Orthogonality, a dimension formula for holomorphic mappings and their applications in Cauchy-Riemann geometry

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Abstract

The first objective of this article is to present a new coordinate-free approach to study the Cauchy-Riemann (CR) maps between the real hyperquadrics in the complex projective space. The central theme is based on a notion of orthogonality on the projective space induced by the Hermitian structure defining the hyperquadrics. There are various kinds of special linear subspaces associated to this orthogonality which are well respected by the relevant CR maps. For the purpose of analyzing how these CR maps interact with linear subspaces, we developed a dimension formula for the local holomorphic mappings between projective spaces, which gives an explicit dimension estimate for the linear spans of the images of linear subspaces in each dimension.

Our method allows us to not only generalize many well-known rigidity theorems for the CR mappings between the hyperquadrics with much simpler arguments, but also give the first proof for the existence of infinitely many gaps conjectured by Huang-Ji-Yin on the *gap phenomenon* for the complex unit balls. In addition, our proof does not distinguish the unit balls from other generalized balls and thus it simultaneously demonstrates the same phenomenon for all generalized balls. Finally, a new degree estimate for rational proper maps between the complex unit balls was obtained as a by-product.

1 Introduction

The study of holomorphic mappings between real hyperquadrics in the complex projective space is a very classical topic in Several Complex Variables, especially in the field of CR (Cauchy-Riemann) Geometry. On the one hand, they are the among

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simplest CR-manifolds (e.g. the boundaries of the unit balls) which can serve as model spaces on which one can formulate or verify various statements or theories. On the other hand, to CR geometry they play a role like what the Euclidean spaces to Riemannian Geometry and what the projective spaces to Algebraic Geometry. For instance, there is the well-known problem about which CR manifolds can be embedded into these hyperquadrics. We refer the reader to [BEH, Da, Fo, HJ, Za1] (and the references therein) for the related works in this area.

The traditional approach to the study is based on Chern-Moser's normal form theory, in which the central theme is that one can choose good coordinates such that the CR manifolds and the relevant holomorphic maps take certain normal forms. On the other hand, we observe that when dealing with hyperquadrics, there are a certain type of orthogonality and a number of related notions, like null spaces, orthogonal complements, which are well respected under CR maps and one can work on these objects directly without any reference to coordinates.

Let $r, s, t \in \mathbb{N}$ and denote by $\mathbb{C}^{r,s,t}$ be the Euclidean space equipped with the standard (possibly degenerate) Hermitian bilinear form whose eigenvalues are $+1$, -1 and 0 with multiplicities r, s and t respectively. Consider its projectivization $\mathbb{P}^{r,s,t} := \mathbb{P}\mathbb{C}^{r,s,t}$. The notions of positive, negative and null points are well defined on $\mathbb{P}^{r,s,t}$. Among these, the set of positive points, denoted by $\mathbb{B}^{r,s,t} \subset \mathbb{P}^{r,s,t}$, is called a *generalized ball* and its boundary $\partial\mathbb{B}^{r,s,t}$, which consists of the set of null points, is a CR hypersurface in $\mathbb{P}^{r,s,t}$ defined by a real quadratic equation. The name "generalized balls" comes from the fact that $\mathbb{B}^{1,s,0}$ is just the ordinary s-dimensional complex unit ball.

Suppose $U \subset \mathbb{P}^{r,s,t}$ is a connected open set, $U \cap \partial\mathbb{B}^{r,s,t} \neq \varnothing$ and $f : U \to \mathbb{P}^{r',s',t'}$ is a holomorphic map such that $f(U \cap \partial\mathbb{B}^{r,s,t}) \subset \partial\mathbb{B}^{r',s',t'}$. A classical problem in CR geometry is under what conditions f is rigid in the sense that it comes from some "standard" map between the projective spaces, e.g. from a linear map. Our starting point is the observation that f preserves the orthogonality induced by the Hermitian bilinear form (Proposition 2.5). (Note that the notion of orthogonality descends naturally to $\mathbb{P}^{r,s,t}$.) This leads to the definition of *local orthogonal maps* (Definition 2.3). Here we collect our major rigidity results for local orthogonal maps:

Theorem 1.1. *Let $U \subset \mathbb{P}^{r,s,t}$ be a connected open set such that $U \cap \partial\mathbb{B}^{r,s,t} \neq \varnothing$ and $f : U \to \mathbb{P}^{r',s',t'}$ be a local orthogonal map. Then f is either null or quasi-linear if one of the conditions below is satisfied:*

$$\begin{array}{llll} (i) & r,s \geq 2 & \textit{and} & \min\{r',s'\} \leq \min\{r,s\}; \quad \text{(Theorem 5.5)} \\ (ii) & t=0 & \textit{and} & \min\{r',s'\} \leq 2\min\{r,s\}-2; \quad \text{(Theorem 5.11)} \\ (iii) & t=0 & \textit{and} & r'+s' \leq 2\dim(\mathbb{P}^{r,s})-1. \quad \text{(Theorem 5.10)} \end{array}$$

In addition, f is quasi-standard if it maps a positive point to a positive point under any of the conditions above, or

$$(iv) \quad r,s \geq 2, \ r=r' \ \textit{ and } \ f(U \cap \mathbb{B}^{r,s,t}) \subset \mathbb{B}^{r',s',t'}. \quad \text{(Theorem 5.9)}$$

Here, we call f *null* if $f(U) \subset \partial\mathbb{B}^{r',s',t'}$. This is a kind of triviality in the current setting analogous to constant maps. On the other hand, we call f *quasi-standard* (resp. *quasi-linear*) if it is in some sense a "direct sum" of two parts, of which one comes from a linear isometry (resp. linear map) from $\mathbb{C}^{r,s,t}$ to $\mathbb{C}^{r',s',t'}$ and one is null. The more precise definitions will be given in Section 2. From these results, we can deduce and generalize many well-known rigidity theorems, including those of Baouendi-Huang [BH] (from (*i*) and (*iv*)); Baouendi-Ebenfelt-Huang [BEH] (from (*ii*)); Faran [Fa1] (from (*iii*)); and Xiao-Yuan [XY] (from (*iii*)).

For the ordinary complex unit balls $\mathbb{B}^s \cong \mathbb{B}^{1,s,0}$ with $s \geq 2$, it is well-known that there is an interesting *gap phenomenon*, for the holomorphic maps between their boundaries. Fix an integer $n \geq 2$. For each $k \in \mathbb{N}^+$ such that $n > k(k+1)/2$, define the closed interval $\mathcal{I}_k := [kn+1, (k+1)n - \frac{k(k+1)}{2} - 1]$. The classical theorem of Faran [Fa1] amounts to saying that when $N \in \mathcal{I}_1 = [n+1, 2n-2]$, any local holomorphic map sending an open piece of $\partial\mathbb{B}^n$ to $\partial\mathbb{B}^N$ actually maps $\partial\mathbb{B}^n$ to a linear section $\partial\mathbb{B}^n \subset \partial\mathbb{B}^N$. In other words, there are no "new" maps when N increases from n to $2n-2$. Then, it was discovered by Huang-Ji-Xu [HJX] that the same phenomenon holds for $N \in \mathcal{I}_2 = [2n+1, 3n-4]$ and later by Huang-Ji-Yin [HJY] for $N \in \mathcal{I}_3 = [3n+1, 4n-7]$. The *Gap Conjecture*, formulated in [HJY2], states that the gap phenomenon holds whenever $N \in \mathcal{I}_k$. Our method enables us to not only prove the existence of infinitely many similar gaps *at once*, but also demonstrate the gap phenomenon actually holds for *all* generalized balls.

Theorem 1.2. *Let $k, n \in \mathbb{N}^+$ such that $n > k(k+1)$. For the local proper holomorphic maps between generalize balls, the gap phenomenon holds over the intervals*

$$\mathcal{J}_k := [kn+k, (k+1)n - (k^2+1)].$$

The theorem above will be reformulated with more detail as Theorem 6.5. Note that although the interval $\mathcal{J}_k$ in our theorem is smaller than the $\mathcal{I}_k$ in the original Gap Conjecture, this is to be expected since our theorem holds for *all generalized balls*. As a matter of fact, the lower bound for $\mathcal{J}_k$ is sharp in the present context, as will be demonstrated after Theorem 6.5.

Besides using orthogonality, another important difference between our method and the traditional normal form method is that we have made heavy use of linear subspaces. The null spaces and orthogonal complements (which are well defined on $\mathbb{P}^{r,s,t}$) are linear subspaces that are well respected by local orthogonal maps. This allows us to obtain information about how the linear subspaces of certain dimensions are mapped to the target. Together with the dimension formula below, we are then able to deduce an explicit dimension estimate for the linear spans of the images of the subspaces in *each* dimension. One may say that many of our arguments are interplays between orthogonality and the dimension formula.

To state our dimension formula, we first bring out the fact that every positive integer A can be written as certain sums of binomial coefficients. For every $n \in \mathbb{N}^+$, there exist unique positive integers $a_n > a_{n-1} > \cdots > a_\delta$, where $\delta \geq 1$ and $a_j \geq j$ for every j, such that $A = \binom{a_n}{n} + \cdots + \binom{a_\delta}{\delta}$. This is called the n-th Macaulay's representation of A and its existence and uniqueness can be proved by a greedy algorithm. These representations originally appeared in Macaulay's work of homogeneous ideals in polynomial rings [Ma]. Using the n-th Macaulay representation of A, we define the operation $A^{-<n>} := \binom{a_n-1}{n-1} + \cdots + \binom{a_\delta-1}{\delta-1}$. In what follows, "span" means the projective linear span:

Theorem 1.3. *Let $f : U \subset \mathbb{P}^n \to \mathbb{P}^M$ be a local holomorphic map such that* $\dim(\mathrm{span}(f(U)) \geq N$*. Then, for a general hyperplane H such that $H \cap U \neq \varnothing$,* $\dim(\mathrm{span}(f(H \cap U)) \geq N^{-<n>}$*.*

The equality in the theorem can hold, for example, when f is a rational map whose components are all the monomials of a fixed degree. Our dimension formula is obtained from combining Green's hyperplane restriction theorem (Theorem 3.1) with a pair of combinatoric identities (especially Lemma 3.2). It holds for any local holomorphic maps between projective spaces and we believe that it will find applications elsewhere.

Finally, in the course of proving Theorem 1.2, we have got a new degree estimate for rational proper maps between the unit balls as a by-product:

Theorem 1.4. *Let f be a rational proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^N$. If $n \geq 3$ and $N < \binom{n+2}{2}$, then,*

$$\deg(f) \leq \left(\frac{2N}{n+3} + 1\right)\left(\frac{4N}{n+3} + 1\right).$$

This can be compared with the estimate obtained by D'Angelo-Lebl [DL] which says that $\deg(f) \leq \dfrac{N(N-1)}{2(2n-3)}$ when there is no restriction for N.

