On conjugacies of C^*-dynamics
– with an introduction to the classification program

Wei Sun

Research Center for Operator Algebras
Department of Mathematics
East China Normal University

9th Korean Operator Algebra Seminar
November 8th, 2015

Definition (C^*-algebra)

A C^*-algebra A is a Banach algebra over $\mathbb{C}$, with an anti-isomorphism $*\colon A \to A$ [that is, $(a+b)^* = a^* + b^*, (ab)^* = b^* a^*, (a^*)^* = a$ and $(\lambda a)^* = \overline{\lambda} a^*$ for all $\lambda \in \mathbb{C}$ and $a, b \in A$] , satisfying $\|aa^*\| = \|a\|^2$.

Definition (Simpleness of C^*-algebra)

A C^*-algebra A is simple if there is no proper two sided ideal of A that is norm closed.

Examples:

$\mathbb{C}$. *Trivial case.*

$C_0(X)$, where X is a locally compact Hausdorff space, with the norm of any $f \in C_0(X)$ being $\|f\| = \sup_{x \in X} |f(x)|$. *All the abelian cases.*

$M_n(\mathbb{C})$ with the operator norm $\|a\| = \sup_{x \in \mathbb{C}^n, \|x\|=1} \|ax\|$. *All the simple and finitely dimensional cases.*

$\bigoplus_{k=1}^{M} M_{n_k}(\mathbb{C})$ with the operator norm. *All the finitely dimensional cases.*

Tensor products. $C(X) \otimes M_n(\mathbb{C}) \cong C(X, M_n(\mathbb{C}))$, etc.

Direct limits. $M_{2^\infty} = \lim M_{2^n}$. $\mathcal{K} = \lim M_n$. $\lim \bigoplus_{k=1}^{D_n} M_{s_{n,k}}(\mathbb{C})$ (AF algebras). $\lim \bigoplus_{k=1}^{D_n} M_{s_{n,k}}(X_n)$ (AH algebras)

Examples continued

$B(H)$, $B(H)/\mathcal{K}$, where H is a separable infinite dimensional Hilbert space.

$A = \lim A_n$, where each A_n is a subalgebra of certain $M_{k_n}(X_n)$. ASH algebras (approximate sub-homogeneous algebras)

The Jiang-Su algebra $\mathcal{Z}$ is an ASH algebra, while it is not an AH algebra.

Some crossed product C^*-algebras $C(X) \rtimes_\alpha \mathbb{Z}$ are ASH algebras.

Definition (finite and stably finite)

A unital C^*-algebra A is finite, if $xx^* = 1_A$ implies that $x^*x = 1_A$. A unital C^*-algebra A is stably finite, if $M_n(A)$ is finite for all $n \in \mathbb{N}_{\geq 1}$.

AH algebras and ASH algebras are all stably finite, that is because each matrix algebra is finite. In fact, if A is finite/stably finite, then so is any subalgebra B with $1_B = 1_A$. If A is finite/stably finite, then so is $C(X, A)$ for any compact Hausdorff space X.

We will focus on stably finite C^*-algebras in this talk.

Current classsification program is mainly on simple, separable amenable unital C^*-algebras. The goal is to give complete invariants (the Elliott Invariant), which is relatively easy to handle, that classifies those C^*-algebras.

Definition (The Elliott Invariant)

The Elliott Invariant of a unital C^*-algebra A consisists of ordered K_0, K_1, the tracial space $T(A)$, and the pairing between $K_0(A)$ and $T(A)$. That is

$$\mathrm{Ell}(A) = (K_0(A), K_0(A)_+, [1_A], K_1(A), T(A), r_A),$$

where $r_A\colon T(A) \to S(K_0(A), K_0(A)_+, [1_A])$ [linear, order preserving, maps $[1_A]$ to 1]

Remark: $K_0(A)$ is generated by projections in $M_\infty(A)$ (where $M_\infty(A)$ is the algebraic limit of $A \to M_2(A) \to M_3(A) \to \cdots$).

Remark: In case A is abelian, i.e, $A = C(X)$, the $K_0(A)$ just corresponds to the topological K-theory of X.

Remark: r_A is always onto when A is exact (tensoring with A preserves short exact sequences).

Remark: r_A is injective if A has real rank zero (invertible elements are dense in $A_{s.a}$, which is $\{a \in A\colon a = a^*\}$).

