2.3 埃尔米特 (Hermite) 插值

朱升峰

数学科学学院

华东师范大学

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重节点均差

根据均差性质，易证

定理 1

设 $f(x) \in C^n[a,b]$，$x_0, x_1, \cdots, x_n$ 为 $[a,b]$ 上互异节点，则 $f[x_0, x_1, \cdots, x_n]$ 是各变量的多元连续函数。

定义重节点的一阶均差

$$f[x_0, x_0] := \lim_{x \to x_0} f[x_0, x] = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

定义重节点的二阶均差

$$f[x_0, x_0, x_1] = \frac{f[x_0, x_1] - f[x_0, x_0]}{x_1 - x_0}$$

当 $x_1 \to x_0$ 时，由性质 $f[x_0, x_1, \cdots, x_n] = \frac{f^{(n)}(\xi)}{n!}$ 得

$$f[x_0, x_0, x_0] = \lim_{\substack{x_1 \to x_0 \\ x_2 \to x_0}} f[x_0, x_1, x_2] = \frac{1}{2} f''(x_0).$$

泰勒（Taylor）插值多项式

定义重节点的 n 阶均差

$$f[x_0, x_0, \cdots, x_0] = \lim_{x_i \to x_0} f[x_0, x_1, \cdots, x_n] = \frac{1}{n!} f^{(n)}(x_0).$$

Newton 均差插值多项式中若令 $x_i \to x_0 (i = 1, 2, \cdots, n)$ 则得泰勒插值多项式

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

满足插值条件

$$P_n^{(k)}(x_0) = f^{(k)}(x_0), \quad k = 0, 1, \cdots, n$$

余项为

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}, \quad \xi \in (a, b).$$

Lagrange 插值 $\longleftrightarrow$ Newton 插值 $\longleftrightarrow$ Taylor 插值

Hermite 插值

$n+1$个含函数值和导数值的插值条件可构造次数 $\leqslant n$ 的 Hermite 插值多项式

- 三点三次 Hermite 插值多项式
- 两点三次 Hermite 插值多项式

三点三次 Hermite 插值多项式: 构造插值多项式 $P = P(x)$ 满足插值条件

$$P(x_i) = f(x_i)\ (i = 0, 1, 2)\ 且 P'(x_1) = f'(x_1)$$

f 的二次 Newton 插值多项式

$$P_2(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

满足 $P_2(x_i) = f(x_i)(i = 0, 1, 2)$ 。因而

$$(P - P_2)(x_i) = 0 \Rightarrow P(x) - P_2(x) = A(x - x_0)(x - x_1)(x - x_2)$$

插值余项

由 $P'(x_1) = f'(x_1)$ 得

$$A = \frac{f'(x_1) - f[x_0, x_1] - (x_1 - x_0) f[x_0, x_1, x_2]}{(x_1 - x_0)(x_1 - x_2)}$$

设

$$R(x) := f(x) - P(x) = k(x)(x - x_0)(x - x_1)^2(x - x_2),$$

其中 $k(x)$ 为待定函数。类似 Lagrange 插值余项的推导得

定理 2

若 $f(x) \in C^4[a, b]$，则对任何 $x \in [a, b]$，插值余项

$$R(x) = \frac{f^{(4)}(\xi)}{4!}(x - x_0)(x - x_1)^2(x - x_2),$$

$\xi \in (a, b)$ 且依赖于 x。

例：给定 $f(x) = x^{3/2}, x_0 = \frac{1}{4}, x_1 = 1, x_2 = \frac{9}{4}$，试求三次 Hermite 插值多项式 $p(x)$，使其满足 $p(x_i) = f(x_i)\ (i = 0, 1, 2)$，$p'(x_1) = f'(x_1)$，并求出余项表达式。

解

$$f_0 = f\left(\frac{1}{4}\right) = \frac{1}{8}, \quad f_1 = f(1) = 1, \quad f_2 = f\left(\frac{9}{4}\right) = \frac{27}{8},$$

$$f[x_0, x_1] = \frac{7}{6},\ f[x_0, x_1, x_2] = \frac{11}{30}$$

令

$$\begin{aligned} P(x) =& \frac{1}{8} + \frac{7}{6}\left(x - \frac{1}{4}\right) + \frac{11}{30}\left(x - \frac{1}{4}\right)(x-1) \\ &+ A\left(x - \frac{1}{4}\right)(x-1)\left(x - \frac{9}{4}\right) \end{aligned}$$

由条件$P'(1) = f'(1) = \frac{3}{2}$ 可得

$$A = -\frac{14}{225}.$$

所求 Hermite 插值多项式

$$P(x) = \frac{1}{8} + \frac{7}{6}\left(x - \frac{1}{4}\right) + \frac{11}{30}\left(x - \frac{1}{4}\right)(x-1) - \frac{14}{225}\left(x - \frac{1}{4}\right)(x-1)\left(x - \frac{9}{4}\right)$$