2 Definitions and basic properties

Let $r, s, t \in \mathbb{N}$ and $n := r+s+t > 0$. Define the (possibly degenerate) indefinite inner product of signature $(r; s; t)$ on $\mathbb{C}^n$:

$$\langle z, w\rangle_{r,s,t} = z_1\bar{w}_1 + \cdots + z_r\bar{w}_r - z_{r+1}\bar{w}_{r+1} - \cdots - z_{r+s}\bar{w}_{r+s},$$

where $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$. We also define the indefinite norm $\|z\|^2_{r,s,t} = \langle z, z\rangle_{r,s,t}$. Then, for any $z \in \mathbb{C}^{r,s,t}$, we call it a *positive point* if $\|z\|^2_{r,s,t} > 0$; a *negative point* if $\|z\|^2_{r,s,t} < 0$ and a *null point* if $\|z\|^2_{r,s,t} = 0$. If $\langle z, w\rangle_{r,s,t} = 0$, we say that z is orthogonal to w and write $z \perp w$. In addition, the *orthogonal complement* of z is defined as

$$z^\perp = \{w \in \mathbb{C}^{r,s,t} \mid \langle z, w\rangle_{r,s,t} = 0\}.$$

We denote by $\mathbb{C}^{r,s,t}$ the $\mathbb{C}^n$ with the Hermitian inner product defined above and by $\mathbb{P}^{r,s,t} := \mathbb{P}\mathbb{C}^{r,s,t}$ its projectivization. We write $\mathbb{C}^{r,s}$ and $\mathbb{P}^{r,s}$ instead of $\mathbb{C}^{r,s,0}$ and $\mathbb{P}^{r,s,0}$. A biholomorphism on $\mathbb{P}^{r,s,t}$ induced by a linear isometry of $\mathbb{C}^{r,s,t}$ is said to be an automorphism of $\mathbb{P}^{r,s,t}$.

Although the norm $\|\cdot\|^2_{r,s,t}$ of course does not descend to $\mathbb{P}^{r,s,t}$, the positivity, negativity or nullity of a line (1-dimensional subspace) remains well defined and thus we can talk about positive points, negative points and null points on $\mathbb{P}^{r,s,t}$. Furthermore, the orthogonality of two points on $\mathbb{P}^{r,s,t}$ and hence the notion of orthogonal complement also make sense on $\mathbb{P}^{r,s,t}$.

More generally, let V be a complex vector space equipped with a Hermitian inner product (possibly degenerate or indefinite) H_V of signature $(r;s;t)$, where $\dim(V) = r+s+t$. Let $\mathbb{P}V$ be its projectivization. The notion of positivity, negativity, nullity and orthogonality can be similarly defined on $\mathbb{P}V$. In addition, any linear isometry $F : \mathbb{C}^{r,s,t} \to V$ induces a biholomorphic map $\tilde{F} : \mathbb{P}^{r,s,t} \to \mathbb{P}V$ preserving all these notions. Sufficient for our purpose, we can simply identify any such projective space $\mathbb{P}V$ with $\mathbb{P}^{r,s,t}$ through any such biholomorphism and we write $\mathbb{P}V \cong \mathbb{P}^{r,s,t}$ for such identification.

Now let H be a complex linear subspace in $\mathbb{C}^{r,s,t}$ and the restriction of $\langle\cdot,\cdot\rangle_{r,s,t}$ on H has the signature $(a;b;c)$. Obviously, we have $0 \le a \le r$, $0 \le b \le s$, $0 \le c \le \min\{r-a, s-b\}+t$ and $a+b+c = \dim(H)$. Then $\mathbb{P}H \cong \mathbb{P}^{a,b,c}$. We call $\mathbb{P}H$ an (a,b,c)-*subspace* of $\mathbb{P}^{r,s,t}$. Usually, we denote an (a,b,c)-subspace by $H^{a,b,c}$.

If $a = b = 0$, $\mathbb{P}H$ is called a *null space*. Similarly, it is called a *positive space* (resp. *negative space*) if $b = c = 0$ (resp. $a = c = 0$). We will also use the terms *null k-plane, positive k-plane* and *negative k-plane* when the $\dim(\mathbb{P}H) = k$. Obviously, the maximum dimension of the null spaces in $\mathbb{P}^{r,s,t}$ is $\min\{r,s\}+t-1$. The null spaces with the maximal dimension are called the *maximal null spaces*.

We now recall the definition of type-I irreducible bounded symmetric domain and with that we can give some useful parametrizations for the positive, negative and null spaces in $\mathbb{P}^{r,s,t}$.

Definition 2.1. *Let $M_{r,s}$ be the set of $r\times s$ complex matrices. The type-I irreducible bounded symmetric domain $\Omega_{r,s}$ is the domain in $M_{r,s} \cong \mathbb{C}^{rs}$ defined by $\Omega_{r,s} = \{A \in M_{r,s} : I - AA^H > 0\}$, where A^H denotes the Hermitian transpose of A.*

In what follows, for a point $[z] = [z_1,\ldots,z_n] \in \mathbb{P}^{r,s,t}$, we split the homogeneous coordinates as $[z] = [z^+, z^-, z^0]$, where $z^+ = [z_1,\cdots,z_r]$, $z^- = [z_{r+1},\cdots,z_{r+s}]$, $z^0 = [z_{r+s+1},\cdots,z_n]$. If $t = 0$, $[z]$ is split as $[z] = [z^+, z^-]$.

Let $A \in M_{r,s}$ and $B \in M_{r,t}$. Consider the $(r-1)$-plane defined by

$$H_{A,B} = \{[z^+, z^-, z^0] \in \mathbb{P}^{r,s,t} \mid z^- = z^+A \text{ and } z^0 = z^+B\} \cong \mathbb{P}^{r-1} \subset \mathbb{P}^{r,s,t}.$$

Using the definition of $H_{A,B}$, we see that $M_{r,s} \times M_{r,t}$ can be identified naturally as an open subset of the Grassmannian $\mathbb{G}(r-1, \mathbb{P}^{r,s,t})$ which is the set of $(r-1)$-planes in $\mathbb{P}^{r,s,t}$. (Note that the dimensions of $M_{r,s} \times M_{r,t}$ and $\mathbb{G}(r-1, \mathbb{P}^{r,s,t})$ are the same.)

The $(r-1)$-plane $H_{A,B}$ is a positive subspace if and only if $\|z^+\|^2 > \|z^-\|^2 = \|z^+A\|^2$ for all $z^+ \in \mathbb{P}^{r-1}$, which is in turn equivalent to $A \in \Omega_{r,s}$. Similarly, when $r \leq s$, we see that $H_{A,B}$ is a null $(r-1)$-plane if and only if $AA^H = I$. We know that latter equation defines precisely the Shilov boundary of $\Omega_{r,s}$ in $M_{r,s} \cong \mathbb{C}^{rs}$, denoted by $S(\Omega_{r,s})$. To summarize, we have the following (cf. [NZ]):

Proposition 2.2. *There is an open embedding of $M_{r,s} \times M_{r,t}$ into $\mathbb{G}(r-1, \mathbb{P}^{r,s,t})$ such that its restriction to $\Omega_{r,s} \times M_{r,t}$ gives a parametrization of all positive $(r-1)$-planes in $\mathbb{P}^{r,s,t}$. In addition, when $r \leq s$, the restriction to $S(\Omega_{r,s}) \times M_{r,t}$ also gives a parametrization of the null $(r-1)$-planes in $\mathbb{P}^{r,s,t}$.*

Many analyses on the mapping problems between CR manifolds begin with the fact that the associated *Segre varieties* are well respected by CR maps. The study of Segre varieties has a very important role in many problems like reflection principle and algebracity [Za2]. For real hyperquadrics on complex projective space, we note that their Segre varieties are just the orthogonal complements (to points) with respect to the orthogonality described previously. Motivated from this, we thus give the following definition.

Definition 2.3. *Let $U \subset \mathbb{P}^{r,s,t}$ be a connected open set containing a null point. We call a holomorphic map $F : U \to \mathbb{P}^{r',s',t'}$ **orthogonal** if $F(p) \perp F(q)$ for any $p, q \in U$ such that $p \perp q$; **sign-preserving** if F maps positive points to positive points and negative points to negative points. We also simply call F a local orthogonal map or local sign-preserving map from $\mathbb{P}^{r,s,t}$ to $\mathbb{P}^{r',s',t'}$ in such cases.*

Remark 1. If U doesn't contain any null point, the orthogonality or sign-preserving condition may become vacuous. For instance, in such case, it could happen that $p^\perp \cap U = \varnothing$ for any point $p \in U$.

Remark 2. It follows easily from continuity that a sign preserving map also maps null points to null points.

Convention. *In what follows, if we say that F maps lines to lines (resp. k-planes to k'-planes), we mean F maps the intersection of any line (resp. k-plane) and U into a line (resp. k'-plane).*

Local orthogonal map preserves null spaces, as demonstrated below.

Proposition 2.4. *Let F be a local orthogonal map from $\mathbb{P}^{r,s,t}$ to $\mathbb{P}^{r',s',t'}$. Then F maps null spaces to null spaces (in particular, null points to null points).*

Proof. For any two points α, β in a null space in $\mathbb{P}^{r,s,t}$, we have $\langle \alpha, \alpha \rangle_{r,s,t} = \langle \beta, \beta \rangle_{r,s,t} = \langle \alpha, \beta \rangle_{r,s,t} = 0$. Since F is a local orthogonal map, we get

$$\langle F(\alpha), F(\alpha) \rangle_{r',s',t'} = \langle F(\beta), F(\beta) \rangle_{r',s',t'} = \langle F(\alpha), F(\beta) \rangle_{r',s',t'} = 0.$$

So the linear span of the image of a null space is a null space. □

There are two kinds of null points in $\mathbb{P}^{r,s,t}$ when $r, s, t > 0$. The first kind of null points is the null points α satisfying $\langle \alpha, \beta \rangle_{r,s,t} = 0$, for any $\beta \in \mathbb{P}^{r,s,t}$. We call these null points *special null points* and call other null points *ordinary null points.* It is easy to see from the defining equations for special and ordinary null points that a *general* null point is ordinary and for $r, s > 0$, whenever an open set contains a null point, it must contain an ordinary null point.

Both orthogonal maps and sign-preserving maps map null points to null points. Conversely, the following proposition in particular implies that a sign-preserving map is locally an orthogonal map.

Proposition 2.5. *Let $r, s > 0$ and $F : U \subset \mathbb{P}^{r,s,t} \to \mathbb{P}^{r',s',t'}$ be a holomorphic map, where U is an open set containing a null point. If F maps null points to null points, then there exists an open set $V \subset U$, such that $F : V \to \mathbb{P}^{r',s',t'}$ is an orthogonal map.*

Proof. Let $n = r+s+t$, $n' = r'+s'+t'$ and write the homogeneous coordinates of $\mathbb{P}^{r,s,t}$ and $\mathbb{P}^{r',s',t'}$ as $[z_1, \ldots, z_n]$ and $[w_1, \ldots, w_{n'}]$ respectively. Since $r, s > 0$, we know that U must contain an ordinary null point and hence by shrinking U if necessary, we may assume without loss of generality that U is contained in the open set $U_1 \cong \mathbb{C}^{n-1} \subset \mathbb{P}^{r,s,t}$ defined by $z_1 \neq 0$ and $F(U)$ is contained in the open set of $U_1' \cong \mathbb{C}^{n'-1} \subset \mathbb{P}^{r',s',t'}$ defined by $w_1 \neq 0$. We write the standard inhomogeneous coordinates in U_1 as $(\zeta_2, \ldots, \zeta_n)$, where $\zeta_j = z_j/z_1$ and similarly write $(\eta_2, \ldots, \eta_{n'})$ for U_1', where $\eta_\ell = w_\ell/w_1$. In terms of these coordinates, the null points in U_1 and U_1' are respectively given by the equations

$$1 + \sum_{j=2}^{r} |\zeta_j|^2 - \sum_{j=r+1}^{r+s} |\zeta_j|^2 = 0 \quad \text{and} \quad 1 + \sum_{\ell=2}^{r'} |\eta_\ell|^2 - \sum_{\ell=r'+1}^{r'+s'} |\eta_\ell|^2 = 0.$$

(Strictly speaking, these equations are for the cases where $r, r' \geq 2$ but other cases can be handled in a similar fashion.)

