Crossed product C^*-algebras from minimal dynamical systems.

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Definition

Let X, Y be two compact Hausdorff spaces. Let (X, α) and (Y, β) be two dynamical systems. They are conjugate if there exists $\sigma \in \text{Homeo}(X, Y)$ such that $\sigma \circ \alpha = \beta \circ \sigma$.

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Definition

Let X, Y be two compact Hausdorff spaces. Let (X, α) and (Y, β) be two dynamical systems. They are weakly approximately conjugate if there exist $\{\sigma_n \in Homeo(X, Y)\}$ and $\{\tau_n \in Homeo(Y, X)\}$, such that $\text{dist}(g \circ \beta, g \circ \tau_n^{-1} \circ \alpha \circ \tau_n) \to 0$ and $\text{dist}(f \circ \alpha, f \circ \sigma_n^{-1} \circ \beta \circ \sigma_n) \to 0$ for all $f \in C(X)$ and $g \in C(Y)$.

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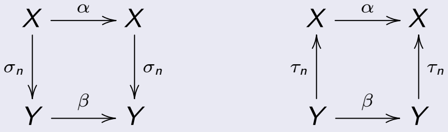

Roughly speaking, the diagrams above "approximately" commute.

Definition

Let $\{\varphi_n : A \to B\}$ be a sequence of positive linear maps. We say that $\{\varphi_n\}$ is an asymptotic morphism if $\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| \to 0$ for all $a, b \in A$.

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Example: Let X and Y be two compact Hausdorff spaces. Suppose that (X, α) and (Y, β) are approximately conjugate. Then we can find $\psi_n : C^*(\mathbb{Z}, Y, \beta) \to C^*(\mathbb{Z}, X, \alpha)$ such that $\{\psi_n\}$ is an asymptotic morphism induced by σ_n.

Definition (Lin)

Let X and Y be two compact Hausdorff spaces. Let (X, α) and (Y, β) be two minimal dynamical systems. Assume that $C^*(\mathbb{Z}, X, \alpha)$ and $C^*(\mathbb{Z}, Y, \beta)$ both have tracial rank zero. We say that (X, α) and (Y, β) are approximately K-conjugate if there exist homeomorphisms $\sigma_n : X \to Y$, $\tau_n : Y \to X$ and unital order isomorphisms $\rho : K_*(C^*(\mathbb{Z}, Y, \beta)) \to K_*(C^*(\mathbb{Z}, X, \alpha))$, such that

$$\sigma_n \circ \alpha \circ \sigma_n^{-1} \to \beta, \ \tau_n \circ \beta \circ \tau_n^{-1} \to \alpha$$

and the associated asymptotic morphisms $\psi_n : C^*(\mathbb{Z}, Y, \beta) \to C^*(\mathbb{Z}, X, \alpha)$ and $\varphi_n : C^*(\mathbb{Z}, X, \alpha) \to C^*(\mathbb{Z}, X, \beta)$ induce the isomorphisms ρ and ρ^{-1}.

Definition (Lin)

Let (X,α) and (X,β) be two minimal dynamical systems such that $\mathrm{TR}(C^*(\mathbb{Z},X,\alpha))=\mathrm{TR}(C^*(\mathbb{Z},X,\beta))=0$, we say that (X,α) and (X,β) are C^*-strongly approximately flip conjugate if there exists a sequence of isomorphisms

$$\varphi_n\colon C^*(\mathbb{Z},X,\alpha)\to C^*(\mathbb{Z},X,\beta),\ \psi_n\colon C^*(\mathbb{Z},X,\beta)\to C^*(\mathbb{Z},X,\alpha)$$

and a sequence of isomorphisms $\chi_n,\lambda_n\colon C(X)\to C(X)$ such that
1) $[\varphi_n]=[\varphi_m]=[\psi_n^{-1}]$ in $KL(C^*(\mathbb{Z},X,\alpha),C^*(\mathbb{Z},X,\alpha))$ for all $m,n\in\mathbb{N}$,
2) $\lim\limits_{n\to\infty}\|\varphi_n\circ j_\alpha(f)-j_\beta\circ\chi_n(f)\|=0$ and $\lim\limits_{n\to\infty}\|\psi_n\circ j_\beta(f)-j_\alpha\circ\lambda_n(f)\|=0$ for all $f\in C(X)$, with j_α,j_β being the injections from $C(X)$ into $C^*(\mathbb{Z},X,\alpha)$ and $C^*(\mathbb{Z},X,\beta)$.

Definition

Let (X, α) and (Y, β) be two minimal Cantor dynamical sytsems. We say that they are orbit equivalent if there exists a homeomorphism $F\colon X \to Y$ such that $F(\text{orbit}_\alpha(x)) = \text{orbit}_\beta(F(x))$ for all $x \in X$. The map F is called an orbit map.

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Definition

Let (X, α) and (Y, β) be two minimal Cantor dynamical sytsems that are orbit equivalent. Two integer-valued functions $m, n\colon X \to \mathbb{Z}$ are called orbit cocyles associated to the orbit map F if $F \circ \alpha(x) = \beta^{n(x)} \circ F(x)$ and $F \circ \alpha^{m(x)}(x) = \beta \circ F(x)$ for all $x \in X$. We say that (X, α) and (Y, β) are strongly orbit equivalent if they are orbit equivalent and the orbit cocycles have at most one point of discontinuity.

