish the rest of the work and show that $\mu_0(D) \leq \sum_{i=1}^{\infty} \mu_0(D_i)$.

Step 2.5

For the μ_0 defined above, show that it is finitely additive. That is, if there are two open sets U and V with $U \cap V = \emptyset$, show that $\mu_0(U \sqcup V) = \mu_0(U) + \mu_0(V)$. If this holds true, then we can get finite additivity on μ_0 simply by induction. It is mostly plain verifications, according to the definition of μ_0. You might want to use the facts like this, "If a compact subset K is in $U \sqcup V$, where both U and V are open sets, then both $K \cap U$ and $K \cap V$ are compact".

Step 2.6

For the μ_0 above, show that it is countably monotonic. If this can be done, combing this with the results we got in Step 2.5, we have proved that μ really extends μ_0. According to the result in Caratheodory Extension Theorem, we just need to show that μ_0 is finitely additive and countably monotone.

As for finite additiveness, for any finite disjoint open sets $D_1, \cdots, D_n$, we need to show that

$$\mu_0\left(\bigsqcup_{i=1}^{n} D_i\right) = \sum_{i=1}^{n} \mu_0(D_i).$$

It is trivial to show that $\sum_{i=1}^{n} \mu_0(D_i) \leq \mu_0\left(\bigsqcup_{i=1}^{n} D_i\right)$ using definitions (why?). It just remains to show

$$\mu_0\left(\bigsqcup_{i=1}^{n} D_i\right) \leq \sum_{i=1}^{n} \mu_0(D_i),$$

which is mainly about checking against definitions. That is, according to the definition of μ_0, $\mu_0\left(\bigsqcup_{i=1}^{n} D_i\right)$ equals .... . Some math you might want to use is 1) every open covering of a compact set contains a finite subcovering, and 2) on a locally compact Hausdorff space X (which implies that X is paracompact), a continuous function f which is defined/supported on a finite union of open sets, say, $\bigcup_{i=1}^{n} U_i$, can be written as the sum of functions f_i, such that each f_i is supported on the corresponding U_i only.

Step 3:

As like the standard process we had done in class, this above defined λ might not be a measure on $\mathcal{P}(X)$, but it will be a measure when restricted to a subset $\mathcal{M}$ of $\mathcal{P}(X)$. Here we have $E \subset \mathcal{M}$ if

$$\lambda(A) = \lambda(A \cap E) + \lambda(A \cap E^c) \quad \text{for all } A \subset X.$$

As is covered in the Caratheodory Extension Theorem, when restriected on $\mathcal{M}$, the λ is really a measure. We use μ to denote this measure. That is,

$$\mu = \lambda\,|_{\mathcal{M}}\ .$$

Now, it remains to show some important properties of this measure $(\mu, \mathcal{M})$.

Step 4:

Check that each Borel set is measurable. That is, for any Borel set D, we have

$$\lambda(A) = \lambda(A \cap D) + \lambda(A \cap D^c)$$

for all subset A of $\mathbb{R}$.

In fact, recalling that those measurable sets (in the sense above) form a σ-algebra (check your notes on Caratheodory Extension Theorem related stuff for details), we just need to show that every open set is measurable. That is, for any open set E and for any subset A of $\mathbb{R}$, we have

$$\lambda(A) = \lambda(A \cap E) + \lambda(A \cap E^c).$$

We can prove the following claim first.

<u>Claim:</u> For any set A and any $\epsilon > 0$, there exists an open set U such that $U \supset A$, and $|\lambda(A \cap B) - \lambda(U \cap B)| < \epsilon$ for all the subsets B.

Sketch of the proof: We can use the definition of λ to find an open set U such that $A \subset U$ and $|\lambda(A) - \lambda(U)| < \epsilon$. Note that the outer measure λ is subadditive, and we can check that this U is the desired one.

With that claim in mind, we just need to show that for any given open set E and any open set U of $\mathbb{R}$, we have

$$\lambda(U) = \lambda(U \cap E) + \lambda(U \cap E^c).$$

If that is true, with the claim above, we can show that $\lambda(A) = \lambda(A \cap E) + \lambda(A \cap E^c)$ for any subset A. Thus any open set U is measurable.

