HW # 1 Due on Oct. 9th 2015.

1. Let $f\colon X \to Y$ be a map and let π_X be a topology on X. Prove that π_Y, the push-out/push-forward topology on Y defined via f and π_X, is really a topology. Also prove that the map f from the topological space (X, π_X) to the topological space (Y, π_Y) is continuous.

2. Prove that $\mathbb{R}$ equipped with the usual topology, which is derived from the distance function $d(x, y) = |x - y| \quad \forall x, y \in \mathbb{R}$, is second countable.

3. In a topological space X, let $K \subset X$ be a compact subset. If $D \subset K$ and D is closed, prove that D is also compact. Note that we do not need X to be Hausdorff.

4. In $\mathbb{R}^2$, let $D = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 \leq 1\}$. Prove that we can not find $X, Y \subset \mathbb{R}$, such that $X \times Y = D$.

5. In a topological space X, for any subset A in X. Prove that $\overline{A}$ is always closed, and $\operatorname{int}(A)$ is always open. If A is open, prove that $\partial A = \overline{A} \setminus A$. If A is closed, prove that $\partial A = A \setminus \operatorname{int}(A)$.

6. Construct a set X and two topologies π_1 and π_2 on X, such that $\pi_1 \not\subset \pi_2$ and $\pi_2 \not\subset \pi_1$.

7. On $\mathbb{R}$, let π_1 be the topology generated by the distance function $d(x, y) = |x - y| \quad \forall x, y \in \mathbb{R}$. Let π_2 be the topology generated by the following topological basis

$$\pi_2^{(0)} = \{(x - r, x + 2r) \colon x \in \mathbb{R}, r \in \mathbb{R}_{>0}\}.$$

Prove that $\pi_1 = \pi_2$.

8. Let $X = \{a, b\}$ and let $\pi_X = \{\emptyset, \{a\}, \{a, b\}\}$. First prove that π_X is a topology on X. Then prove that this topology π_X is not metrizable.

9. In $\mathbb{R}^2$, let $S = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 = 1\}$.

a) For any straight line $L \subset \mathbb{R}^2$, prove that $L \cap S$ contains at most two points. (This should be junior high school math).

b) Prove that we can not find countably many straight lines $L_1, L_2, \cdots$ in $\mathbb{R}^2$, such that

$$\bigcup_{i=1}^{\infty} L_i = \mathbb{R}^2.$$

10. For $\mathbb{R}$, let $\pi_{\mathbb{R}}$ be the usual topology derived from the distance function $d(x, y) = |x-y| \quad \forall x, y \in \mathbb{R}$. As for $\mathbb{Q}$, the set of all rational numbers, which is a subset of $\mathbb{R}$, let $\pi_{\mathbb{Q}}$ be the topology on $\mathbb{Q}$ derived from $\pi_{\mathbb{R}}$ via restriction (i.e, the restricted topology). Prove the following:

1) $(\mathbb{R}, \pi_{\mathbb{R}})$ is locally compact. *Note: You can use the following fact directly without proof: In $\mathbb{R}$, for any $a, b \in \mathbb{R}$ with $a < b$, $[a, b]$ is compact with respect to the topology $\pi_{\mathbb{R}}$.*

2) $(\mathbb{Q}, \pi_{\mathbb{Q}})$ is not locally compact.

3) $(\mathbb{Q}, \pi_{\mathbb{Q}})$ is σ-compact.