HW # 2 Due on Oct. 23rd 2015.

1. Let (X, π_X) be a topological space. Let $Y = X \sqcup \{\infty\}$ for some abstract point ∞. Define π_Y to be

$$\pi_Y = \pi_X \sqcup \{(X \setminus K) \sqcup \{\infty\} \colon K \subset X,\ K \text{ is both compact and closed in } X\}.$$

a) Show that π_Y is really a topology on Y.

b) In (Y, π_Y), prove that X is dense. In other words, prove that $\overline{X} = Y$.

c) Prove that the topological space (Y, π_Y) is compact.

2. Consider $\mathbb{R}$ with the usual topology. Use $\mathbb{R}^+$ to denote the one point compactification of $\mathbb{R}$. Let S^1 be the unit circle in $\mathbb{R}^2$, with the restricted topology, where the topology on $\mathbb{R}^2$ is the usual one. Prove that $\mathbb{R}^+$ is homeomorphic to S^1.

3. Let X and Y be two non-compact topological spaces. If X is homeomorphic to Y, prove that X^+ is homeomorphic to Y^+, where X^+ and Y^+ are the one point compactifications of X and Y correspondingly.

4. Let (X, π_X) be a topological space and let $Y \subset X$. Let π_Y be the restricted topology (of X on Y). Then we know that (Y, π_Y) is also a topological space. Prove that the following are equivalent:

1) Y is a compact subset of the topological space (X, π_X).

2) The topological space (Y, π_Y) is compact.

5. In the topological space $\mathbb{R}$ with the usual topology, for any subset X of $\mathbb{R}$, we have the restricted topology on X, which makes X a topological space.

1) Assume X is connected and contains two distinct points a, b (without loss of generality, we assume that $a < b$). Prove that for any $x \in (a, b)$, we have $x \in X$. *Hint: Just recall how we prove that $\mathbb{Q}$ is not connected in the class.*

2) If a subset X of $\mathbb{R}$ is connected, prove that it is also path connected.

3) Prove that $[0, 1]$ is connected. *Warning: The following approach is not considered to be acceptable. "In order to show that $[0, 1]$ is connected, we can prove, using the definition directly, that $[0, 1]$ is path connected. As any path connected space must be connected, it follows that $[0, 1]$ is connected." The reason is as follows. During the class, we did prove such theorem that "any path connected space must be*

connected". However, in our proof, we implicitly used the well-known fact that $[0,1]$ *is connected. Here in 3), you are asked explictly to prove this well-known fact that* $[0,1]$ *is connected.* **Hint:** You can feel free to use all the properties on the topological space $\mathbb{R}$ that you have learned.

4) Prove the following: Let X be a subset of $\mathbb{R}$ equipped with the usual topology. Then X is connected if and only if X is path connected. *Remark: If we change* $\mathbb{R}$ *to* $\mathbb{R}^2$ *in the statement, then it is no longer true.*

5) Let $[0,1]^2 = [0,1] \times [0,1]$ be a subset of $\mathbb{R}^2$, where the topology on $\mathbb{R}^2$ is the usual one. Prove that $[0,1]^2$ is connected under the restricted topology from $\mathbb{R}^2$.

6. Prove that the one point compactification of $[0,1)$ is homemorphic to $[0,1]$, where the topologies on $[0,1)$ and $[0,1]$ are the restricted topology from $\mathbb{R}$ (with the usual topology).

7. Let $\{0,1\}$ be a space with the discrete topology. Use $\{0,1\}_B^\infty$ to denote the product space $\prod_1^\infty \{0,1\}$ equipped with the box topology.

1) Prove that $\{0,1\}$ is compact.

2) Prove that $\{0,1\}_B^\infty$ is not compact.

8. For the space $[0,1]$ with the usual topology, consider the product $\prod_1^\infty [0,1]$ equipped with the box topology, which will be denoted by $[0,1]_B^\infty$. Define

$$f\colon [0,1] \longrightarrow [0,1]_B^\infty, \quad t \mapsto (t, t, \cdots).$$

Is this map f continuous or not? Prove your conjecture.

9. Let S^1 be the circle with the usual topology, and let $\mathbb{R}$ be the real number line with the usual topology. Prove that we can not find a continuous map f from S^1 to $\mathbb{R}$, such that f is both continuous and one-to-one (we do not require f to be onto though).

10. Prove that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.

11. In the class, while defining a ring of sets, instead of requiring that it is closed under taking unions and intersections, we require that this ring is closed under taking symmetric differences and intersections.

1) For any two sets A and B, express the set $A - B$ using A, B and finite many steps of symmetric

differences and intersections.

2) Construct two sets A and B, such that, starting from A and B, and via finitely many rounds of unions and intersections, we can never arrive at the set $A - B$. A proof is needed. **Hint:** It is not difficult.

12. For the topological space $\mathbb{R}$ with the usual topology, find a subset X of $\mathbb{R}$ such that X is not in the σ-algebra generated by all the open sets of $\mathbb{R}$. *Note: You need to prove that the X you constructed really satisfies such property.*

13. Does there exist an infinite σ-algebra that has only countably many members? That is, in formal mathematical language, let Σ be a subset of the power set of some set X such that Σ is a σ-algebra. Is it possbile that $|\Sigma| = |\mathbb{N}|$? If so, give an example. If not, give a proof.

14. Define

$$\mu\colon \{[a,b)\colon a, b \in \mathbb{R}, a \leq b\} \longrightarrow [0,\infty], \ [a,b) \mapsto b-a.$$

Let $\lambda\colon \mathcal{P}(\mathbb{R}) \to [0,\infty]$ be the outer measured induced by μ, where $\mathcal{P}(\mathbb{R})$ denotes the power set of $\mathbb{R}$.

1) Suppose that $[0,1] \subset \bigcup_{i=1}^{\infty}(a_i, b_i)$. Prove that $\sum_{i=1}^{\infty}(b_i - a_i) \geq 1$.

2) Suppose that $[0,1] - \mathbb{Q} \subset \bigcup_{i=1}^{\infty}(a_i, b_i)$. Prove that $\sum_{i=1}^{\infty}(b_i - a_i) \geq 1$.

3) Without using such heavy machinery as the Caratheodory Extension Theorem, following the definition of outer measures, prove that $\lambda([0,1] - \mathbb{Q}) = 1$.