余项

$$\begin{aligned} R(x) = f(x) - P(x) &= \frac{f^{(4)}(\xi)}{4!}\left(x - \frac{1}{4}\right)(x-1)^2\left(x - \frac{9}{4}\right). \\ &= \frac{1}{4!}\frac{9}{16}\xi^{-5/2}\left(x - \frac{1}{4}\right)(x-1)^2\left(x - \frac{9}{4}\right), \quad \xi \in \left(\frac{1}{4}, \frac{9}{4}\right). \end{aligned}$$

两点三次 Hermite 插值多项式

已知插值节点 x_k，x_{k+1} 及节点上函数值和导数值求插值多项式 $H_3(x)$，满足

$$
\begin{aligned}
H_3\left(x_k\right) = y_k, \quad & H_3\left(x_{k+1}\right) = y_{k+1}, \\
H_3'\left(x_k\right) = m_k, \quad & H_3'\left(x_{k+1}\right) = m_{k+1}.
\end{aligned}
$$

插值基函数方法: 基函数 $\alpha_k(x)$, $\alpha_{k+1}(x)$, $\beta_k(x)$, $\beta_{k+1}(x)$

$$
H_3(x) = \alpha_k(x) y_k + \alpha_{k+1}(x) y_{k+1} + \beta_k(x) m_k + \beta_{k+1}(x) m_{k+1},
$$

满足插值条件

$$
\begin{aligned}
&\alpha_k\left(x_k\right) = 1, \quad \alpha_k\left(x_{k+1}\right) = 0, \quad \alpha_k'\left(x_k\right) = \alpha_k'\left(x_{k+1}\right) = 0 \\
&\alpha_{k+1}\left(x_k\right) = 0, \quad \alpha_{k+1}\left(x_{k+1}\right) = 1, \quad \alpha_{k+1}'\left(x_k\right) = a_{k+1}'\left(x_{k+1}\right) = 0;
\end{aligned}
$$

$$
\begin{aligned}
&\beta_k\left(x_k\right) = \beta_k\left(x_{k+1}\right) = 0, \quad \beta_k'\left(x_k\right) = 1, \quad \beta_k'\left(x_{k+1}\right) = 0; \\
&\beta_{k+1}\left(x_k\right) = \beta_{k+1}\left(x_{k+1}\right) = 0, \quad \beta_{k+1}'\left(x_k\right) = 0, \quad \beta_{k+1}'\left(x_{k+1}\right) = 1
\end{aligned}
$$

由 $\alpha_k(x_{k+1}) = 0, \alpha_k'(x_{k+1}) = 0$, 可设

$$\alpha_k(x) = (ax + b)(x - x_{k+1})^2$$

又由

$$1 = \alpha_k(x_k) = (ax_k + b)(x_k - x_{k+1})^2$$
$$0 = \alpha_k'(x_k) = 2(ax_k + b)(x_k - x_{k+1}) + a(x_k - x_{k+1})^2$$

解得

$$a = -\frac{2}{(x_k - x_{k+1})^3}, \quad b = \left(1 + \frac{2x_k}{x_k - x_{k+1}}\right)\frac{1}{(x_k - x_{k+1})^2},$$

$$\alpha_k(x) = \left(1 + 2\frac{x - x_k}{x_{k+1} - x_k}\right)\left(\frac{x - x_{k+1}}{x_k - x_{k+1}}\right)^2$$

同理

$$\alpha_{k+1}(x) = \left(1 + 2\frac{x - x_{k+1}}{x_k - x_{k+1}}\right)\left(\frac{x - x_k}{x_{k+1} - x_k}\right)^2$$

求 $\beta_k(x)$: 可令

$$\beta_k(x) = a\,(x - x_k)\,(x - x_{k+1})^2$$

由 $\beta_k'(x_k) = 1$ 得 $a = 1/(x_k - x_{k+1})^2$，从而

$$\beta_k(x) = (x - x_k)\left(\frac{x - x_{k+1}}{x_k - x_{k+1}}\right)^2$$

同理，

$$\beta_{k+1}(x) = (x - x_{k+1})\left(\frac{x - x_k}{x_{k+1} - x_k}\right)^2$$

$$\begin{aligned} H_3(x) = &\left(1 + 2\frac{x - x_k}{x_{k+1} - x_k}\right)\left(\frac{x - x_{k+1}}{x_k - x_{k+1}}\right)^2 y_k + \left(1 + 2\frac{x - x_{k+1}}{x_k - x_{k+1}}\right)\left(\frac{x - x_k}{x_{k+1} - x_k}\right)^2 y_{k+1} \\ &+ (x - x_k)\left(\frac{x - x_{k+1}}{x_k - x_{k+1}}\right)^2 m_k + (x - x_{k+1})\left(\frac{x - x_k}{x_{k+1} - x_k}\right)^2 m_{k+1} \end{aligned}$$

余项 $R_3(x) = f(x) - H_3(x)$

$$R_3(x) = \frac{1}{4!} f^{(4)}(\xi)\,(x - x_k)^2\,(x - x_{k+1})^2, \quad \xi \in (x_k, x_{k+1})$$