Using the inhomogeneous coordinates above, write $F = (F_2, \cdots, F_{n'})$. If F maps null points to null points, then there exist a connected open set $V \subset U$ containing a null point, $k \in \mathbb{N}^+$ and a real analytic function ρ on V such that

$$1 + \sum_{\ell=2}^{r'} |F_\ell|^2 - \sum_{\ell=r'+1}^{r'+s'} |F_\ell|^2 = \left(1 + \sum_{j=2}^{r} |\zeta_j|^2 - \sum_{j=r+1}^{r+s} |\zeta_j|^2\right)^k \rho \tag{1}$$

holds on V. Hence, by shrinking V if necessary, we can polarize the equation, i.e. for any $\zeta, \xi \in V$, we have

$$1 + \sum_{\ell=2}^{r'} F_\ell(\zeta)\overline{F_\ell(\xi)} - \sum_{\ell=r'+1}^{r'+s'} F_\ell(\zeta)\overline{F_\ell(\xi)} = \left(1 + \sum_{j=2}^{r} \zeta_j\overline{\xi_j} - \sum_{j=r+1}^{r+s} \zeta_j\overline{\xi_j}\right)^k \rho(\zeta, \bar{\xi}),$$

where $\zeta = (\zeta_2, \ldots, \zeta_n)$, $\xi = (\xi_2, \ldots, \xi_n)$.

Now, ζ as a point in $\mathbb{P}^{r,s,t}$ has homogeneous coordinates $[1, \zeta_2, \ldots, \zeta_n]$ and thus $\zeta^\perp \cap V$ consists precisely the points $\xi = [1, \xi_2, \ldots, \xi_n] \in V$ satisfying $1 + \sum_{j=2}^{r} \zeta_j \overline{\xi_j} - \sum_{j=r+1}^{r+s} \zeta_j \overline{\xi_j} = 0$. With a similar consideration for $F(\zeta)^\perp$, we conclude from the above equation that $F(\zeta^\perp) \subset (F(\zeta))^\perp$ for $\zeta \in V$. □

It has been known that for any local proper holomorphic map f between two generalized balls (excluding the unit balls), there exist $k, k' \in \mathbb{N}^+$ such that f maps k-planes to k'-planes ([Ng1] Proposition 4.1 therein). The following proposition is a generalization of this result to local orthogonal maps, which is very crucial in our study.

Proposition 2.6. *If F is a local orthogonal map from $\mathbb{P}^{r,s,t}$ to $\mathbb{P}^{r',s',t'}$, then F maps $(\min\{r,s\}-1)$-planes to $(\min\{r',s'\}+t'-1)$-planes.*

Proof. By symmetry, it suffices to prove the case for $r \le s$ and $r' \le s'$. The case $r = 1$ is trivial and so we let $r \ge 2$. Recall from Proposition 2.2 that the null $(r-1)$-planes in $\mathbb{P}^{r,s,t}$ can be parametrized by $S(\Omega_{r,s}) \times M_{r,t} \subset \mathbb{G}(r-1, \mathbb{P}^{r,s,t})$, where $S(\Omega_{r,s})$ is the Shilov boundary of the type-I bounded symmetric domain $\Omega_{r,s}$. Now from Proposition 2.4, the map F maps null spaces to null spaces and thus the image of every null $(r-1)$-plane in $\mathbb{P}^{r,s,t}$ is contained in a maximal null space, which is an $(r'+t'-1)$-plane in $\mathbb{P}^{r',s',t'}$.

Note that for any point $p \in \mathbb{G}(r-1, \mathbb{P}^{r,s,t})$, there exists a set of local holomorphic functions in a neighborhood $\mathcal{U} \ni p$ such that they vanish at a point $q \in \mathcal{U}$ precisely when the image under F of the $(r-1)$-plane given by q is contained in an $(r'+t'-1)$-plane. (For instance, one can consider a set of determinants given by the Taylor coefficients of F. For more detail, see [NZ], Proof of Theorem 1.1 therein.) From the previous paragraph, we know that when choose $p \in S(\Omega_{r,s}) \times M_{r,t}$, these functions vanish at $\mathcal{U} \cap (S(\Omega_{r,s}) \times M_{r,t})$. The Shilov boundary is the distinguished boundary of $\Omega_{r,s}$ and from this we deduce that these functions actually vanish identically on the open set $\mathcal{U}$ (see [Ng2] or [NZ]). Thus, F maps $(r-1)$-planes to $(r'+t'-1)$-planes. □

3 Macaulay representation and a dimension formula for holomorphic mappings

Every positive integer A can be written as certain sums of binomial coefficients. For every $n \in \mathbb{N}^+$, there exist unique positive integers $a_n > a_{n-1} > \cdots > a_\delta$, where $\delta \ge 1$ and $a_j \ge j$ for every j, such that $A = \binom{a_n}{n} + \cdots + \binom{a_\delta}{\delta}$. This is called the n-th Macaulay's representation of A. These representations naturally appeared in the works of Macaulay [Ma] and Green [Gr] on homogeneous ideals in polynomial rings.

There are several operations pertaining to the Macaulay's representations, as follows. Let $A = \binom{a_n}{n} + \cdots + \binom{a_\delta}{\delta}$ be the n-th Macaulay's representation of A, define

$$A^{<n>} = \binom{a_n+1}{n+1} + \cdots + \binom{a_\delta+1}{\delta+1}; \quad A^{-<n>} = \binom{a_n-1}{n-1} + \cdots + \binom{a_\delta-1}{\delta-1};$$

$$A_{<n>} = \binom{a_n-1}{n} + \cdots + \binom{a_\delta-1}{\delta}; \quad A_{-<n>} = \binom{a_n+1}{n} + \cdots + \binom{a_\delta+1}{\delta}.$$

Here, we employ the convention that $\binom{a}{b} = 0$ whenever $a < b$ or $b = 0$. The seemingly peculiar choice of notations between $A_{<n>}$ and $A_{-<n>}$ are due the fact that the operations $A^{<n>}$ and $A_{<n>}$ were used by Macaulay and Green and the other two are in some sense the reverse operations. We will need the following Green's hyperplane restriction theorem, which has already been used in [GLV] to study CR mappings between real hyperquadrics.

Theorem 3.1 ([Gr])**.** *Let W be a complex vector subspace of $H^0(\mathcal{O}_{\mathbb{P}^n}(d))$ of codimension c. Let $W_H \subset H^0(\mathcal{O}_H(d))$ be the restriction of W to a general hyperplane H and c_H be its codimension. Then, $c_H \leq c_{<d>}$.*

We begin by stating two combinatoric lemmas related to these operations. The first one is our key, which connects Green's hyperplane theorem and our Theorem 3.4. The second one is much simpler, and has already been discovered in [GLV]. Their proofs are contained in the Appendix section.

Lemma 3.2. *Suppose $m, k \geq 1$ and $A, B \geq 0$. If $A + B = \binom{m+k}{k} - 1$, then*

$$A^{-<m>} + B_{<k>} = \binom{m+k-1}{k} - 1.$$

Here, we make the convention that $0^{-<m>} = 0_{<k>} = 0$.

Lemma 3.3 ([GLV])**.** *Let $n, d, K \in \mathbb{N}^+$ such that $K \leq \binom{n+d}{d}$. Then, the number*

$$\binom{n+d-1}{d} - \left(\binom{n+d}{d} - K\right)_{<d>}$$

is independent of d.

We can now prove our dimension formula. In what follows, "span" means the projective linear span.

Theorem 3.4. *Let $f : U \subset \mathbb{P}^n \to \mathbb{P}^M$ be a local holomorphic map such that* $\dim(\mathrm{span}(f(U)) \geq N$*. Then, for a general hyperplane H such that $H \cap U \neq \varnothing$,* $\dim(\mathrm{span}(f(H \cap U)) \geq N^{-<n>}$*.*

Proof. We first prove the theorem for rational maps. Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^M$ be a rational map such that $\dim(\operatorname{span}(f(\mathbb{P}^n)) \geq N$. Let $f = [f_0, \ldots, f_M]$, where $f_j \in \mathbb{C}[z_0, \ldots, z_n]$, $0 \leq j \leq M$, are homogeneous polynomials of degree d without non-constant common factors.

Let $\mathcal{H}_{d,n} \subset \mathbb{C}[z_0, \ldots, z_n]$ be the vector subspace of homogeneous polynomials of degree d and $W \subset \mathcal{H}_{d,n}$ be the subspace spanned by $f_0, \ldots, f_M$. Thus, we have

$$N + 1 \leq \dim(W) \leq \dim(\mathcal{H}_{d,n}) = \binom{n+d}{d}.$$

For a general hyperplane $H \subset \mathbb{P}^n$, choose a set of homogeneous coordinates on H and let $W_H \subset \mathcal{H}_{d,n-1}$ be the subspace spanned by restrictions of $f_0, \ldots, f_M$ on H. By Green's Theorem 3.1, we have

$$\dim(\mathcal{H}_{d,n-1}) - \dim(W_H) \leq (\dim(\mathcal{H}_{d,n}) - \dim(W))_{<d>} \leq (\dim(\mathcal{H}_{d,n}) - N - 1)_{<d>},$$

in which the last inequality follows from the fact that $c_{<d>} \leq c'_{<d>}$ if $c \leq c'$. Thus,

$$\dim(W_H) \geq \binom{n+d-1}{d} - \left(\binom{n+d}{d} - N - 1\right)_{<d>} \tag{2}$$

Now choose d' such that $\binom{n+d'-1}{d'-1} < N + 1 \leq \binom{n+d'}{d'}$ and let $A = N - \binom{n+d'-1}{d'-1}$, $B = \binom{n+d'}{d'} - N - 1$. Then $A, B \geq 0$ and

$$A + B = \binom{n+d'}{d'} - \binom{n+d'-1}{d'-1} - 1 = \binom{n+d'-1}{d'} - 1.$$

By Lemma 3.3, the right hand side of Eq.(2) is independent of d and thus together with Lemma 3.2, we have

$$\begin{aligned}
\dim(W_H) &\geq \binom{n+d'-1}{d'} - \left(\binom{n+d'}{d'} - N - 1\right)_{<d'>} \\
&= \binom{n+d'-1}{d'} - B_{<d'>} \\
&= \binom{n+d'-1}{d'} - \left(\binom{n+d'-2}{d'} - 1 - A^{-<n-1>}\right) \\
&= \binom{n+d'-2}{d'-1} + \left(N - \binom{n+d'-1}{d'-1}\right)^{-<n-1>} + 1 \\
&= \binom{n+d'-2}{n-1} + \left(N - \binom{n+d'-1}{n}\right)^{-<n-1>} + 1
\end{aligned}$$

Since $\binom{n+d'-1}{d'-1} < N+1 \leq \binom{n+d'}{d'}$, we have $\binom{n+d'-1}{n} \leq N < \binom{n+d'}{n}$ and it follows that $\binom{n+d'-1}{n}$ is the leading term of the n-th Macaulay representation of N. Therefore, by writing $N = \binom{n+d'-1}{n} + (N - \binom{n+d'-1}{n})$, we see that

$$N^{-<n>} = \binom{n+d'-2}{n-1} + \left(N - \binom{n+d'-1}{n}\right)^{-<n-1>}.$$

Hence, we get that $\dim(W_H) \geq N^{-<n>} + 1$ and thus $\dim(\mathrm{span}(f(H)) \geq N^{-<n>}$. We have now proved the theorem for rational maps.