In fact, as the outer measure μ is subadditive, we just need to check that

$$\lambda(U \cap E) + \lambda(U \cap E^c) \leq \lambda(U).$$

This is relatively easy to check. Just note that $U \cap E$ is also open, and we can find, for any given $\epsilon > 0$, a compact subset K in $U \cap E$, such that there exists a continuous function $f \colon X \to [0, 1]$ such that the support of f is inside K, and

$$\lambda(U \cap E) - \epsilon \leq L(f) \leq \mu_0(U \cap E) = \lambda(U \cap E).$$

As X is Hausdorff, the compact subset K is also closed. Thus K^c is open. As $U \cap K^c$ is open and $U \cap K^c \supset U \cap E^c$, according to the definition of $\lambda(U \cap E^c)$, we can find a compact subset F in $U \cap K^c$ and a continuous function $g \colon X \to [0, 1]$ such that the support of g is inside F and

$$\lambda(U \cap E^c) - \epsilon \le L(g) \le \mu_0(U \cap E^c) = \lambda(U \cap E^c).$$

Now, it should be easy to show that $\lambda(U \cap E) + \lambda(U \cap E^c) \le \lambda(U)$.

Step 5:

For each compact subset K, show that $\mu(K) < +\infty$. We can borrow the proof of the same fact from Rudin book, and it will work without problem. That is because the proof only needs the fact that as L is a positive linear functional, it is automatically bounded on the set of continuous functions with any **given** compact support. A sketchy proof of this fact can be found in Step 0.

Also, following the sketchy proof in Step 0, we can directly deduce the that $\mu(K) < +\infty$ for any compact subset D, without having to borrowing anything from the proof on Riesz Representation Theorem in Rudin book.

Note that we can use $\mu(K)$ as we can now safely claim that every compact set K is measurable in the above sense as in Step 3. This is because that in this Hausdorff space X, every compact space is closed, thus lies in the σ-algebra generated by the open sets.

Step 6:

In the proof of the Riesz Representation Theorem in Rudin book, the set of measurable subsets of X is definely "differently" compared with the definition here of $\mathcal{M}$ in Step 3. This difference is not essential. These two definitions of "measurability" are equivalent, as we will show in this step.

For simplicity, we just assume the total space X is compact. If not, with slightly more work, parallel arguments will get the job done.

First, if a subset D is measurable in the sense as the part of Riesz Representation Theorem of Rudin book, we have

$$\underline{\lambda}(D) = \overline{\lambda}(D),$$

where

$$\underline{\lambda}(D) = \sup\{\lambda(K) \colon K \text{ is compact and } K \subset D\}$$

and

$$\overline{\lambda}(D) = \inf\{\lambda(U) \colon U \text{ is open and } U \supset D\}.$$

Note that this $\overline{\lambda}$ is the same as the λ we defined above. We use this notation $\overline{\lambda}$ to better indicate its relation with $\underline{\lambda}$.

Now, assume a subset D is measurable in the sense of Rudin book, we will show that it is also measurable in the sense of our defintion above. That is, for any subset A, we have

$$\overline{\lambda}(A) = \overline{\lambda}(A \cap D) + \overline{\lambda}(A \cap D^c).$$

As $\overline{\lambda}$ is an outer measure, we just need to show

$$\overline{\lambda}(A \cap D) + \overline{\lambda}(A \cap D^c) \leq \overline{\lambda}(A).$$

Note that, in general, we shall not expect any of the following three equations to hold true: $\overline{\lambda}(A \cap D) = \underline{\lambda}(A \cap D)$, $\overline{\lambda}(A \cap D^c) = \underline{\lambda}(A \cap D^c)$ and $\overline{\lambda}(A) = \underline{\lambda}(A)$.

As D is measurable in the sense of Rudin book, and as X is compact, it follows (why?) immediately that D^c is also measurable in the sense of Rudin book. Also, it is proved in Rudin book that every open set is measurable in the sense of Rudin book. That is, for every open set E in X, we have $\overline{\lambda}(E) = \underline{\lambda}(E)$. Besides, if two subsets are measurable in the sense of Rudin book, it is proved in Rudin book that their intersection is also measurable in the sense of the Rudin book.

Now, back to what we need to do: prove that $\overline{\lambda}(A \cap D) + \overline{\lambda}(A \cap D^c) \leq \overline{\lambda}(A)$.

Key step: Following the observation in Step 4, we just need to prove $\overline{\lambda}(A \cap D) + \overline{\lambda}(A \cap D^c) \leq \overline{\lambda}(A)$ in case A is an open subset.

