VI. HAAR WAVELET AND MULTIRESOLUTION ANALYSIS

In this chapter we will look at a specific Multiresolution Analysis, which is related to the Haar wavelet introduced in Chapter 3. Recall that in Example 1 of Chapter 5, we defined a sequence of subspaces $V_n$ of $L^2(\mathbb{R})$. In fact, for any $n \in \mathbb{Z}$, we defined a subspace $V_n$ of $L^2(\mathbb{R})$ such that $V_n = \{ f \in L^2(\mathbb{R}) | \text{for any } j \in \mathbb{Z}, f|_{\left[\frac{j}{2^n}, \frac{j+1}{2^n}\right]} \text{ is constant} \}$.

We discussed some of their properties in Lemma 1 of Chapter 5. By definition of Multiresolution Analysis given in Chapter 5, we see that the sequence of subspaces defined above is indeed a Multiresolution Analysis.

For such a sequence $\ldots V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots$, we can define the orthogonal complement of $V_0$ in $V_1$ and call it $W_0$. Namely, $W_0 = V_1 \ominus V_0$. Equivalently, $V_1 = V_0 \oplus W_0$. In general, since by Lemma 1 of Chapter 5, for any $n \in \mathbb{Z}$, we have $V_n \subset V_{n+1}$, so we can define the orthogonal complement of $V_n$ in $V_{n+1}$, we call it $W_n$. Namely, $W_n = V_{n+1} \ominus V_n$. Equivalently, $V_{n+1} = V_n \oplus W_n$.

We want to investigate the orthogonal projections of any function in $L^2(\mathbb{R})$ onto subspaces $V_n$ and $W_n$. To this end, we need to find for each of these subspaces a complete orthonormal system. So far, we have already obtained a complete orthonormal system for $V_0$. We repeat the result here.

**Lemma 1.** For any $n \in \mathbb{Z}$, let $V_n = \{ f \in L^2(\mathbb{R}) | \text{for any } j \in \mathbb{Z}, f|_{\left[\frac{j}{2^n}, \frac{j+1}{2^n}\right]} \text{ is constant} \}$. If

$$\varphi(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

then $\varphi(x) \in V_0$ and $\{ \varphi(x-l) | l \in \mathbb{Z} \}$ is a complete orthonormal system of $V_0$.

Next lemma shows that Haar wavelet has something to do with a complete orthonormal system for $W_0$.

**Lemma 2.** For any $n \in \mathbb{Z}$, let $V_n = \{ f \in L^2(\mathbb{R}) | \text{for any } j \in \mathbb{Z}, f|_{\left[\frac{j}{2^n}, \frac{j+1}{2^n}\right]} \text{ is constant} \}$ and $W_n = V_{n+1} \ominus V_n$. 

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If
\[ H(x) = \begin{cases} 
1 & 0 \leq x < \frac{1}{2} \\
-1 & \frac{1}{2} \leq x < 1 \\
0 & \text{otherwise}
\end{cases} \]
then \( H(x) \in W_0 \) and \( \{H(x-l)|l \in \mathbb{Z}\} \) is a complete orthonormal system of \( W_0 \).

Proof. We first compute to get that for each \( l \in \mathbb{Z} \),
\[ H_0,l = H(x-l) = \begin{cases} 
1 & l \leq x < l + \frac{1}{2} \\
-1 & l + \frac{1}{2} \leq x < l + 1 \\
0 & \text{otherwise}
\end{cases} \]
First of all, by definition of \( V_n \), we see that \( H(x-l) \in V_1 \) for each \( l \in \mathbb{Z} \). Secondly, for any \( f \in V_0 \), by definition, \( f|_{[j,j+1)} \) is a constant for each \( j \in \mathbb{Z} \). Let \( f|_{[j,j+1)} = a_j \) for each \( j \in \mathbb{Z} \), then
\[ \langle f, H_0,l \rangle = \int_{l}^{l+\frac{1}{2}} a_l dx + \int_{l+\frac{1}{2}}^{l+1} (-a_l) dx = \frac{1}{2}(a_l - a_l) = 0 \]
Hence \( H(x-l) \in V_1 \ominus V_0 = W_0 \) for each \( l \in \mathbb{Z} \). Lastly, since \( \{H(x-l)|l \in \mathbb{Z}\} = \{H_0,l|l \in \mathbb{Z}\} \) is a subset of a orthonormal system \( \{H_{n,l}|l, n \in \mathbb{Z}\} \) we discussed in Chapter 3, so we only need to show that \( \{H(x-l)|l \in \mathbb{Z}\} = \{H_0,l|l \in \mathbb{Z}\} \) is complete.