For the general case, if $f : U \subset \mathbb{P}^n \to \mathbb{P}^M$ is a local holomorphic map such that $\dim(\mathrm{span}(f(U)) \geq N$, then for a sufficiently large k, the k-th order jet of f can be represented by a rational map $f^\flat : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ (e.g. a truncated Taylor polynomial of f at point in U after homogenization) such that $\dim(\mathrm{span}(f^\flat(\mathbb{P}^n)) \geq N$. Since the restriction of $f^\flat$ to a hyperplane $H \subset \mathbb{P}^n$ represents the k-th order jet of $f|_{U\cap H}$, we see from the proven case of rational maps that for a general hyperplane H, we have $\dim(\mathrm{span}(f(H \cap U)) \geq \dim(\mathrm{span}(f^\flat(H)) \geq N^{-<n>}$. □

Remark. The equality in Theorem 3.4 can hold since the equality can hold in Green's theorem [Gr]. One can also see directly that the equality holds when F is the rational map whose components are all the monomials (with unit coefficients) of a fixed degree in $\mathbb{C}[z_0, \ldots, z_n]$.

Sometimes it is convenient to use the following counterpart of Theorem 3.4 and we will also elaborate a couple of special cases for later use.

Theorem 3.5. *Let $g : U \subset \mathbb{P}^m \to \mathbb{P}^{m'}$ be a local holomorphic map and $\ell \in \mathbb{N}^+$ such that $\ell \leq m - 1$. If g maps ℓ-planes to ℓ'-planes, then it maps $(\ell + 1)$-planes to $((\ell' + 1)^{<\ell>} - 1)$-planes. In particular,*

(i) if $\ell' \leq \ell - 1$, then the image of g is contained in an ℓ'-plane;

(ii) if $\ell \leq \ell' \leq 2\ell - 1$, then g maps $(\ell + k)$-planes to $(\ell' + k)$-planes for $k \geq 0$;

Proof. Suppose on the contrary the image of a general $(\ell + 1)$-plane under g is not contained in any $((\ell'+1)^{<\ell>} - 1)$-plane. Since $((\ell'+1)^{<\ell>})^{-<\ell+1>} = \ell'+1$, Theorem 3.4 implies that the image of a general ℓ-plane is not contained in any ℓ'-plane.

If $\ell' \leq \ell - 1$, then $\ell'+1 = \binom{\ell}{\ell} + \binom{\ell-1}{\ell-1} + \cdots + \binom{\delta}{\delta}$ for some $\delta \geq 1$, so $(\ell'+1)^{<\ell>} - 1 = \ell'$. Therefore we deduce inductively that the image of g is contained in an ℓ'-plane.

If $\ell \leq \ell' \leq 2\ell - 1$, then $\ell' + 1 = \binom{\ell+1}{\ell} + \binom{\ell-1}{\ell-1} + \binom{\ell-2}{\ell-2} + \cdots + \binom{\delta}{\delta}$ for some $\delta \geq 1$. Thus, $(\ell'+1)^{<\ell>} - 1 = \ell'+1$ and so g maps $(\ell+1)$-planes to $(\ell'+1)$-planes. Moreover, as $\ell + 1 \leq \ell' + 1 < 2(\ell + 1) - 1$, we can proceed inductively and the desired result follows. □

Remark. One can apply Theorem 3.5 repeatedly to get the following simple formula. Under the same hypotheses, if the ℓ-th Macaulay's representation of $\ell' + 1$ is $\binom{\lambda_\ell}{\ell} + \cdots + \binom{\lambda_\delta}{\delta}$, then for any $k \in \mathbb{N}^+$, g maps every $(\ell + k)$-plane to some linear subspace of dimension $\binom{\lambda_\ell+k}{\ell+k} + \cdots + \binom{\lambda_\delta+k}{\delta+k} - 1$.

Proposition 3.6. *Under the hypotheses of Theorem 3.5, if $\ell' \leq \ell$ and the image of g is not contained in an ℓ'-plane, then g extends to a linear rational map.*

Proof. If $\ell' \leq \ell$ and the image g is not contained in an ℓ'-plane, by considering a general pair of ℓ-planes such that their intersection is an $(\ell-1)$-plane, we see g maps $(\ell-1)$-planes to $(\ell'-1)$-planes. Inductively, we deduce that g maps lines to lines (or lines to points) and hence g is linear. For a proof of the last fact, see [Ng3] (Lemma 4.1 therein). □

4 Gap phenomenon for local orthogonal maps

Proposition 4.1. *Let $f : U \subset \mathbb{P}^{r,s,t} \to \mathbb{P}^{r',s'}$ be a local orthogonal map. Then, for every linear subspace $E \subset \mathbb{P}^{r,s,t}$ such that $E \cap U \neq \varnothing$ and $E^{\perp} \cap U \neq \varnothing$,*

$$\dim(\mathrm{span}(f(E \cap U)) + \dim(\mathrm{span}(f(E^{\perp} \cap U)) \leq \dim(\mathbb{P}^{r',s'}) - 1.$$

Proof. By orthogonality, $f(E^{\perp} \cap U) \subset (f(E \cap U))^{\perp}$ and since the Hermitian form on $\mathbb{C}^{r',s'}$ is non-degenerate, we have

$$\dim(\mathrm{span}(f(E^{\perp} \cap U)) + 1 \leq \dim(\mathbb{C}^{r',s'}) - (\dim(\mathrm{span}(f(E \cap U)) + 1),$$

and the desired result follows. □

Let $n, N \in \mathbb{N}$ such that $n+1 \leq N < \binom{n+2}{2} = \binom{n+2}{n}$. By considering the n-th Macaulay representation of N, we deduce that N is of the following form:

$$N = N(n;a,b) := \binom{n+1}{n} + \cdots + \binom{n-a+1}{n-a} + b$$

for some integers $a, b \geq 0$ such that $b \leq n-a-1$. In fact, the n-th Macaulay's representation of $N(n;a,b)$ is

$$N(n;a,0) := \binom{n+1}{n} + \cdots + \binom{n-a+1}{n-a}$$

and

$$N(n;a,b) := \binom{n+1}{n} + \cdots + \binom{n-a+1}{n-a} + \binom{n-a-1}{n-a-1} + \cdots + \binom{n-a-b}{n-a-b}$$

for $b \geq 1$.

Lemma 4.2. $N(n;a,b)^{-<n>} = \begin{cases} N(n-1;a,b) & \text{if} \quad n-a-b \geq 2; \\ N(n-1;a,b-1) & \text{if} \quad n-a-b = 1 \text{ and } b \geq 1; \\ N(n-1;a-1,0) & \text{if} \quad n-a-b = 1 \text{ and } b = 0; \end{cases}$

Proof. It is an immediate consequence of the n-th Macaulay representation of $N(n; a, b)$ described above. □

Proposition 4.3. *Let a, b, n be non-negative integers such that $n - a - b \geq 2$. Let $g : U \subset \mathbb{P}^n \to \mathbb{P}^{N(n;a,b)}$ be a local holomorphic map whose image is not contained in a proper linear subspace. Let $D_m = \dim(\mathrm{span}(g(M \cap U)))$ for a general m-dimensional linear subspace M intersecting U. Then,*

$$D_m \geq \begin{cases} N(m; a, b) & \text{if} \quad m \in [a+b+1, n-1] \\ N(m; a, m-a-1) & \text{if} \quad m \in [a+1, a+b] \\ N(m; m-1, 0) & \text{if} \quad m \in [1, a] \end{cases}$$

Proof. We will apply Theorem 3.4 and Lemma 4.2 repeatedly and proceed from $m = n-1$ down to $m = 1$.

As we start with $n - a - b \geq 2$, from the first line of Lemma 4.2, we get that $D_m \geq N(m; a, b)$ for $m = n-1, n-2, \ldots, a+b+1$, When we reach $m = a+b+1$, we have $N(m; a, b) = N(m; a, m-a-1)$. Thus, if $b \geq 1$, we deduce from the second line of Lemma 4.2 that for $m = a+b, \ldots, a+1$, we always have $D_m \geq N(m; a, m-a-1)$. In particular, $D_{a+1} \geq N(a+1; a, 0)$, therefore by using the third line of Lemma 4.2, it follows that for $m = a, \ldots, 1$, we have $D_m \geq N(m; m-1, 0)$. If $b = 0$, the previous argument is basically the same except the second line of Lemma 4.2 is never needed. □

Theorem 4.4. *Let a, n be non-negative integers such that $n \geq a^2 + 3a + 3$. Let f be a local orthogonal map from $\mathbb{P}^{r,s}$ to $\mathbb{P}^{r',s'}$, where $n = \dim(\mathbb{P}^{r,s})$ and let $n' := \dim(\mathbb{P}^{r',s'})$. If*

$$(a+1)(n+1) \leq n' \leq (a+2)n - (a^2 + 2a + 2),$$

then the image of f lies in a hyperplane of $\mathbb{P}^{r',s'}$

Proof. We will prove by contradiction. Suppose n' satisfies the hypotheses and the image of f is not contained in any hyperplane in $\mathbb{P}^{r',s'}$.

