Egorov's Theorem, a detailed proof.

Theorem: Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $\{f_n\}$ be a sequence of measurable functions on X and let f be a measurable function on X. Assume that $f_n \to f$ a.e. pointwise. Then for any $\epsilon > 0$, there exists a measurable set D of X, such that $\mu(D) < \epsilon$ and $f_n \to f$ uniformly on $X - D$.

Proof.

As $f_n \to f$ pointwise a.e. on X, without loss of generality, we can just assume that $f_n \to f$ pointwise on X.

From the ϵ-N definition of $f_n(x) \to f(x)$, and note that $f_n \to f$ on X, for any $\epsilon > 0$, we have

$$X = \bigcup_{N \in \mathbb{N}_{\geq 1}} \bigcap_{n > N} \{x \colon |f_n(x) - f(x)| < \epsilon\}.$$

Use $E_{n,\epsilon}$ to denote $\{x \colon |f_n(x) - f(x)| < \epsilon\}$. Note that $1/k \to 0$ as $k \to \infty$, we then translate the ϵ-N language of pointwise convergence on X to

$$X = \bigcup_{N \in \mathbb{N}_{\geq 1}} \bigcap_{n > N} E_{n,1/k} \quad \text{for all} \quad k \in \mathbb{N}_{\geq 1}.$$

Note that

$$\bigcap_{n>1} E_{n,1/k} \subset \bigcap_{n>2} E_{n,1/k} \subset \cdots .$$

By the continuity of measures, we have

$$\mu(X) = \lim_{N \to \infty} \mu \left(\bigcap_{n>N} E_{n,1/k} \right).$$

As $\mu(X) < \infty$, it follows that (as no such case as $\infty - \infty$ is involved)

$$\mu((\bigcap_{n>N} E_{n,1/k})^c) = \mu(X - \bigcap_{n>N} E_{n,1/k}) = \mu(X) - \mu(\bigcap_{n>N} E_{n,1/k}) \to 0 \quad \text{as} \quad N \to \infty.$$

Then for all $k \in \mathbb{N}_{\geq 1}$, we can choose $N_k \in \mathbb{N}_{\geq 1}$, such that

$$\mu(X - \bigcap_{n > N_k} E_{n,1/k}) < \epsilon/2^k.$$

Use X_k to denote $\bigcap_{n>N_k} E_{n,1/k}$. As $\mu(X - X_k) < \epsilon/2^k$ for all k, and note that

$$X - \bigcap_{k\geq 1} X_k = \left(\bigcap_{k\geq 1} X_k\right)^c = \bigcup_{k\geq 1} X_k^c = \bigcup_{k\geq 1}(X - X_k),$$

it follows that

$$\mu\left(X - \bigcap_{k\geq 1} X_k\right) \leq \sum_{k\geq 1} \mu(X - X_k) \leq \sum_{k\geq 1} \epsilon/2^k = \epsilon.$$

Let $D = X - \bigcap_{k\geq 1} X_k$. Then D is measurable and $\mu(D) < \epsilon$. It remains to show that $f_n \to f$ uniformly on $X - D = \bigcap_{k\geq 1} X_k$.

In fact, for any integer $m \geq 1$, note that $\bigcap_{k\geq 1} X_k \subset X_m$. Thus for any $x \in \bigcap_{k\geq 1} X_k$, we also have $x \in X_m$. According to the definition of X_m, for any $n > N_m$, we have

$$|f_n(x) - f(x)| < 1/m,$$

which finishes the proof. Q.E.D.