Theorem (Giordano, Putnam, Skau)

For minimal Cantor dynamical systems (X, α) and (Y, β), $C^*(\mathbb{Z}, X, \alpha)$ and $C^*(\mathbb{Z}, Y, \beta)$ are isomorphic if and only if (X, α) and (Y, β) are strongly orbit equivalent.

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Remark 2: For Cantor minimal dynamical systems, strongly orbit equivalent is an equivalence relationship.

Theorem (Lin, Matui)

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Corollary

For two Cantor minimal dynamical systems (X, α) and (Y, β), they are approximately K-conjugacy if and only if they are strongly orbit equivalent.

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$$M_{\alpha \times \varphi} \to M_{\alpha},$$

$$\tau \mapsto \widetilde{\tau}.\ \widetilde{\tau}(D) = \tau(D \times \mathbb{T}) \text{ for all Borel subset } D \subset X.$$

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Theorem (Lin, Phillips)

If the minimal dynamical system $(X \times \mathbb{T}, \alpha \times \varphi)$ is rigid, then the tracial rank of $C^*(\mathbb{Z}, X \times \mathbb{T}, \alpha \times \varphi)$ is zero. Furthermore, if the tracial rank of A is zero, then the dynamical system $(X \times \mathbb{T}, \alpha \times \varphi)$ if is rigid.

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The minimal dynamical system $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$ is rigid if the following map is one-to-one:

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Remark: Under this definition, if minimal dynamical system $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$ is rigid, then the crossed product C^*-algebra has tracial rank zero.

Do we have lots of minimal dynamical system of type $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$?

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Lemma

Given any minimal dynamical system $(X \times \mathbb{T}, \alpha \times \mathrm{R}_\xi)$, there exist uncountably many $\theta \in [0, 1]$ such that if we use θ to denote the constant function in $C(X, \mathbb{T})$ defined by $\theta(x) = \theta$ for all $x \in X$ (identifying $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$), then the dynamical system $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\theta)$ is still minimal.

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$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{\varphi_0} & \mathbb{T} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ \mathbb{T} & \xrightarrow[\mathrm{R}_\theta]{} & \mathbb{T} \end{array} \quad .$$

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Let $\varphi = \varphi_0 \,|_X$.

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X \\ {\scriptstyle h|_X}\downarrow & & \downarrow{\scriptstyle h|_X} \\ \mathbb{T} & \xrightarrow[\mathrm{R}_\theta]{} & \mathbb{T} \end{array} \quad .$$

Consider $\gamma\colon \mathbb{T}^3 \to \mathbb{T}^3, (s, t_1, t_2) \mapsto (s + \theta, t_1 + \xi(s), t_2 + \eta(s))$.

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$$\begin{array}{ccc} X\times\mathbb{T}\times\mathbb{T} & \xrightarrow{\alpha\times \mathrm{R}_{\xi\circ h}\times \mathrm{R}_{\eta\circ h}} & X\times\mathbb{T}\times\mathbb{T} \\ {\scriptstyle h|_X\times id_{\mathbb{T}}\times id_{\mathbb{T}}}\downarrow & & \downarrow{\scriptstyle h|_X\times id_{\mathbb{T}}\times id_{\mathbb{T}}} \\ \mathbb{T}\times\mathbb{T}\times\mathbb{T} & \xrightarrow{\gamma} & \mathbb{T}\times\mathbb{T}\times\mathbb{T} \end{array} .$$

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X \times \mathbb{T} \times \mathbb{T} & \xrightarrow{\alpha \times \mathrm{R}_{\xi\circ h} \times \mathrm{R}_{\eta\circ h}} & X \times \mathbb{T} \times \mathbb{T} \\
{\scriptstyle h|_X \times id_{\mathbb{T}} \times id_{\mathbb{T}}}\downarrow & & \downarrow{\scriptstyle h|_X \times id_{\mathbb{T}} \times id_{\mathbb{T}}} \\
\mathbb{T} \times \mathbb{T} \times \mathbb{T} & \xrightarrow{\gamma} & \mathbb{T} \times \mathbb{T} \times \mathbb{T}
\end{array}.$$

Proposition

For the minimal dynamical systems as in diagram above, if $(\mathbb{T} \times \mathbb{T} \times \mathbb{T}, \gamma)$ is a minimal dynamical system, then $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_{\xi\circ h} \times \mathrm{R}_{\eta\circ h})$ is also a minimal dynamical system. Also, there is a one-to-one correspondence between γ-invariant probability measures on $\mathbb{T}^3$ and $\alpha \times \mathrm{R}_{\xi\circ h} \times \mathrm{R}_{\eta\circ h}$-invariant probability measures on $X \times \mathbb{T} \times \mathbb{T}$.