Since

$$(a+1)(n+1) = \binom{n+1}{n} + \cdots + \binom{n-a+1}{n-a} + \frac{(a+1)a}{2}$$

and

$$(a+2)n - (a^2 + 2a + 2) = \binom{n+1}{n} + \cdots + \binom{n-a+1}{n-a} + \left(n - \frac{a^2 + 5a + 6}{2}\right),$$

we see that

$$n' = \binom{n+1}{n} + \cdots + \binom{n-a+1}{n-a} + b = N(n; a, b),$$

for some b satisfying

$$\frac{(a+1)a}{2} \leq b \leq n - \frac{a^2 + 5a + 6}{2}. \tag{3}$$

By hypotheses, $n \geq a^2 + 3a + 3$, thus if we let $n_1 := \left[\frac{n-1}{2}\right]$, then

$$\frac{(a+2)(a+1)}{2} \leq n_1 \leq \frac{n-1}{2} \leq n - \frac{a^2+3a+4}{2},$$

which implies

$$\frac{(a+1)a}{2} \leq n_1 - a - 1 \leq n - \frac{a^2+5a+6}{2}.$$

Comparing with Eq.(3), we see that there are two possibilities for b, either

$$\begin{array}{rcccl} \textbf{Case I}: \quad \frac{(a+1)a}{2} & \leq & b & \leq & n_1 - a - 1; \qquad \text{or} \\ \textbf{Case II}: \quad n_1 - a & \leq & b & \leq & n - \frac{a^2+5a+6}{2}. \end{array}$$

Define also $n_2 = \begin{cases} n_1 & \text{if } n \text{ is odd;} \\ n_1 + 1 & \text{if } n \text{ is even.} \end{cases}$ Then $n_1 + n_2 + 1 = n$ and for an n_1-dimensional linear subspace in $\mathbb{P}^{r,s}$, its orthogonal complement is of dimension n_2 and conversely any n_2-dimensional linear subspace is the orthogonal complement of some n_1-dimensional linear subspace. Let D_m be the dimension of the linear span of the image under f of a general m-dimensional linear subspace intersecting the domain of definition of f. We are going to use Proposition 4.1 to reach a contradiction by showing that $D_{n_1} + D_{n_2} \geq n'$.

In **Case I**, since $n - 1 \geq n_2 \geq n_1 \geq a + b + 1$, so by Proposition 4.3 and Eq.(3),

$$\begin{aligned} D_{n_1} + D_{n_2} &\geq N(n_1; a, b) + N(n_2; a, b) \\ &= (a+1)(n+1-a) + 2b \\ &= (a+1)(n+1-\frac{a}{2}) + b + \left(b - \frac{(a+1)a}{2}\right) \\ &\geq (a+1)(n+1-\frac{a}{2}) + b \\ &= N(n; a, b) = n', \end{aligned}$$

which contradicts Proposition 4.1.

In **Case II**, we have $n_1 - a \leq b \leq 2n_1 - \frac{a^2+5a+2}{2}$, which is equivalent to

$$\frac{b}{2} + \frac{a^2+5a+2}{4} \leq n_1 \leq a + b.$$

In particular, $a + 1 \leq n_1 \leq a + b$. Now by Proposition 4.3,

$$D_{n_1} \geq N(n_1; a; n_1 - a - 1) = (a+2)n_1 - \frac{a^2+a}{2}$$

and

$$D_{n_2} \geq \begin{cases} N(n_2; a; b) & \text{if} \quad n_1 = a + b \quad \text{and} \quad n_2 = n_1 + 1; \\ N(n_2; a; n_2 - a - 1) & \text{if} \quad n_1 < a + b \quad \text{or} \quad n_2 = n_1. \end{cases}$$

Therefore,

$$D_{n_1} + D_{n_2} \geq \begin{cases} (a+2)(n_1 + n_2 + 1) - (a^2 + a) + b - n_2 - 1 & \text{if} \quad n_1 = a + b \quad \text{and} \quad n_2 = n_1 + 1; \\ (a+2)(n_1 + n_2) - (a^2 + a) & \text{if} \quad n_1 < a + b \quad \text{or} \quad n_2 = n_1, \end{cases}$$

which simplifies to

$$D_{n_1} + D_{n_2} \geq \begin{cases} (a+2)n - (a^2 + 2a + 2) & \text{if} \quad n_1 = a + b \quad \text{and} \quad n_2 = n_1 + 1; \\ (a+2)n - (a^2 + 2a + 2) & \text{if} \quad n_1 < a + b \quad \text{or} \quad n_2 = n_1. \end{cases}$$

Thus, we always have $D_{n_1} + D_{n_2} \geq n'$, which again contradicts Proposition 4.1. □

5 Rigidity of local orthogonal maps

In order to simplify the presentation, we will give a couple of definitions. In what follows, when we say that a local holomorphic map F between projective spaces is *linear*, we mean F extends to a linear rational map.

Definition 5.1. *Let F be a local holomorphic map from $\mathbb{P}^{r,s,t}$ to $\mathbb{P}^{r',s',t'}$. We call F **standard** if it is linear and comes from a linear isometry from $\mathbb{C}^{r,s,t}$ into $\mathbb{C}^{r',s',t'}$. We call F **null** if its image is contained in a null space in $\mathbb{P}^{r',s',t'}$.*

For any non-trivial orthogonal direct sum decomposition $\mathbb{C}^{r,s,t} = A \oplus B$, there are two canonical projections $\pi_A : \mathbb{P}^{r,s,t} \dashrightarrow \mathbb{P}A$ and $\pi_B : \mathbb{P}^{r,s,t} \dashrightarrow \mathbb{P}B$ (as rational maps). Using these, we make the following definition.

Definition 5.2. *Let F be a local holomorphic map from $\mathbb{P}^{r,s,t}$ to $\mathbb{P}^{r',s',t'}$. We call F **quasi-standard** (resp. **quasi-linear**) if either F is standard (resp. linear) or there exists a non-trivial orthogonal decomposition $\mathbb{C}^{r',s',t'} = A \oplus B$ for some subspaces A, B such that $\pi_A \circ F$ is standard (resp. linear) and $\pi_B \circ F$ is null.*

Since the projection from $\mathbb{P}^{r',s',t'}$ to $\mathbb{P}^{r',s'}$ is also an orthogonal map (see the lemma below), therefore a local orthogonal map F from $\mathbb{P}^{r,s,t}$ to $\mathbb{P}^{r',s',t'}$ naturally gives rise to a local orthogonal map from $\mathbb{P}^{r,s,t}$ to $\mathbb{P}^{r',s'}$ (unless the image lies entirely in the space of special null vectors, which is the set of indeterminacy for the projection). The same thing happens for local sign-preserving maps. This allows us to reduce the problem to the case for $t' = 0$ in many situations. Moreover, the orthogonality of mappings also interacts nicely with orthogonal decompositions. The following two lemmas should be evident to the reader and we omit their proofs.

Lemma 5.3. *The projection $\pi : \mathbb{P}^{r,s,t} \dashrightarrow \mathbb{P}^{r,s}$ is standard. In particular, it is sign-preserving map and orthogonal.*

Lemma 5.4. *Let F be a local holomorphic map from $\mathbb{P}^{r,s,t}$ to $\mathbb{P}^{r',s',t'}$ and $\mathbb{C}^{r',s',t'} = A \oplus B$ be an orthogonal decomposition such that the image of F does not lie entirely in $\mathbb{P}A$ or $\mathbb{P}B$. Let $\pi_A : \mathbb{P}^{r',s',t'} \dashrightarrow \mathbb{P}A$ and $\pi_B : \mathbb{P}^{r',s',t'} \dashrightarrow \mathbb{P}B$ be the canonical projections. If F and $\pi_A \circ F$ are orthogonal, then so is $\pi_B \circ F$.*

We are now ready to prove our rigidity theorems for local orthogonal maps.

Theorem 5.5. *Let F be a local orthogonal map from $\mathbb{P}^{r,s,t}$ to $\mathbb{P}^{r',s',t'}$, where $r, s \geq 2$. If*

$$\min\{r', s'\} \leq \min\{r, s\},$$

then F is either null or quasi-linear (linear for $t' = 0$). If in addition F preserves the sign of any single positive or negative point, then F is quasi-standard (standard for $t' = 0$).

Proof. We first consider the case $t' = 0$. From Proposition 2.6, we know that F maps $(\min\{r, s\} - 1)$-planes into $(\min\{r', s'\} - 1)$-planes. Suppose F is not linear. Since $r, s \geq 2$ and $\min\{r', s'\} \leq \min\{r, s\}$, by Proposition 3.6, the linear span the image of F, denoted by S, is of dimension at most $(\min\{r', s'\} - 1)$.

We claim that S is a null space. If on the contrary S is not null, then $S \cong \mathbb{P}^{a,b,c}$ for some a, b, c and the dimension of the maximal null space of $\mathbb{P}^{a,b,c}$ is $\min\{a, b\} + c - 1$, which is strictly less than $\dim(S)$. Now F can be regarded as a local orthogonal map from $\mathbb{P}^{r,s,t}$ into $\mathbb{P}^{a,b,c}$ which is not linear, and thus by Propositions 2.6 and 3.6 again, the image of F is contained in a linear subspace of dimension $\min\{a, b\} + c - 1 < \dim(S)$, contradicting to the fact that S is the linear span of the image of F. So S is a null space and hence F is null. We have thus shown that F is either linear or null. If in addition F preserves the sign of a positive or negative point, then F cannot be null and from Lemma 5.6 below, we see that F must be standard.

Since $\mathbb{C}^{r',s',t'} = \mathbb{C}^{r',s'} \oplus \mathbb{C}^{0,0,t'}$ is an orthogonal direct sum, the desired result for the general case now follows directly from the case $t' = 0$ and Definition 5.2. $\square$

Lemma 5.6. *Let $G : \mathbb{C}^{r,s,t} \to \mathbb{C}^{r',s',t'}$ be linear and it maps null vectors to null vectors. Then, there exists $\lambda \in \mathbb{R}$ such that $\langle G(u), G(v)\rangle_{r',s',t'} = \lambda \langle u, v\rangle_{r,s,t}$ for any $u, v \in \mathbb{C}^{r,s,t}$. Moreover, if G preserves the sign of any single positive or negative vector, then $r \leq r'$, $s \leq s'$ and $\lambda > 0$.*

Proof. The proof is just standard linear algebra. For the detail, we refer the reader to [NZ], Lemma 4.2 therein. $\square$

We can say more when $\min\{r', s'\} < \min\{r, s\}$:

Theorem 5.7. *If $\min\{r', s'\} < \min\{r, s\}$, then any local orthogonal map F from $\mathbb{P}^{r,s,t}$ to $\mathbb{P}^{r',s',t'}$ is null.*

Proof. When $\min\{r,s\}=1$, the result is obvious. Now, suppose $\min\{r,s\}\geq 2$. We start from the case $t'=0$. By Proposition 2.6, F maps $(\min\{r,s\}-1)$-planes to $(\min\{r',s'\}-1)$-planes. Then Theorem 3.5(i) says that the linear span of the image of F, denoted by S is of dimension at most $(\min\{r',s'\}-1)$. Exactly the same argument as in Theorem 5.5 shows that S is a null space and hence F is null.

Now, when $t'>0$, we know that $\pi\circ F$ is null, where π is the projection from $\mathbb{P}^{r',s',t'}$ to $\mathbb{P}^{r',s'}$. Hence, F is also null. □

Proposition 5.8. *If $r'<r$ or $s'<s$, then there is no local sign-preserving map from $\mathbb{P}^{r,s,t}$ to $\mathbb{P}^{r',s',t'}$.*

Proof. By symmetry, it suffices to prove the statement for $r'<r$. The statement is trivial if $r'=0$. Suppose $1\leq r'<r$ and $F:U\subset\mathbb{P}^{r,s,t}\to\mathbb{P}^{r',s',t'}$ is a local sign-preserving map.