Examples:

For $A = M_n(\mathbb{C})$,

$$(K_0(A), K_0(A)_+, [1_A]) = (\mathbb{Z}, \mathbb{Z}_{\geq 0}, n).$$

For $A = C(\mathbb{T})$,

$$(K_0(A), K_0(A)_+, [1_A]) = (\mathbb{Z}, \mathbb{Z}_{\geq 0}, 1).$$

For $A = C(\mathbb{T}^2)$,

$$(K_0(A), K_0(A)_+, [1_A]) = \left(\mathbb{Z}^2, \left\{(m, n) \in \mathbb{Z}^2 \colon m > 0\} \cup \{(0, 0)\right\}, (1, 0)\right).$$

For $A = C(X)$, where X is the Cantor set,

$$(K_0(A), K_0(A)_+, [1_A]) = \left(C(X, \mathbb{Z}), C(X, \mathbb{Z})_{\geq 0}, 1_X\right).$$

For $A = M_{2^\infty}(\mathbb{C})$,

$$(K_0(A), K_0(A)_+, [1_A]) = \left(\mathbb{Z}\left[\frac{1}{2}\right], \mathbb{Z}\left[\frac{1}{2}\right]_{\geq 0}, 1\right),$$

where

$$\mathbb{Z}\left[\frac{1}{2}\right] = \left\{\sum_{i=1}^{n} \frac{k_i}{2^i} \colon n \in \mathbb{N}_{\geq 1}, k_i \in \mathbb{Z}\right\}.$$

It is important to choose the right invariants/descriptions during classification.

While dealing with separable AF algebras, from the naive defintion, $M_{2^\infty} = \lim M_{2^n}$. Given $A = \lim A_n$, where each A_n is finitely dimensional (finite direct sum of matrices), how can we determine whether A is isomorphic to M_{2^∞} or not?

The Elliott invariant turns out to suit the purpose (classifying unital separable AF-algebras) perfectly, even in case the AF-algebras are not simple.

Definition (nuclear/amenable C^*-algebras)

A C^*-algebra A is nuclear, if for any $\epsilon > 0$ and any finite subse $\mathcal{F}$ of A, there exists completely contractive positive maps $\varphi\colon A \to M_n(\mathbb{C})$ and $\psi\colon M_n(\mathbb{C}) \to A$, such that

$$(\psi \circ \varphi)(x) \approx_\epsilon x, \ \forall\, x \in \mathcal{F}.$$

In other words, the following diagram approximately commutes on $\mathcal{F}$.

$$\begin{tikzcd} A \arrow[rr, "\mathrm{id}"] \arrow[dr, "\varphi"'] & & A \\ & M_n(\mathbb{C}) \arrow[ur, "\psi"'] & \end{tikzcd}$$

The general strategy for classification program of C^*-algebras: the existence theorem and the uniqueness theorem.

The Existence Theorem

Given a homomorphism $\rho\colon \mathrm{Ell}(A) \to \mathrm{Ell}(B)$, lift ρ to a homomorphism (or completely positive contractive) map $\varphi\colon A \to B$, such that φ will induce ρ.

The Uniqueness Theorem

Given two homomorphism (or completely positive contractive) maps $\varphi_i\colon A \to B$, $i = 1, 2$, such that the induced homomorphisms $(\varphi_i)_*\colon \mathrm{Ell}(A) \to \mathrm{Ell}(B)$ are the same, prove that φ_1 is unitarily equivalent to (or "approximatley" unitarily equivalent to) φ_2.

If these two above mentioned theorems can be established for certain C^*-algebras, just combine them together, and we can get the classification program for those C^*-algebras.

Remark: In actual cases, oftentimes, it is not easy to lift an homomorphism of Elliott invariants to homomorphisms or c.p.c maps from A to B. Typically, we use the local structures of A and B. That is, assume $A = \lim A_n$ and $B = \lim B_n$, where A_n and B_n has relateive simple structure and the lifting among "local" algebras is easier to find. This will lead to the intertwining technique/argument.

A sketch of how typical intertwining argument works together with existence theorem and uniqueness theorem

$$
\begin{tikzcd}
A_1 \arrow[r] \arrow[dr, "\varphi_1"'] & A_2 \arrow[r] & A_3 \arrow[r] & A_4 \arrow[r] \arrow[drr, "\varphi_4"] & A_5 \arrow[r] & A_6 \arrow[r] & \cdots & A \arrow[d, shift right, "\varphi"'] \\
B_1 \arrow[r] & B_2 \arrow[r] \arrow[urr, "\psi_2"'] & B_3 \arrow[r] & B_4 \arrow[r] & B_5 \arrow[r] & B_6 \arrow[r] \arrow[ur, "\psi_6"] & \cdots & B \arrow[u, shift right, "\psi"']
\end{tikzcd}
$$

1. Starting from the isomorphism maps between $\mathrm{Ell}(A)$ and $\mathrm{Ell}(B)$, try to lift to (existence theorem) homomorphisms or c.p.c maps between A_ns and B_ns.