Theorem (S)

Let X, Y be Cantor sets and let $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times R_{\xi_1} \times R_{\eta_1})$, $(Y \times \mathbb{T} \times \mathbb{T}, \beta \times R_{\xi_2} \times R_{\eta_2})$ be two minimal rigid dynamical systems. Use A and B to denote the corresponding crossed product C^*-algebra, and use j_A, j_B to denote the canonical embedding of $C(X \times \mathbb{T} \times \mathbb{T})$ into A and B. Then the following are equivalent:

a) $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times R_{\xi_1} \times R_{\eta_1})$ and $(Y \times \mathbb{T} \times \mathbb{T}, \beta \times R_{\xi_2} \times R_{\eta_2})$ are approximately K-conjugate.

b) There exists a unital order isomorphism ρ such that $\rho(K_i(j_B(C(X \times \mathbb{T} \times \mathbb{T})))) = K_i(j_A(C(Y \times \mathbb{T} \times \mathbb{T})))$ for $i = 0, 1$.

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$$
\begin{array}{ccc}
K_i(B) & \xrightarrow{\quad\rho\quad} & K_i(A) \\
\uparrow{\scriptstyle (j_B)_*} & & \uparrow{\scriptstyle (j_A)_*} \\
K_i(C(X \times \mathbb{T} \times \mathbb{T})) & & K_i(C(Y \times \mathbb{T} \times \mathbb{T}))
\end{array}
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Definition

Use A to denote the crossed prouduct C^*-algebra $C^*(\mathbb{Z}, X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_{\xi} \times \mathrm{R}_{\eta})$. Define A_x to be the sub-algebra generated by $C(X \times \mathbb{T} \times \mathbb{T})$ and $uC_0((X \backslash \{x\}) \times \mathbb{T} \times \mathbb{T})$.

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Remark: As we cutting off one fiber $x \times \mathbb{T} \times \mathbb{T}$ instead of one single ponit, we shall no longer expect to have $K_i(A_x) \cong K_i(A)$.

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Lemma

For the A_x defined above, $TR(A_x) \leq 1$.

Lemma

Let j be the inclusion of A_x in A, then $(j_*)_i \colon K_i(A_x) \to K_i(A)$ is injective for $i = 0, 1$.

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Yes, we do. There are examples of minimal dynamical system $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$ such that it is not rigid, and in the corresponding crossed product C^*-algebra, the projection does not separate traces (and the crossed product C^*-algebra has tracial rank one).

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Definition

A map $F\colon \mathbb{T}^2 \to \mathbb{T}^2$ is called a Furstenberg transformation of degree d if there exist $\theta \in \mathbb{T}$ and continuous function $f\colon \mathbb{R} \to \mathbb{R}$ satisfying $f(x+1) - f(x) = d$ for all $x \in \mathbb{R}$ such that (identifying $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$)

$$F(t_1, t_2) = (t_1 + \theta, t_2 + f(t_1)).$$

For the F above, d is called the degree of Furstenberg transform F, and is denoted by $\deg(F)$. The number d is also called the degree of f, and denoted by $\deg(f)$.

Proposition

For the minimal dynamical system $(X \times \mathbb{T}^2, \alpha \times \varphi)$ *with cocycles being Furstenberg transformations, use* A *to denote the crossed product* C^**-algebra of this dynamical system and use* $K^0(X, \alpha)$ *to denote* $C(X, \mathbb{Z})/\{f - f \circ \alpha \colon f \in C(X, \mathbb{Z})\}$.

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1) If $[\deg(\varphi(x))] \neq 0$ *in* $K^0(X, \alpha)$*, then*

$$K_0(A) \cong C(X, \mathbb{Z}^2)/\{f - f \circ \alpha \colon f \in C(X, \mathbb{Z}^2)\} \oplus \mathbb{Z}$$

and $K_1(A)$ *is isomorphic to*

$$C(X, \mathbb{Z}^2)/\{(f, g) - (f, g) \circ \alpha - (\deg(\varphi) \cdot (g \circ \alpha), 0) \colon f, g \in C(X, \mathbb{Z})\} \oplus \mathbb{Z}^2.$$

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$$C(X, \mathbb{Z}^2)/\{(f, g) - (f, g) \circ \alpha - (\deg(\varphi) \cdot (g \circ \alpha), 0) \colon f, g \in C(X, \mathbb{Z})\} \oplus \mathbb{Z}^2.$$

2) If $[\deg(\varphi(x))] = 0$ *in* $K^0(X, \alpha)$, *then*

$$K_0(A) \cong C(X, \mathbb{Z}^2)/\{f - f \circ \alpha \colon f \in C(X, \mathbb{Z}^2)\} \oplus \mathbb{Z}^2$$

and $K_1(A)$ *is isomorphic to*

$$C(X, \mathbb{Z}^2)/\{(f, g) - (f, g) \circ \alpha - (\deg(\varphi) \cdot (g \circ \alpha), 0) \colon f, g \in C(X, \mathbb{Z})\} \oplus \mathbb{Z}^2.$$

Question 1: How can we extend of the results to more general topological base space?

Question 2: When the crossed product C^*-algebra has tracial rank one (say, for the non-rigid cases), what is the relationship between approximately K-conjugacy and isomorphism of the crossed-product C^*-algebras?