For a null point $x\in U$, we have $x\in x^{\perp}\cap U$ and so if we take a positive point $p_1\in U$ close enough to x, then $p_1^{\perp}\cap U$ is also non-empty. Note that $p_1^{\perp}$ is an $(r-1,s,t)$-subspace $H^{r-1,s,t}$ and by the sign-preserving property $(F(p_1))^{\perp}$ is an $(r'-1,s',t')$-subspace $H^{r'-1,s',t'}$. So the restriction of F on $U_1:=H^{r-1,s,t}\cap U$ is a local sign-preserving map from $H^{r-1,s,t}$ to $H^{r'-1,s',t'}$. By choosing a convex U with respect to the standard coordinates, we may assume that U_1 remains convex and connected. Now, as $r-1\geq 1$ and $s\geq 1$, we can always choose p_1 such that U_1 contains both positive and negative points and hence also null points.

We repeat the same procedures on U_1. Inductively, after taking r' positive points $p_1,\ldots,p_{r'}$, we get from the restriction of F a local sign-preserving map from $H^{r-r',s,t}$ to $H^{0,s',t'}$. This is a contradiction since $H^{r-r',s,t}$ contains positive points but $H^{0,s',t'}$ does not. □

Theorem 5.9. *Let F be a local sign-preserving map from $\mathbb{P}^{r,s,t}$ to $\mathbb{P}^{r',s',t'}$, where $r,s\geq 2$. If $r=r'$ or $s=s'$ then F is quasi-standard (standard for $t'=0$).*

Proof. By symmetry, it suffices to prove the theorem for $r=r'$. We begin with the case $t'=0$. If $r\leq s$, then everything follows from Theorem 5.5. Suppose now $r>s$.

By following the same procedures of taking orthogonal complements as in the proof of Proposition 5.8, we can pick $(r-s)$ orthogonal positive points in U such that the restriction of F on their orthogonal complement is a local sign-preserving map from some (s,s,t)-subspace $H^{s,s,t}\subset\mathbb{P}^{r,s,t}$ to an (s,s')-subspace $H^{s,s'}\subset\mathbb{P}^{r',s'}$.

As $s\geq 2$, by Theorem 5.5, $F|_{H^{s,s,t}}$ is standard. In particular, $F|_{H^{s,s,t}}$ maps the positive $(s-1)$-planes in $H^{s,s,t}$ to positive $(s-1)$-planes in $\mathbb{P}^{r',s'}$. Note that our argument would give the same conclusion for any other (s,s,t)-subspace $\tilde{H}^{s,s,t}$ close enough to $H^{s,s,t}$ (as points in the Grassmannian). Since the set of positive $(s-1)$-planes is open in the Grassmannian $\mathbb{G}(s-1,\mathbb{P}^{r,s,t})$, it follows that F maps $(s-1)$-planes to $(s-1)$-planes. Thus, by Proposition 3.6, F is either linear or the image of F is contained in an $(s-1)$-plane. The latter is impossible since we already know that $F|_{H^{s,s,t}}$ is standard. So F is linear and together with the sign-preserving hypothesis, we conclude that F is standard by Lemma 5.6.

Finally, if $t' \geq 1$, by considering again the projection from $\mathbb{P}^{r',s',t'}$ to $\mathbb{P}^{r',s'}$ as in Theorem 5.5, we see that F is quasi-standard. □

Theorem 5.10. *Let F be a local orthogonal map from $\mathbb{P}^{r,s}$ to $\mathbb{P}^{r',s',t'}$. If*

$$r' + s' \leq 2\dim(\mathbb{P}^{r,s}) - 1,$$

then F is either null or quasi-linear. If in addition F preserves the sign of any single positive or negative point, then F is quasi-standard.

Proof. The statement is trivial if $\dim(\mathbb{P}^{r,s}) = 1$. We assume $\dim(\mathbb{P}^{r,s}) \geq 3$ for the moment and the two dimensional case will be included later in the argument.

Suppose F is not null and $r' + s' \leq 2\dim(\mathbb{P}^{r,s}) - 1$. Let $F_1 := \pi_1 \circ F$, where $\pi_1 : \mathbb{P}^{r',s',t'} \dashrightarrow \mathbb{P}^{r',s'}$ is the standard projection. If F_1 is not linear, then by Proposition 3.6, either image of F_1 is contained in a line or the linear span S of the image of a general line L under F_1 is of dimension $d \geq 2$. In the latter situation, since $\dim(L^{\perp}) = \dim(\mathbb{P}^{r,s}) - 2$ and $\dim(S^{\perp}) = \dim(\mathbb{P}^{r',s'}) - d - 1$ (here $S^{\perp}$ is the orthogonal complement of S in $\mathbb{P}^{r',s'}$) and any $(\dim(\mathbb{P}^{r,s})-2)$-plane in $\mathbb{P}^{r,s}$ is the orthogonal complement of some line, we see by orthogonality that F_1 maps $(\dim(\mathbb{P}^{r,s})-2)$-planes to $(\dim(\mathbb{P}^{r',s'})-d-1)$-planes. (This in particular also implies that we must have $\dim(\mathbb{P}^{r',s'}) \geq 3$.)

As $\dim(\mathbb{P}^{r,s}) - 2 \geq 1$ and

$$\dim(\mathbb{P}^{r',s'}) - d - 1 \leq r' + s' - 4 \leq 2\dim(\mathbb{P}^{r,s}) - 5 = 2(\dim(\mathbb{P}^{r,s}) - 2) - 1,$$

Theorem 3.5(*ii*) implies that the linear span H_1 of the image of F_1 is of dimension at most

$$\dim(\mathbb{P}^{r',s'}) - d - 1 + 2 = \dim(\mathbb{P}^{r',s'}) - d + 1 \leq \dim(\mathbb{P}^{r',s'}) - 1.$$

Therefore, we have shown that if F_1 is not linear, then the image of F_1 lies in a hyperplane of $\mathbb{P}^{r',s'}$.

If we write $H_1 \cong \mathbb{P}^{r'_1,s'_1,t'_1}$, then we have $r'_1 + s'_1 \leq r' + s' \leq 2\dim(\mathbb{P}^{r,s}) - 1$. So the hypotheses of the theorem still hold for the local orthogonal map F_1 from $\mathbb{P}^{r,s}$ to H_1 and thus by similarly considering the standard projection $\pi_2 : H_1 \dashrightarrow K_1$ for some (r'_1, s'_1)-subspace $K_1 \subset H_1$ and $F_2 := \pi_2 \circ F_1$, etc., we deduce inductively there exists some (r'', s'')-subspace $\Phi \subset \mathbb{P}^{r',s',t'}$ with $F_\Phi := \pi_\Phi \circ F$, where $\pi_\Phi : \mathbb{P}^{r',s',t'} \dashrightarrow \Phi$ is the associated projection, such that we have two possibilities:

(*i*) F_Φ is orthogonal and linear; or

(*ii*) F_Φ is orthogonal and there is a line $L \subset \Phi$ containing the image of F_Φ.

(We note here that if $\dim(\mathbb{P}^{r,s}) = 2$ (which implies also $\dim(\mathbb{P}^{r',s'}) \leq 2$), by replacing a general line L in the argument above by a general point, we see immediately by orthogonality that F_1 maps lines to lines and so we still end up with the two possibilities above (with F_Φ being F_1).)

We are going to first show that case (ii) will lead to a contradiction. If $L \cong \mathbb{P}^{1,0,1}$ or $\mathbb{P}^{0,1,1}$, then F_Φ must be constant (and hence null) since there is only one null point in L. However, F is assumed to be not null and F_Φ is obtained from F by composing a number of projections which are standard each time, thus F_Φ cannot be null neither. So we can only have $L \cong \mathbb{P}^{1,1}$. Consequently, F_Φ can be regarded as a (non-null) local orthogonal map from $\mathbb{P}^{r,s}$ to $\mathbb{P}^{1,1}$ and by Theorem 5.7, we get that $\min\{r,s\} = 1$. If $\max\{r,s\} \geq 2$, for any point $p \in \mathbb{P}^{r,s}$ at which F_Φ is defined, we have $\dim(p^\perp) = \max\{r,s\} - 1 \geq 1$. However, we also have $\dim(F_\Phi(p)^\perp) = 0$ since $F_\Phi(p) \in \mathbb{P}^{1,1}$, so by orthogonality F_Φ maps lines to points. Therefore, F_Φ is constant and hence null, contradicting our assumption at the beginning that F is not null. Therefore, we must have $\max\{r,s\} = 1$ and hence $r = s = 1$. This is again a contradiction since $\dim(\mathbb{P}^{r,s}) \geq 2$.

For case (i), if the image of F is contained in Φ, then F is just F_Φ and is linear. Otherwise, consider the map $F_{\Phi^\perp} := \pi_{\Phi^\perp} \circ F$, where $\pi_{\Phi^\perp} : \mathbb{P}^{r',s',t'} \dashrightarrow \Phi^\perp$ is the canonical projection. Then, $F_{\Phi^\perp}$ is a local orthogonal map from $\mathbb{P}^{r,s}$ to $\Phi^\perp \cong \mathbb{P}^{r'-r'',s'-s'',t'}$ by Lemma 5.4. If $F_{\Phi^\perp}$ is null, then F is quasi-linear. If not, we can repeat the entire argument above on $F_{\Phi^\perp}$ and use induction to conclude that F is quasi-linear.

Finally, if F maps some positive (resp. negative) point to a positive (resp. negative) point, then the linear part of F is standard by Lemma 5.6 and so F is quasi-standard. □

Theorem 5.11. *Let F be a local orthogonal map from $\mathbb{P}^{r,s}$ to $\mathbb{P}^{r',s',t'}$. If*

$$\min\{r',s'\} \leq 2\min\{r,s\} - 2,$$

then F is either null or quasi-linear. If in addition F preserves the sign of any single positive or negative point, then F is quasi-standard.

Proof. The theorem is trivial if $\min\{r,s\} \leq 1$ since it would imply $\min\{r',s'\} = 0$. Suppose $\min\{r,s\} \geq 2$. As before, we just need to prove the case $t' = 0$. By Proposition 2.6, F maps $(\min\{r,s\}-1)$-planes to $(\min\{r',s'\}-1)$-planes. By hypotheses $\min\{r',s'\} - 1 \leq 2(\min\{r,s\}-1) - 1$, and since $\dim(\mathbb{P}^{r,s}) = \min\{r,s\} - 1 + \max\{r,s\}$, we deduce from Theorem 3.5(ii) that the image of F is contained in a linear subspace $\Xi \subset \mathbb{P}^{r',s'}$ such that $\dim(\Xi) \leq \min\{r',s'\} - 1 + \max\{r,s\}$. If we write $\Xi \cong \mathbb{P}^{a_1,b_1,c_1}$ for some non-negative integers a_1, b_1, c_1, then we can regard F as a local orthogonal map from $\mathbb{P}^{r,s}$ to $\mathbb{P}^{a_1,b_1,c_1}$. Note that

$$a_1 + b_1 \leq \dim(\Xi) + 1 \leq \min\{r',s'\} + \max\{r,s\} \leq 2\min\{r,s\} - 2 + \max\{r,s\}.$$

Thus, $a_1 + b_1 < 2(r+s) - 3 = 2\dim(\mathbb{P}^{r,s}) - 1$ and now the desired results follows directly from Theorem 5.10. □

Theorem 5.12. *Let F be a local orthogonal map from $\mathbb{P}^{1,s}$ to $\mathbb{P}^{1,s',t'}$. If $s' \leq 2s-2$, then F either null or quasi-standard. If in addition if $t'=0$, then F is either constant or standard.*

Proof. As usual, we just need to prove the theorem for $t'=0$. Let $F : U \subset \mathbb{P}^{1,s} \to \mathbb{P}^{1,s'}$ be a non-constant orthogonal map and $s' \leq 2s-2$. Take a null point $x \in U$. Since we are in $\mathbb{P}^{1,s}$, it follows that $x^\perp$ is a semi-negative hyperplane and x is the only null point in $x^\perp$. By orthogonality, $F(x^\perp \cap U) \subset (F(x))^\perp$ and $F(x)$ is also the only null point in the semi-negative hyerplane $F(x)^\perp \subset \mathbb{P}^{1,s'}$. As F is non-constant, we can always choose x such that F is not constant on $x^\perp \cap U$ and hence F preserves the sign of some negative point on $x^\perp$.

