HW # 4 Due on Dec. 9th 2015.

1. Let X be a metric space.

(1) let A_1 and A_2 be two disjoint closed subset of X, and let f_i $(i = 1, 2)$ be continuous functions from X_i to $\mathbb{R}$, where the topologies on X_i are just the restricted topology from X to X_i. We define

$$f_1 \bigsqcup f_2 \colon A_1 \sqcup A_2 \to \mathbb{R},\ x \mapsto\ f_i(x) \text{ if } x \in A_i.$$

Prove that $f_1 \bigsqcup f_2$ is continuous, where the topology on $A_1 \sqcup A_2$ is the one restricted from X. *Note: You can easily extend this result to the case of finitely many A_is.*

(2) Let $A_1, A_2, \cdots$ be a sequence of disjoint subsets of X and let $f_i \colon A_i \to Y$ be continuous for all $i \in \mathbb{N}_{\geq 1}$, where each A_i is equipped with the restricted topology from X. Define $\bigsqcup_{i=1}^{\infty} f_i$ as

$$\bigsqcup_{i=1}^{\infty} f_i \colon \sqcup_{i=1}^{\infty} A_i \to Y,\ x \mapsto f_i(x) \text{ if } x \in A_i,$$

with the topology on $\sqcup_{i=1}^{\infty} A_i$ being the restricted topology of X onto the subset $\sqcup_{i=1}^{\infty} A_i$.

Give such an example of those X_is and f_is as above, such that although each f_i is continuous, but $\bigsqcup_{i=1}^{\infty} f_i$ is not continuous.

2. Let $f \in C[0, 1]$. That is, f is a continuous function on $[0, 1]$. Let u^* be the Lebesgue measure on $[0, 1]$. Prove that

$$\int_0^1 f(x)\,\mathrm{d}x = \int_{[0,1]} f\,\mathrm{d}\mu^*,$$

where the left hand side is the Riemann integration and the right hand side is the Lebesgue integration.

Hint: For a continuous funciton, just recall that the Riemann integration equals the infimum of Darboux upper sums, and it also equals the supremum of the Darboux lower sums.

3. If $f \colon [0, 1] \to \mathbb{R}$ is a function which is Riemann integrable, prove that f is also Lebesgue integrable (in the sense of upper sum equals lower sum) with respect to the Lebesgue measure.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space and let f be a bounded function from X to $\mathbb{R}$. Assume that f

is Lebesgue integrable in the sense that $U(f,\mu) = L(f,\mu)$, where

$$U(f,\mu) = \inf\left\{\int_X s\,\mathrm{d}\mu \colon s \text{ is simple and } s \geq f\right\}$$

and

$$L(f,\mu) = \sup\left\{\int_X s\,\mathrm{d}\mu \colon s \text{ is simple and } s \leq f\right\}.$$

Prove that $f = g$ a.e. (with respect to μ), where g is a measurable function. In extra, if the measure μ is complete, further prove that f must be a measurable function. *Note: This result, combined with the result of problem 5.5, shows that, for bounded functions on a complete measure space, being Lebesgue measurable is just what is needed for being Lebesgue integrable in the sense of upper sum equals lower sum.*

Hint: Just recall the definition of being Lebesgue measurable, and note the following facts: i) A simple function is measurable. ii) The inf, sup, lim inf, and lim sup of a sequence of measurable functions are still measurable. iii) Assume f and g are measurable and $f \geq h \geq g$. If $f = g$ a.e., then h equals a measurable function a.e. (why? You will need to prove it)

5. In the class, we learned that on a measure space X with $\mu(X) < \infty$, if f is a bounded measurable function, then f is Lebesgue integrable. f being bounded is not necessary though. Let $f\colon (0,1] \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 0 & x \text{ is irrational} \\ q & x = \frac{p}{q},\ p \text{ and } q \text{ are coprime.} \end{cases}$$

And assume the measure on $(0,1]$ is the standard Lebesgue measure μ.

Use the definiton to prove that this f, although being unbounded, is integrable. Besides, show that $\int_{(0,1]} f\,\mathrm{d}\mu = 0$.

Note: The f above is not bounded on $(0,1]$, but it is bounded a.e. on $(0,1]$.

5.5 Let f and g be two functions on the measure space $(X, \mathcal{M}, \mu)$, such that $f = g$ a.e. with respect to μ. Note that the measure μ need not be complete. Prove that f is Lebesgue integrable on X with respect to μ if and only if g is. Besides, if both (or equivalently, one) are Lebesgue integrable, then $\int_X f\,\mathrm{d}\mu = \int_X g\,\mathrm{d}\mu$.

6. For the function

$$f\colon [0,1] \to \mathbb{R},\, x \mapsto \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{else.} \end{cases}$$

Prove that f is Lebesgue integrable but not Riemann integrable.

7. In Egorov's Theorem, we require the measure space (X, π, μ) to have finite measure. In fact, X being of finite measure is really necessary for Ergorov's Theorem to hold. To be more precise, even X being σ-finite cannot ensure the correctness of Ergorov's Theorem. Construct such an example. That is, find a measure space (X, π, μ) which is σ-finite, and find $\{f_n\}_{n=1}^{\infty}$ such that $f_n \to f$ pointwise on X, but there exists $\epsilon > 0$ such that for any measurable set D of X with $\mu(D) < \epsilon$, we do *not* have that f_n converges to f uniformly on $X \setminus D$.

