Review Problems for "Applied Functional Analysis"

1. Let M be an inner product space, on which the norm is derived from inner products via

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

Prove that for any $x, y \in M$, we have the following Parallelogram identify:

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

2. On $\mathbb{C}^2$, define two norms as

$$\|(x,y)\|_1 = \sqrt{|x|^2 + |y|^2} \quad \text{and} \quad \|(x,y)\|_2 = |x| + |y|.$$

Prove that these two norms are equivalent. In other words, prove that there exist $0 < C_1 < C_2$, such that for all $(x, y) \in \mathbb{C}^2$,

$$C_1 \|(x,y)\|_2 \leq \|(x,y)\|_1 \leq C_2 \|(x,y)\|_2 .$$

3. Prove that $l^1 = \{(x_1, x_2, \cdots) \colon \sum_{i=1}^{\infty} |x_i| < \infty\}$ is not an inner product space. In other words, show that the norm on l^1, which is defined as

$$\|x\| = \sum_{i=1}^{\infty} |x_i|,$$

is not a norm derived from certain inner product.

4. On $\mathbb{C}^2$, for any $x = (x_1, x_2)$ and $y = (y_1, y_2)$, define $\rho(x, y) = |x_1 - y_1|^2 + |x_2 - y_2|^2$. Prove that ρ is not a metric on $\mathbb{C}^2$ (thus it cannot be a norm on $\mathbb{C}^2$).

5. Define

$$C^b(\mathbb{R}) = \{f \in C(\mathbb{R}) \colon \sup_{x\in\mathbb{R}} |f(x)| < \infty\}.$$

For any $f \in \mathbb{C}^(\mathbb{R})$, define its norm to be

$$\|f\| = \sup_{x\in\mathbb{R}} f(x).$$

Prove that $(C^b(\mathbb{R}), \|\|)$ is a normed space, but NOT an inner product space.

6. On $\mathbb{C}^2$, define the norm of $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2$ to be $\|(x, y)\| = \sqrt{|x|^2 + |y|^2}$. For a 2×2 matrix, regard it as an linear operator on $\mathbb{C}^2$ defined as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

Prove that this linear operator $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a bounded linear operator, and the operator norm satisfies

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| \leq |a| + |b| + |c| + |d|.$$

7. This is about finding a basis for finite dimensional Hilbert spaces, and about the properties of such basis.

On $\mathbb{R}^2$, define the inner product as

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2, \ \forall\, x, y \in \mathbb{R}^2, \text{ with } x = (x_1, x_2) \text{ and } y = (y_1, y_2).$$

It is easy to check that $\mathbb{R}^2$ is a Hilbert space under this inner product. Assume we have $e_1 = (\frac{3}{5}, \frac{4}{5})$. Easy to check that $\|e_1\| = 1$.

a) Find $e_2 \in \mathbb{R}^2$ such that

$$\|e_2\| = 1 \quad \text{and} \quad \langle e_1, e_2 \rangle = 0.$$

b) For the e_1, e_2 we have so far, prove that for any $x \in \mathbb{R}^2$,

$$\|x\|^2 = \langle x, x \rangle = \langle x, e_1 \rangle^2 + \langle x, e_2 \rangle^2.$$