Now since $1+s' \leq 2\dim(\mathbb{P}^{1,s})-1$, from Theorem 5.10 we have that F is quasi-standard. Finally, as the orthogonal complement of a $(1,s)$-subspace in $\mathbb{P}^{1,s'}$ is a $(0,s'-s)$-subspace which does not contain any null point, we see from the definition of quasi-standard maps that F is actually standard. □

6 Proper maps between generalized balls

On $\mathbb{P}^{r,s,t}$, the set of positive points $\mathbb{B}^{r,s,t} \subset \mathbb{P}^{r,s,t}$ (or $\mathbb{B}^{r,s} \subset \mathbb{P}^{r,s}$) has been called a *generalized ball* in the literature since $\mathbb{B}^{1,s}$ is just the ordinary s-dimensional complex unit ball $\mathbb{B}^s$ embedded in $\mathbb{P}^s$. In additon, the boundary $\partial\mathbb{B}^{r,s,t}$ of $\mathbb{B}^{r,s,t}$ is simply the set of null points on $\mathbb{P}^{r,s,t}$.

A local holomorphic map $f : U \subset \mathbb{P}^{r,s,t} \to \mathbb{P}^{r',s',t'}$, defined on a connected open set U such that $U \cap \partial\mathbb{B}^{r,s,t} \neq \varnothing$, is called a *local proper holomorphic map* from $\mathbb{B}^{r,s,t}$ to $\mathbb{B}^{r',s',t'}$ if $f(U \cap \mathbb{B}^{r,s,t}) \subset \mathbb{B}^{r',s',t'}$ and $f(U \cap \partial\mathbb{B}^{r,s,t}) \subset \partial\mathbb{B}^{r',s',t'}$.

The following statement is a direct consequence of Proposition 2.5.

Proposition 6.1. *By shrinking the domain of definition if necessary, a local proper holomorphic map from $\mathbb{B}^{r,s,t}$ to $\mathbb{B}^{r',s',t'}$ is a local orthogonal map from $\mathbb{P}^{r,s,t}$ to $\mathbb{P}^{r',s',t'}$.*

Consequently, our results for local orthogonal maps in the previous sections also hold for local proper holomorphic maps among generalized balls, from which we can obtain and generalize many well-known results in the literature.

6.1 Rigidity theorems

Theorem 6.2. *When $r,s \geq 2$, every local proper holomorphic map from $\mathbb{B}^{r,s,t}$ to $\mathbb{B}^{r,s',t'}$ is quasi-standard (standard for $t'=0$).*

Proof and remarks. The statement follows from Theorem 5.9. Here we note that although Theorem 5.9 is for local sign-preserving maps, its proof actually only assumes that positive points are mapped to positive points.

The case for $r \leq s$ and $t = t' = 0$ has been obtained by Baouendi-Huang [BH] (Theorem 1.2 therein) while the case for $r \leq s$ has been obtained by Ng-Zhu [NZ] (Theorem 1.2 therein). □

Theorem 6.3. *When $r' + s' \leq 2(r+s) - 3$, every local proper holomorphic map from $\mathbb{B}^{r,s}$ to $\mathbb{B}^{r',s',t'}$ is quasi-standard (standard for $t' = 0$). In particular, every local proper holomorphic map from $\mathbb{B}^s$ to $\mathbb{B}^{s'}$ is standard if $s' \leq 2s - 2$.*

Proof and remarks. It follows from Theorem 5.10. The special case for $r + 1 = r' = 2$ and $t' = 0$ has been proven by Xiao-Yuan [XY] (Theorem 3.2 therein). The special case for the ordinary complex unit balls is a classical theorem by Faran [Fa1]. □

Theorem 6.4. *When $r \leq s$ and $r' \leq 2r - 2$, every local proper holomorphic map from $\mathbb{B}^{r,s} \to \mathbb{B}^{r',s',t'}$ is quasi-standard.*

Proof and remarks. It is a direct consequence of Theorem 5.11. The case $t' = 0$ has been obtained by Baouendi-Ebenfelt-Huang [BEH] (Theorem 1.1 and Corollary 1.6 therein) □

6.2 Gap phenomenon

Let f be a local proper holomorphic map from $\mathbb{B}^{r,s}$ to $\mathbb{B}^{R,S}$. If there exists an (r', s')-subspace $H^{r',s'} \subset \mathbb{P}^{R,S}$, with canonical projections $\pi : \mathbb{P}^{R,S} \dashrightarrow H^{r',s'}$ and $\pi^{\perp} : \mathbb{P}^{R,S} \dashrightarrow (H^{r',s'})^{\perp}$ such that either (i) the image of f is contained in $H^{r',s'}$; or (ii) $\pi \circ f$ is a local proper holomorphic map from $\mathbb{B}^{r,s}$ to $\mathbb{B}^{R,S} \cap H^{r',s'} \cong \mathbb{B}^{r',s'}$ and $\pi^{\perp} \circ f$ is null, then we say that f is a ***null prolongation*** of $\pi \circ f$.

Example. If $f = [f_1, \ldots, f_{r'+s'}]$ is a rational proper holomorphic map from $\mathbb{B}^{r,s}$ to $\mathbb{B}^{r',s'}$, where each $f_j \in \mathbb{C}[z_1, \ldots, z_{r+s}]$ is a degree-d homogeneous polynomial, then for any homogeneous $\psi, \phi \in \mathbb{C}[z_1, \ldots, z_{r+s}]$ with $\deg(\phi) = \deg(\psi) + d$, the map

$$F := [\psi f_1, \ldots, \psi f_{r'}, \phi, \psi f_{r'+1}, \ldots, \psi f_{r'+s'}, \phi],$$

which is locally proper from $\mathbb{B}^{r,s}$ to $\mathbb{B}^{r'+1,s'+1}$, is a null prolongation of f. It is also easy to see that for maps between unit balls, a null prolongation is simply a local proper holomorphic map whose image lies in a smaller dimensional ball.

Theorem 6.5. *Let $k, n \in \mathbb{N}^+$ such that $n \geq k^2 + k + 1$. Let f be a local proper holomorphic map from $\mathbb{B}^{r,s}$ to $\mathbb{B}^{R,S}$, where $n = \dim(\mathbb{B}^{r,s})$ and $N := \dim(\mathbb{B}^{R,S})$. If*

$$kn + k \leq N \leq (k+1)n - (k^2 + 1),$$

then there exists an (r', s')-subspace $H^{r',s'} \subset \mathbb{P}^{R,S}$, with $\dim(H^{r',s'}) = kn + k - 1$, such that f is a null prolongation of some local proper holomorphic map from $\mathbb{B}^{r,s}$ to $\mathbb{B}^{R,S} \cap H^{r',s'} \cong \mathbb{B}^{r',s'}$.

Proof and remarks. In Theorem 4.4, by substituting $k := a+1$, we see that whenever the hypotheses are satisfied, the image of f is contained in a hyperplane $H \subset \mathbb{P}^{r,s}$. Write $H \cong \mathbb{P}^{r_1,s_1,t_1}$. Let $H^{r_1,s_1} \subset H$ be any (r_1, s_1)-subspace and $\pi : H \dashrightarrow H^{r_1,s_1}$ be the canonical projection. Then, it follows that f is a null prolongation of $f_1 := \pi \circ f$ from $\mathbb{B}^{r,s}$ to $\mathbb{B}^{R,S} \cap H^{r_1,s_1} \cong \mathbb{B}^{r_1,s_1}$. The desired result then follows if we repeat the argument for a finite number of times. (Note that if f_1 is a null prolongation of another local proper holomorphic map f_2 from $\mathbb{B}^{r,s}$ to $\mathbb{B}^{r_1,s_1} \cap H^{r_2,s_2} \cong \mathbb{B}^{r_2,s_2}$ for some (r_2, s_2)-subspace H^{r_2,s_2}, then f is also a null prolongation of f_2.)

For proper holomorphic maps from $\mathbb{B}^{1,n} \cong \mathbb{B}^n$ to $\mathbb{B}^{1,N} \cong \mathbb{B}^N$, Faran's result [Fa1] is essentially the statement that the conclusion of the theorem holds for $n+1 \le N \le 2n-2$. Moreover, the same conclusion has been shown by Huang-Ji-Xu [HJX] to hold for $2n+1 \le N \le 3n-4$ and by Huang-Ji-Yin [HJY] for $3n+1 \le N \le 4n-7$.

The lower bound $kn+k$ of our gap is actually optimal. This can be seen by considering the expansion of $\left(\sum_{j=1}^{k} |z_j|^2 - \sum_{j=k+1}^{n+1} |z_j|^2\right)\left(\sum_{j=1}^{k} |z_j|^4\right)$. The expansion is a sum of (plus or minus) norm squares of $kn+k$ linearly independent cubic monomials. Using these monomials as components we get a rational proper map from $\mathbb{B}^{k,n+1-k}$ to $\mathbb{B}^{k^2,kn+k-k^2}$ whose image does not lie in any hyperplane. Note that $\dim(\mathbb{B}^{k,n+1-k}) = n$ and $\dim(\mathbb{B}^{k^2,kn+k-k^2}) = kn+k-1$. □

6.3 Degree estimate

Recall from Section 4 that if $n, N \in \mathbb{N}^+$ are such that $n+1 \le N < \binom{n+2}{2}$, then by using the n-th Macaulay's representation of N we can write $N = N(n; a, b) := \binom{n+1}{n} + \cdots + \binom{n-a+1}{n-a} + b$ for some $a, b \ge 0$ such that $b \le n-a-1$.

Lemma 6.6. $N(n;a,b) - N(n;a,b)^{-<n>} = \begin{cases} a+2 & \text{if} \quad n-a-b=1; \\ a+1 & \text{if} \quad n-a-b \ge 2. \end{cases}$

Proof. For $b = 0$, since $N(n;a,0) := \binom{n+1}{n} + \cdots + \binom{n-a+1}{n-a}$, it follows immediately that $N(n;a,0)^{-<n>} = N(n;a,0) - (a+2)$ if $n-a=1$; and $N(n;a,0)^{-<n>} = N(n;a,0) - (a+1)$ if $n-a \ge 2$. The case for $b \ge 1$ is similar. □

Theorem 6.7. *Suppose $n \ge 3$ and f is a rational proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^{N(n;a,b)}$. Then,* $\deg(f) \le (a+2)(2a+3)$.