2. Generally speaking, those triangles above might not commute. By adjoining with unitary elements, we make them commutes (or "approximately" commutes). This uses the uniqueness theorem.

3. As those triangles commutes or almost commutes (can let the error terms be controlled by $\epsilon/2^n$), standard argument will yield ψ and φ which are inverse of each other.

Remark: Typically, the uniqueness theorem is harder to achieve.

As for the classification purpose of C^*-algebras of C^*-algebras using the above defined Elliott invariants, we focus on unital, simple, separable, amenable C^*-algebras.

Why simple? In short, to make the goal obtainable. Else, just think about $C(X)$.

Remark: For certain non-simple C^*-algebras whose ideal structure can be somewhat recovered from the Elliott invariants, the Elliott conjecture might still hold. For example, all the separable AF-algebras can be classified using the Elliott invariants.

Why separable? There exists separable AF algebra A and non separable AF algebra B whose K_* are isomorphic (but they are not). Furthermore, there is a set-theoretic argument that ensures that Elliott invariant is not enough while dealing with non-separable C^*-algebras.

Why amenable? A requirement to ensure the uniqueness theorem in classification.

Fact: For any minimal homemorphism α on the separable compact Hausdorff space X, the crossed product C^*-algebra $C(X) \rtimes_\alpha \mathbb{Z}$ is unital, simple, separable and amenable.

Main classification results along the line.

Theorem (Elliott)

Let A and B be two unital separable AF algebras, then $A \cong B$ if and only if

$$(K_0(A), K_0(A)_+, [1_A]) \cong (K_0(B), K_0(B)_+, [1_B]).$$

Remark: *The main tools for classification such as the "uniqueness theorem + existence theorem" approach and the intertwining technique are used in the proof of this result.*

Theorem (Elliott)

Let A and B be two unital simple $A\mathbb{T}$ algebras of real rank zero, then $A \cong B$ if and only if

$$(K_0(A), K_0(A)_+, [1_A], K_1(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(A)).$$

Remark: For these two results, we do not need such information as the tracial space $T(A)$ and the pairing between $K_0(A)$ and $T(A)$, because they are about real rank zero case, in which those information can be recovered from $(K_0(A), K_0(A)_+, [1_A])$.

Theorem (Elliott-Gong-Li)

The Elliott conjecture (two unital simple separable amenable C^*-algebras are isomorphic if and only if they have the same Elliott invariant) holds for AH-algebras of no dimension growth (there is a global bound on the dimension of the base space X_is).

Remark: If we do not assume "no dimension growth" condition on AH-algebras, the Elliott conjecture will fail.

Theorem (Lin)

The Elliott conjecture holds for all unital simple nuclear C^*-algbras that has tracial rank no more than one and satisfies the UCT.

It is known that for all those unital simple nuclear C^*-algbras that has tracial rank no more than one and satisfies the UCT, they are exactly those simple AH-algebras with no dimension growth. Lin's result, however, make the classification result independent of the local structure of AH-algebras ($A = \lim A_n$). It is generally easier to check the tracial approximation structure than to write the C^*-algebra as direct limit of certain "building blocks".

Definition (tracial rank)

For a unital separable simple C^*-algebra A, we say that A has tracial rank no more than k, denoted as $TR(A) \leq k$, if for any finite subset $\mathcal{F}$ of A, any $c \in A_+ \setminus \{0\}$, and any $\epsilon > 0$, there exists a unital subalgebra D such that D is isomorphic to finite direct sum of $M_{n_i}(X_i)$, such that

1) $\|x1_B - 1_B x\| < \epsilon$

2) $\mathrm{dist}(1_B x 1_B, B) < \epsilon$

3) $1 - 1_B$ is murry von-Neumann equivalent to a projection in $\mathrm{Her}(c)$.

Tracial rank zero C^*-algebra can be tracially approximated by finite dimensional C^*-algebras.

Tracial rank one C^*-algebra can be tracially approximated by interval algebras.

Fact: Any simple C^*-algebra of finite tracial rank has property (SP). That is, there exists non trivial projections in any non-degenerate hereditary subalgebras.

How to classify C^*-algebras without so many projections (not having property SP)?

Can we expect the Elliott conjecture to hold for more than those above mentioned AH-algebras?