8. This one is from Rudin's book. Guess the value of the following integration first, then prove your conjecture:

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2}\, \mathrm{d}x.$$

9. This is about how to approximate measurable functions with simple functions.

(1) Let X be a measure space and let $f\colon X \to \mathbb{R}$ be a bounded measurable function. Prove that there exist a sequence of simple functions $\{f_n\}_{n=1}^{\infty}$ such that f_n converges to f uniformly.

(2) Construct a measurable function $f\colon X \to \mathbb{R}$ on a measure space X such that no sequence of simple functions on X can converge to f uniformly. Note that you need to prove the non-existence of such simple functions.

(3) Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f be a measurable function. Prove that f can be written as the uniform convergence limit of simple functions if and only if f is bounded.

Note: In this case, assume f_n converges to f uniformly. If we furthur assume that X is of finite measure, then $\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$.

(4) Let X be a measure space and let $f\colon X \to \mathbb{R}$ be a measurable function (not necessarily bounded). Prove that there always exists a sequence of simple functions $\{f_n\}_{n=1}^{\infty}$ such that f_n converges to f pointwise on X. That is, $f_n(x)$ converges to $f(x)$ as $n \to \infty$ for all $x \in X$.

Note: As is covered in class, if measurable functions f_n converge pointwise to a function f, then f

is also measurable. Combined with the result in (4), we know that a function f is measurable if and only if it is the pointwise limit of simple functions (that is, there exist simples functions s_n such that s_n converge to f pointwise).

10. This is a classical exercise problem for integration. Let f be a bounded measurable function on $[0, 1]$. Let μ be the Lebesgue measure on $[0, 1]$. We then know that f is Lebesgue integrable on $[0, 1]$ with respect to the measure μ (why?). For any $n \in \mathbb{Z}$, consider $\sin(nx)$, which is bounded and continuous. It then follows that $f(x) \cdot \sin(nx)$ is measurable (why?) and bounded. Thus $f(x) \cdot \sin(nx)$ is integrable for all $n \in \mathbb{Z}$. Now, your turn. Prove that

$$\lim_{n\to\infty} \int_{[0,1]} f(x) \cdot \sin(nx) \, \mathrm{d}\mu = 0 \ .$$

Hint: You can start with simple cases first. For example, the case when f is a constant. I believe you all know how to do this case. Then consider the case f is a simple function. In this case, when restricted on certain measurable set, f is still a constant. You might argue that measurable set is not as friendly as $[0, 1]$ or $[a, b]$. True, but just recall (from somewhere in HW # 3) that as for measurable sets on $\mathbb{R}$, we do have certain descriptions of them. I will stop here and leave the rest of adventures to you.

10.5 Let f be a bounded measurable function on $[0, 1]$. Let g be a periodic bounded measurable function on $\mathbb{R}$ with L being the period of g. Let μ be the Lebesgue measure on $\mathbb{R}$.

(1) For any $x \in \mathbb{R}$, prove that

$$\int_{\mathbb{R}} \chi_{[0,L]} \cdot g \, \mathrm{d}\mu = \int_{\mathbb{R}} \chi_{[x,x+L]} \cdot g \, \mathrm{d}\mu,$$

where $\chi_{[0,L]}$ and $\chi_{[x,x+L]}$ are the characteristic functions on $[0, L]$ and $[x, x + L]$ respectively.

(2) If we furthur assume that $\int_{[0,L]} g \, \mathrm{d}\mu = 0$, prove that

$$\lim_{n\to\infty} \int_{[0,1]} f(x) \cdot g(nx) \, \mathrm{d}\mu = 0,$$

where we abuse the notation and also use μ to denote the restriction of the Lebesgue measure μ on $[0, 1]$.

11. This is another classical exercise problem. Let f be a continuous function on $[a, b]$ and let μ be the Lebesgue measure on $[a, b]$.

(1) If $\int_{[a,b]} |f| \, \mathrm{d}\mu = 0$, prove that $f = 0$ everywhere on $[a, b]$.

(2) Find another measure μ^* on $[a, b]$ with $\mu^*([a, b]) = 1$, such that there exists a continuous function f on $[a, b]$ with $\int_{[a,b]} |f| \, \mathrm{d}\mu^* = 0$ but f is not always zero on $[a, b]$.

12. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{f_n\}_{n \in \mathbb{N}_{\geq 1}}$ be a sequence of measurable functions. For two measurable funcitons g and h, if $f_n \xrightarrow{\mu} g$ and $f_n \xrightarrow{\mu} h$, prove that $g = h$ a.e. with respect to μ. *Note: This problem is about the uniqueness of limit of functions in the sense of convergence by measure.*

13. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let f and $\{f_n\}_{n \in \mathbb{N}_{\geq 1}}$ be measurable functions. If $f_n \to f$ pointwise a.e., prove that $f_n \xrightarrow{\mu} f$.