Proof. We first assume that the image of f is not contained in a hyperplane. Let $m \in \{1, \ldots, n\}$ and $D_m = \dim(\mathrm{span}(f(M)))$, where $M \subset \mathbb{B}^n$ is a general m-dimensional linear subspace. By applying Lemma 6.6 twice (c.f. Lemma 4.2), we have $D_{n-2} \ge N - 2(a+2)$. Then, by the orthogonality of f (Proposition 4.1),

$$D_1 \le N - D_{n-2} - 1 \le 2a+3.$$

That is, f maps lines to $(2a+3)$-planes.

Meylan ([Me], Theorem 4.1 therein) has shown that if a rational proper holomorphic map g from $\mathbb{B}^2$ to $\mathbb{B}^m$ maps lines to ℓ-planes, then $\deg(g) \le \dfrac{\ell(\ell+1)}{2}$. Since the restriction of f to a general 2-plane has the same degree as f itself, we thus deduce that

$$\deg(f) \le \frac{(2a+3)(2a+4)}{2} = (a+2)(2a+3).$$

Now if the image of f is contained in a hyperplane, then f simply maps into a lower dimensional ball $\mathbb{B}^{N'}$. If $N' \le 2n-2$, we know that $\deg(f) = 1$ by Theorem 6.3. Otherwise we have $N' = N(n; a', b')$ for some a', b'. Since the lexicographic ordering of the Macaulay's representation is the same as the usual ordering of integers, we see from n-th Macaulay's representation of $N(n; a, b)$ that $a' \le a$ and hence in all cases we always have $\deg(f) \le (a+2)(2a+3)$. $\square$

Corollary 6.8. *Let f be a rational proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^N$. If $n \ge 3$ and $N < \binom{n+2}{2}$, then,*

$$\deg(f) \le \left(\frac{2N}{n+3} + 1\right)\left(\frac{4N}{n+3} + 1\right).$$

Proof. We just need to prove the estimate for $N = N(n; a, b)$ since $\deg(f) = 1$ when $N \le 2n-2$. Note that as $n - a - 1 \ge 0$, we have

$$N(n; a, b) = \left(n + 1 - \frac{a}{2}\right)(a+1) + b \ge \left(n + 1 - \frac{n-1}{2}\right)(a+1) = \frac{(n+3)(a+1)}{2}.$$

We then get that $\dfrac{2N(n; a, b)}{n+3} \ge a+1$. Therefore, if $n \ge 3$ and $N < \binom{n+2}{2}$, the desired degree estimate then follows from Theorem 6.7. $\square$

7 Appendix

Proof of Lemma 3.2. We will prove by induction and first show that the lemma is true for $m = 1$ or $k = 1$. Suppose $k = 1$ and $A + B = \binom{m+1}{1} - 1 = m$. If $A = 0$ and $B = m$, then $A^{-<m>} + B_{<1>} = 0 + (m-1) = m - 1 = \binom{m}{1} - 1$. If $1 \le A \le m$, then $A = \binom{m}{m} + \binom{m-1}{m-1} + \cdots + \binom{m-A+1}{m-A+1}$. Hence, $A^{-<m>} + B_{<1>} = A + B - 1 = \binom{m}{1} - 1$. Suppose $m = 1$ and $A + B = \binom{1+k}{k} - 1$. Then $B \le k$ and hence $B_{<k>} = 0$. Thus, $A^{-<1>} + B_{<k>} = 0 + 0 = \binom{k}{k} - 1$.

Suppose $A+B = \binom{m+k}{k} - 1 = \binom{m+k}{m} - 1 = \binom{m+k-1}{m} + \binom{m+k-1}{k} - 1$. Then, either $A \ge \binom{m+k-1}{m}$ or $B \ge \binom{m+k-1}{k}$ (otherwise we would have $A+B \le \binom{m+k-1}{m} - 1 + \binom{m+k-1}{k} - 1$).

If $A \ge \binom{m+k-1}{m}$, then the m-th Macaulay's representation of A is of the form $A = \binom{m+k-1}{m} + \binom{a_{m-1}}{m-1} + \cdots + \binom{a_\delta}{\delta}$. Since $(A - \binom{m+k-1}{m}) + B = \binom{m+k-1}{k} - 1$, by the induction hypothesis we get

$$\begin{aligned}
\left(A - \binom{m+k-1}{m}\right)^{-<m-1>} + B_{<k>} &= \binom{m+k-2}{k} - 1 \\
\Rightarrow \binom{a_{m-1}-1}{m-2} + \cdots + \binom{a_\delta - 1}{\delta - 1} + B_{<k>} &= \binom{m+k-2}{k} - 1 \\
\Rightarrow \binom{m+k-2}{m-1} + \binom{a_{m-1}-1}{m-2} + \cdots + \binom{a_\delta - 1}{\delta - 1} + B_{<k>} &= \binom{m+k-2}{m-1} + \binom{m+k-2}{k} - 1 \\
\Rightarrow A^{-<m>} + B_{<k>} &= \binom{m+k-1}{k} - 1
\end{aligned}$$

If $B \geq \binom{m+k-1}{k}$, then the k-th Macaulay's representation of B is of the form $B = \binom{m+k-1}{k} + \binom{b_{k-1}}{k-1} + \cdots + \binom{b_\epsilon}{\epsilon}$. Since $A + (B - \binom{m+k-1}{k}) = \binom{m+k-1}{k-1} - 1$, by the induction hypothesis we get

$$\begin{aligned}
A^{-<m>} + \left(B - \binom{m+k-1}{k}\right)_{<k-1>} &= \binom{m+k-2}{k-1} - 1 \\
\Rightarrow A^{-<m>} + \binom{b_{k-1}-1}{k-1} + \cdots + \binom{b_\epsilon - 1}{\epsilon} &= \binom{m+k-2}{k-1} - 1 \\
\Rightarrow \binom{m+k-2}{k} + A^{-<m>} + \binom{b_{k-1}-1}{k-1} + \cdots + \binom{b_\epsilon - 1}{\epsilon} &= \binom{m+k-2}{k} + \binom{m+k-2}{k-1} - 1 \\
\Rightarrow A^{-<m>} + B_{<k>} &= \binom{m+k-1}{k} - 1
\end{aligned}$$

□

Proof of Lemma 3.3. It suffices to prove that

$$\binom{n+d-1}{d} - \left(\binom{n+d}{d} - K\right)_{<d>} = \binom{n+d}{d+1} - \left(\binom{n+d+1}{d+1} - K\right)_{<d+1>}.$$

Since $\binom{n+d+1}{d+1} - K = \binom{n+d}{d+1} + \binom{n+d}{d} - K$ and $0 \leq \binom{n+d}{d} - K < \binom{n+d}{d}$, we have

$$\left(\binom{n+d+1}{d+1} - K\right)_{<d+1>} = \binom{n+d-1}{d+1} + \left(\binom{n+d}{d} - K\right)_{<d>},$$

which is equivalent to the previous equation. □

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References

[BH] Baouendi, M. S., Huang X.,: Super-rigidity for holomorphic mappings between hyperquadrics with positive signature. *J. Diff. Geom.* **69** (2005), 379-398.

[BEH] Baouendi, M. S., Ebenfelt, P., Huang, X.: Holomorphic mappings between hyperquadrics with small signature difference. *Amer. Math. J.* **133** (2011), 1633-1661.

[Da] D'Angelo, J. P.: Polynomial proper maps between balls. *Duke Math. J.* **57** (1988), 211-219.

[DL] D'Angelo, J. P., Lebl, J.: On the complexity of proper mappings between balls. *Complex Var. Elliptic Equ.* **54**(3) (2009), 187204.

[Fa1] Faran, J.: The linearity of proper holomorphic maps between balls in the low codimension case. *J. Diff. Geom.* **24** (1986), 15-17.

[Fa2] Faran, J.: Maps from the 2 ball to the 3 ball. *Invent. Math.* **68** (1982), 441-475.

[Fo] Forstnerič, F.: Proper holomorphic maps between balls, *Duke Math. J.* **53** (1986) 427440.

[Gr] Green, M.: Restrictions of linear series to hyperplanes, and some results of Macaulay and Gotzmann. *Algebraic Curves and Projective Geometry* (1988), Trento Lecture Notes in Math., Vol. 1389, Springer, pp. 76-86

[GLV] Grundmeier, D., Lebl, J., Vivas, L.: Bounding the rank of Hermitian forms and rigidity for CR mappings of hyperquadrics. *Math. Ann.* **358** (2014), 1059-1089.

[HJ] Huang, X., Ji, S.: Mapping B^n into B^{2n-1}. *Invent. Math.* **145** (2001), 219-250.

[HJX] Huang, X., Ji, S., Xu, D.: A new gap phenomenon for proper holomorphic mappings from B^n into B^N. *Math. Res. Lett.* **13**(4) (2006), 515-529.

[HJY] Huang, X., Ji, S., Yin, W.: On the third gap for proper holomorphic maps between balls. *Math. Ann.* **358** (2014), 115-142.

[HJY2] Huang, X., Ji, S., Yin, W.: Recent Progress on Two Problems in Several Complex Variables. *ICCM.* Vol.I, (2007) 563-575.

[Ma] Macaulay, F. S.:: Some properties of enumeration in the theory of modular systems, *Proc. London Math. Soc.* **26** (1927), 531555.

[Me] Meylan, F.: Degree of a holomorphic map between unit balls from $\mathbb{C}^2$ to $\mathbb{C}^n$. *Proc. Amer. Math. Soc.* **134**(4) (2006), 1023-1030.

[Ng1] Ng, S.-C.: Proper holomorphic mappings on flag domains of $SU(p,q)$-type on projective spaces . *Michigan Math. J.* **62** (2013), 769-777.

[Ng2] Ng, S.-C.: Cycle spaces of flag domains on Grassmannians and rigidity of holomorphic mappings. *Math. Res. Lett.* **19**(6) (2012), 1219-1236.

[Ng3] Ng, S.-C.: Holomorphic double fibration and the mapping problems of classical domains. *Int. Math. Res. Not.* **215** Issue 2, (2015), 291-324.

[NZ] Ng, S.-C., Zhu, Y.: Rigidity of proper holomorphic maps among generalized balls with Levi-degenerate boundaries. **http://arxiv.org/abs/2104.06318**

[XY] Xiao, M., Yuan, Y.: Holomorphic maps from the complex unit ball to Type IV classical domains. *J. Math. Pures Appl.* **133** (2020), 139-166.

[Za1] Zaitsev, D.: Obstructions to embeddability into hyperquadrics and explicit examples. *Math. Ann.*, **342**(3) (2008), 695-726.

[Za2] Zaitsev, D.: Algebraicity of local holomorphisms between real-algebraic submanifolds of complex spaces. *Acta. Math.* **183** (1999), 273-305.