Factorizations of a \(d\)-cycle and multi-noded rooted trees

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This is joint work with Fu Liu.
Outline

- Factorizations: Definitions and Backgrounds
- Multi-noded Rooted Trees
- Bijections Between Factorizations and Multi-noded Rooted Trees
- Future work
PART I:

Factorizations: Definitions and Backgrounds
Factorizations of a $d$-cycle

Factorizations of a $d$-cycle
Definition 1. Assume $d, r \geq 1, e_1, \ldots, e_{r-1} \geq 2$ are integers satisfying $\sum_{i=1}^{r-1} (e_i - 1) = d - 1$. Fix a $d$-cycle $\tau \in S_d$. We say $(\sigma_1, \ldots, \sigma_{r-1})$ is a factorization of $\tau$ of type $(e_1, \ldots, e_{r-1})$ if the followings are satisfied:

1. For each $i$, $\sigma_i$ is an $e_i$-cycle in $S_d$.

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1. For each $i$, $\sigma_i$ is an $e_i$-cycle in $S_d$.
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Example 2. Let $d = 5$, $r = 4$, $(e_1, e_2, e_3) = (2, 2, 3)$, $\tau = (1 2 3 4 5)$, $\sigma_1 = (2 3)$, $\sigma_2 = (4 5)$, $\sigma_3 = (1 3 5)$. It is easy to check that $(\sigma_1, \sigma_2, \sigma_3)$ is a factorization of $\tau$ of type $(2, 2, 3)$:

$$(2 3)(4 5)(1 3 5) = (1 2 3 4 5).$$
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Question 3. Given a $d$-cycle $\tau$ and integers $e_1, \ldots, e_{r-1} \geq 2$, how many factorizations are there of $\tau$ of type $(e_1, \ldots, e_{r-1})$?
Main Result

Theorem 4. Suppose $\sum_{i=1}^{r-1} (e_i - 1) = d - 1$. Then the number of factorizations of a $d$-cycle of type $(e_1, \ldots, e_{r-1})$ is

$$\text{fac}(d, r; e_1, \ldots, e_{r-1}) = d^{r-2}.$$
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**Example 5.** There are $3 = 3^1$ factorizations of $(1\ 2\ 3)$ of type $(2, 2)$:

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**Example 5.** There are \(3 = 3^1\) factorizations of \((1 2 3)\) of type \((2, 2)\):

\[(12)(23) \quad (23)(13) \quad (13)(12)\]

**Example 6.** There are \(25 = 5^2\) factorizations of \((1 2 3 4 5)\) of type \((2, 2, 3)\):

\[
(12)(23)(345) \quad (23)(34)(451) \quad (34)(45)(512) \quad (45)(51)(123) \quad (51)(12)(234) \\
(23)(13)(345) \quad (34)(24)(451) \quad (45)(35)(512) \quad (51)(41)(123) \quad (12)(52)(234) \\
(13)(12)(345) \quad (24)(23)(451) \quad (35)(34)(512) \quad (41)(45)(123) \quad (52)(51)(234) \\
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Hurwitz Factorizations

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Given $d \geq 1$, $g \geq 0$, $r \geq 0$, and $\lambda^1, \ldots, \lambda^r$ partitions of $d$, the *Hurwitz number* $h(d, r, g; \lambda^1, \ldots, \lambda^r)$ counts the number of degree-$d$-and-genus-$g$ covers of the projective line with $r$ branch points where the monodromy over the $i$th branch point has cycle type $\lambda^i$. 
This question arises from algebraic geometry. A more general question is to find the Hurwitz number:

Given \( d \geq 1, g \geq 0, r \geq 0 \), and \( \lambda^1, \ldots, \lambda^r \) partitions of \( d \), the Hurwitz number \( h(d, r, g; \lambda^1, \ldots, \lambda^r) \) counts the number of degree-\( d \)-and-genus-\( g \) covers of the projective line with \( r \) branch points where the monodromy over the \( i \)th branch point has cycle type \( \lambda^i \).

**Definition 7.** Let \( \iota(\lambda) = \sum_i (\lambda_i - 1) \) and assume

\[
\sum_{i=1}^{r} \iota(\lambda^i) = 2d - 2 + 2g.
\]

A Hurwitz factorization of type \((d, r, g; \lambda^1, \ldots, \lambda^r)\) is a tuple \((\sigma_1, \ldots, \sigma_r)\) satisfying:

1. \( \sigma_i \in S_d \) has cycle type \( \lambda^i \);
2. \( \sigma_1 \cdots \sigma_r = 1 \);
3. the \( \sigma_i \)'s generate a transitive subgroup of \( S_d \).
When $e_1 = \cdots = e_{r-1} = 2$, from $\sum_{i=1}^{r-1} (e_i - 1) = d - 1$ we have $d = r$. Then Theorem 4 gives the following well-known result:

**Corollary 8.** The number of factorizations of a $d$-cycle into $d - 1$ transpositions are $d^{d-2}$. 
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**Corollary 8.** The number of factorizations of a \( d \)-cycle into \( d - 1 \) transpositions are \( d^{d-2} \).