There are unital separable simple amenable C^*-algebras A and B, with $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$, but A is not isomorphic to B. (Rordam, Toms)

Theorem (Toms)

There are stably finite, unital, separable, simple amenable C^*-algebras A and B, with $F(A) \cong F(B)$, but A is not isomorphic to B, where F contains 1) All the homotopy invariant functors that commutes with inductive limits. 2) real rank. 3) The Elliott invariants.

Two options.

Option 1, to use finer invariants that is not homotopy invariant. For example, the Cuntz semigroup $Cu(A)$. (This option had been tried.)

Option 2, to add some regularity requirements for the C^*-algebras.

Theorem (Toms, Winter)

All the previous classification results on stably finite, unital, simple, separable amenable C^*-algebras are for those that absorbs the Jiang-Su algebra $\mathcal{Z}$.

As for option 2, just to deal with stably finite, unital, simple, separable amenable C^*-algebras that is $\mathcal{Z}$-absorbing.

Definition (dimension drop C^*-algebras)

A dimension drop C^*-algebra is of the form

$$I[m_0, m, m_1] = \{f \in C([0,1], M_m)\colon f(0) \in M_{m_0} \otimes 1_{m/m_0}, f(1) \in M_{m_1} \otimes 1_{m/m_1}\}.$$

Definition

The Jiang-Su algebra is *the* direct limit of dimension drop C^*-algebras that is unital, simple, amenable and has the same Elliott invariant as $\mathbb{C}$.

Definition (rational tracial rank)

For a unital simple separable C^*-algebra A, we say that the rational tracial rank of A is no more than one, if $TR(A \otimes \mathbb{Q}) \leq 1$, where $\mathbb{Q}$ is $\lim M_{n!}(\mathbb{C})$, with the connecting maps being the diagonal embeddings.

Example: The Jiang-Su algebra $\mathcal{Z}$ has finite rational tracial rank ($\mathcal{Z} \otimes \mathbb{Q} \cong \mathbb{Q}$), but it does not have finite tracial rank (as it allows no proper projections).

Theorem (Winter, Lin, Niu)

The Elliott conjecture holds for unital simple separable amenable C^*-algebras that has rational tracial rank no more than one, satisfies the UCT and absorbs the Jiang-Su algebra $\mathcal{Z}$.

The result above expanded our chart of classifiable C^*-algebras significantly. For example, it makes the Jiang-Su algebra $\mathcal{Z}$ classifiable.

However, the Elliott invariants of all those above (stably finite, simple unital separable amenable C^*-algebras that has finite rational tracial rank, satisfies the UCT and absorbs $\mathcal{Z}$) cannot exhaust the Elliott invariant of all those stably finite, simple unital separable amenable C^*-algebras that absorbs $\mathcal{Z}$.

Theorem (Gong, Lin, Niu)

The Elliott conjecture holds for a class $\mathcal{N}_1$ of stably finite, simple, unital, separable, amenable C^*-algebras that absorbs $\mathcal{Z}$. Moreover, the Elliott invariant of $\mathcal{N}_1$ exhausts the Elliott invariant of stably finite, simple, unital, separable, amenable C^*-algebras that absorbs $\mathcal{Z}$.

As for stably finite, simple, unital, separable, amenable C^*-algebras that absorbs $\mathcal{Z}$, you shall not expect the Elliott conjecture to hold for a class larger than $\mathcal{N}_1$.

Applications of the classificatoin program (results, methods, etc)

AF embeddings.

Can we embed a classifiable simple C^*-algebra A into a given AF-algebra B? Just check whether there is a map (preserving the structure) from $\mathrm{Ell}(A)$ to $\mathrm{Ell}(B)$. If that is doable, check whether we have the suitable existence theorem that can lead to a homomorphism from A to B. If so, we are done as A is simple.

Structure of irrational rotation algebra A_θ (generated by two untiaries u and v with $uv = e^{2\pi i\theta} vu$).

We can show first that they have tracial rank no more than one and satisfies the UCT, thus they are classifiable. By checking that their Elliott invariants can be realized by A$\mathbb{T}$-algebras, and note that A$\mathbb{T}$-algebras are classifiable, we proved that A_θ is an A$\mathbb{T}$-algebra without bothering to write $A = \lim A_n$, where each A_n is a finite sum of $M_{k_i}(C(\mathbb{T}))$.

When are A_{θ_1} and A_{θ_2} isomorphic?

When are $C(X) \rtimes_\alpha \mathbb{Z}$ and $C(X) \rtimes_\beta \mathbb{Z}$ isomorphic?