Note that A, D and D^c are all measurable in the sense of Rudin book, thus so is A, $A \cap D$ and $A \cap D^c$. Then we have

$$\overline{\lambda}(A \cap D) + \overline{\lambda}(A \cap D^c) \leq \overline{\lambda}(A) \quad \Longleftrightarrow \quad \underline{\lambda}(A \cap D) + \underline{\lambda}(A \cap D^c) \leq \underline{\lambda}(A)$$

Note that $(A \cap D) \bigcap (A \cap D^c) = \emptyset$ and $(A \cap D) \bigcup (A \cap D^c) = A$, from the definition of $\underline{\lambda}$, it follows (why?) that

$$\underline{\lambda}(A \cap D) + \underline{\lambda}(A \cap D^c) \leq \underline{\lambda}(A),$$

for every open set A (thus eventually for every subset A. See Step 4 for details).

So far, we have shown that if a subset is measurable in the sense of the Rudin book, then it is measurable in the sense of our Caratheodory Extension Theorem approach here, as stated in Step 3.

Now, we will show that if a subset D is measurable in the sense of our Caratheodory Extension Theorem approach as in Step 3, then it is measurable in the sense of the Rudin book (i.e, $\overline{\lambda}(D) = \underline{\lambda}(D)$).

From the definition of $\overline{\lambda}$ and $\underline{\lambda}$, we have the following claim, whose proof is just checking against definitions.

<u>Claim:</u> For *any* subset H of X, we have $\overline{\lambda}(H) + \underline{\lambda}(H^c) = \lambda(X)$.

Now, the proof. Assume that E is measurable in the sense of Step 3. Will show that $\overline{\lambda}(D) = \underline{\lambda}(D)$.

As E is measurable in the sense of Step 3, we have

$$\overline{\lambda}(X) \geq \overline{\lambda}(D) + \overline{\lambda}(D^c).$$

As X is assumed to be compact, we can prove (why?) that

$$\underline{\lambda}(X) = \overline{\lambda}(X) = \lambda(X).$$

To show that $\underline{\lambda}(D) = \overline{\lambda}(D)$, we just need to show $\underline{\lambda}(X) \leq \underline{\lambda}(D) + \underline{\lambda}(D^c)$. In fact, if so, then

$$\begin{aligned}
\overline{\lambda}(X) &\geq \overline{\lambda}(D) + \overline{\lambda}(D^c) \\
&\geq \underline{\lambda}(D) + \underline{\lambda}(D^c) \\
&\geq \underline{\lambda}(X) \\
&= \overline{\lambda}(X).
\end{aligned}$$

Thus it follows that $\overline{\lambda}(D) = \underline{\lambda}(D)$, which finishes the proof. In fact, the reasoning above also indicates that $\overline{\lambda}(D^c) = \underline{\lambda}(D^c)$.

It only remains to show that $\underline{\lambda}(X) \leq \underline{\lambda}(D) + \underline{\lambda}(D^c)$. A stupid proof is like this: According to the assumption, we have $\overline{\lambda}(X) \geq \overline{\lambda}(D) + \overline{\lambda}(D^c)$. According to the definition of $\overline{\lambda}$, for any $\epsilon > 0$, we can find open sets E_1 and E_2, such that $E_1 \supset D$, $E_2 \supset D^c$, and $\lambda(E_1) + \lambda(E_2) \leq \lambda(X) + \epsilon$. As X is compact, we know that E_1^c and E_2^c are compact subsets in X. Besides, $E_1^c \subset D^c$ and $E_2^c \subset D$. According to the

Claim above, we have

$$
\begin{aligned}
\underline{\lambda}(E_1^c) + \underline{\lambda}(E_2^c) &= \lambda(X) - \overline{\lambda}(E_1) + \lambda(X) - \overline{\lambda}(E_2) \\
&= \lambda(X) + \big(\lambda(X) - \overline{\lambda}(E_1) - \overline{\lambda}(E_2)\big) \\
&\geq \lambda(X) - \epsilon \\
&= \underline{\lambda}(X) - \epsilon.
\end{aligned}
$$

Let $\epsilon \to 0$, and we are done.

Note: A smarter proof can be done as follows: As $\overline{\lambda}(X) \geq \overline{\lambda}(D) + \overline{\lambda}(D^c)$, according to the Claim above, we have

$$\lambda(X) - \overline{\lambda}(\emptyset) \leq (\lambda(X) - \overline{\lambda}(D^c)) + (\lambda(X) - \overline{\lambda}(D)).$$

Thus

$$\overline{\lambda}(D) + \overline{\lambda}(D^c) \leq \lambda(X) = \overline{\lambda}(X).$$