Note that \( d^{d-2} \) is also the number of trees on \( d \) vertices. Different bijective proofs of this result were given by Dénes (1959), Moszkowski (1989), Goulden-Pepper (1993) and Goulden-Yong (2002).
Special Case

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**Main Idea to prove Theorem 4:** Construct a class of combinatorial objects that are counted by \(d^{r-2}\), and then find a bijection between factorizations and them.
PART II:

Multi-noded Rooted Trees
Definition 9. Suppose $f_0, f_1, \ldots, f_n$ are positive integers. We say $G = (T, \beta)$ is a multi-noded rooted tree on $S \cup \{0\}$ of vertex data $(f_0, f_1, \ldots, f_n)$ if $T = (S \cup \{0\}, E)$ is a rooted tree in $\mathcal{R}_S$ and $\beta : E \rightarrow \mathbb{N}$ is a function satisfying that for any edge $e \in E$, if $s_i$ is the parent of $e$, then $\beta(e) \in \{1, 2, \ldots, f_i\}$. We denote by $\mathcal{M} \mathcal{R}_S(f_0, f_1, \ldots, f_n)$ the set of multi-noded rooted trees.
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Example 10. A multi-noded rooted tree of vertex data $(1, 1, 2, 1, 2, 2, 3, 3, 1, 4)$ and $\beta(\{0, s_3\}) = 1, \beta(\{0, s_5\}) = 1, \beta(\{s_3, s_8\}) = 1, \beta(\{s_3, s_2\}) = 1, \beta(\{s_5, s_9\}) = 1, \beta(\{s_2, s_6\}) = 2, \beta(\{s_9, s_4\}) = 1, \beta(\{s_9, s_1\}) = 3, \beta(\{s_9, s_7\}) = 3$.
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Counting Multi-noded Rooted Trees

**Theorem 11.** $|MR_S(f_0, f_1, \ldots, f_n)| = f_0 \left( \sum_{i=0}^{n} f_i \right)^{n-1}$. 
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Corollary 12. Suppose $\sum_{j=1}^{r-1} (e_j - 1) = d - 1.$ Then

$$|\mathcal{MR}_S(1, e_1 - 1, \ldots, e_{r-1} - 1)| = d^{r-2}.$$
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\[
\begin{pmatrix}
s_3 & s_9 \\
1 & 3
\end{pmatrix}
\]
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PART III:

Bijection between Factorizations and Multi-noded Rooted Trees
Factorizations of a $d$-cycle and multi-noded rooted trees

Rosena R. X. Du

Factorizations Graphs
A factorization of $\tau = (1 \ 2 \ \cdots \ 20)$:

$$(10 \ 11)(14 \ 15 \ 19)(1 \ 19)(3 \ 4 \ 5)(1 \ 2 \ 13)(15 \ 16 \ 17 \ 18)(7 \ 8 \ 9 \ 11)(19 \ 20)(2 \ 5 \ 6 \ 11 \ 12)$$
A factorization of \( \tau = (1 \ 2 \ \cdots \ 20) \):

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Characterization of factorization graphs

Proposition 13. Suppose $G \in \mathcal{G}_S(d, r; e_1, \ldots, e_{r-1})$.

Then $\sum_{j=1}^{r-1}(e_j - 1) = d - 1$ and $G \in \mathcal{G}_S^*(d, r, \tau; e_1, \ldots, e_{r-1})$ if and only if $G$ satisfies the following conditions:

1. $G$ is a tree.

2. For each $S$-vertex $s$ of $G$, if we remove $s$ and all its incident edges from $G$, the set of $[d]$-vertices of the subtrees we obtain partition the circle $\tau$ into consecutive pieces.

3. For each $[d]$-vertex $\nu$ of $G$, suppose we get $t$ subtrees after removing $\nu$ and all its incident edges, then

   (a) The $[d]$-vertices of the $t$ subtrees partition $[d] \setminus \{\nu\}$ into consecutive pieces.

   (b) If we order the pieces in counterclockwise order on $\tau$ starting from $\nu$, then the $m$-th piece is exactly the subtree that contains vertex $s_{jm}$ for any $1 \leq m \leq t$. 
Factorization Graphs to Labeled Multi-noded Rooted Trees

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PART IV:

Future Work?
Hurwitz factorizations of other types?
Factorizations of a $d$-cycle and multi-noded rooted trees

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Hurwitz (1891) and Goulden-Jackson (1997) showed that if $\lambda^1 = \cdots = \lambda^{r-1} = (2, 1, \ldots, 1)$ and $\lambda^r = (\tau_1, \ldots, \tau_n)$, then

$$h(d, r, 0; \lambda^1, \ldots, \lambda^r) = \frac{(r - 1)!d^{n-3}\prod_{i=1}^{n} \tau_i^{\tau_i}/\tau_i!}{m_1!m_2! \cdots m_d!},$$
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Liu and Osserman (2008) showed that

$$h(d, 4, 0; e_1, e_2, e_3, e_4) = \min\{e_i(d + 1 - e_i)\}.$$
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What are the Hurwitz numbers of other types?