By Lemma 4 in Chapter 1, we only need to show that for any \( f \in V_1 \ominus V_0 = W_0 \),
\[ \|f\|_2^2 = \sum_{l \in \mathbb{Z}} |\langle f, H_0,l \rangle|^2. \]
Since \( f \in V_1 \), we can let \( f|_{\left[\frac{j}{2}, \frac{j+1}{2}\right)} = b_j \) for each \( j \in \mathbb{Z} \). By Lemma 1, \( \{\varphi(x-l)|l \in \mathbb{Z}\} \) is a complete orthonormal system of \( V_0 \), so for any \( l \in \mathbb{Z} \), \( \langle f, \varphi(x-l) \rangle = 0 \). This means \( \frac{b_{2l} + b_{2l+1}}{2} = 0 \), \( \frac{b_{2l+1} + b_{2l+1}}{2} = 0 \), ..., in general, for any \( l \in \mathbb{Z} \), we have
\[ \frac{b_{2l} + b_{2l+1}}{2} = 0. \]
On the other hand, we can compute to get that \( \|f\|_2^2 = \sum_{l \in \mathbb{Z}} \frac{1}{2}|b_j|^2 \), \( \langle f, H_{0,1} \rangle = \frac{b_{2l} - b_{2l+1}}{2} \), \( \langle f, H_{0,0} \rangle = \frac{b_{2l+1} - b_{2l+1}}{2} \), ..., in general, for any \( l \in \mathbb{Z} \),
\[ \langle f, H_{0,l} \rangle = \frac{b_{2l} - b_{2l+1}}{2}. \]
Thus, \( |\langle f, H_{0,0}\rangle|^2 = |b_0|^2 \), \( |\langle f, H_{0,1}\rangle|^2 = |b_2|^2 \), ..., in general, for any \( l \in \mathbb{Z} \),
\[ |\langle f, H_{0,l}\rangle|^2 = |b_{2l}|^2. \]
Hence, for any $f \in V_1 \ominus V_0 = W_0$,

$$||f||^2_2 = \sum_{l \in \mathbb{Z}} |\langle f, H_{0,l} \rangle|^2.$$ 

□

We still need to get complete orthonormal systems for each $W_n$ and $V_n$. Recall by Lemma 1 of Chapter 5, for any $n \in \mathbb{Z}$, $f(x) \in V_n \iff f(2x) \in V_{n+1}$. We will see that $W_n$’s share the same property.

**Lemma 3.** Let $V_n$ and $W_n$ be defined above, then for any $n \in \mathbb{Z}$,

$$f(x) \in W_n \iff f(2x) \in W_{n+1}.$$ 

**Proof.** Fix any $n \in \mathbb{Z}$. We see that $f(x) \in W_n = V_{n+1} \ominus V_n \iff f(x) \in V_{n+1}, f(x) \perp V_n$.

By Lemma 1 of Chapter 5, certainly $f(x) \in V_{n+1} \iff f(2x) \in V_{n+2}$. We are going to show that $f(x) \perp V_n \iff f(2x) \perp V_{n+1}$. Indeed, note that for any $h \in L^2(\mathbb{R})$, $h(x) \in V_n \iff h(2x) \in V_{n+1}$, so any function in $V_{n+1}$ can be written as $h(2x)$ with some $h(x) \in V_n$. Now suppose $f(x) \perp V_n$, then

$$\langle f(2x), h(2x) \rangle = \int_{-\infty}^{\infty} f(2x)\overline{h(2x)}dx = \frac{1}{2} \int_{-\infty}^{\infty} f(u)\overline{h(u)}du = \frac{1}{2} \langle f, h \rangle = 0.$$ 

So $f(2x) \perp V_{n+1}$. The other direction is similar. Hence

$$f(x) \in W_n \iff f(2x) \in V_{n+2}, f(2x) \perp V_{n+1} \iff f(2x) \in W_{n+1} = V_{n+2} \ominus V_{n+1}.$$ 

□

If some subspaces in $L^2(\mathbb{R})$ have the above property, then their complete orthonormal systems are related in the following way.

**Lemma 4.** Let $K_1$ and $K_2$ be subspaces of $L^2(\mathbb{R})$. If $f(x) \in K_1 \iff f(2x) \in K_2$, then the following are equivalent:

(a) $\{f_n(x)\}_{n \in \mathbb{Z}}$ is a complete orthonormal system of $K_1$.

(b) $\{\sqrt{2}f_n(2x)\}_{n \in \mathbb{Z}}$ is a complete orthonormal system of $K_2$.

**Proof.** (a) $\implies$ (b) From the fact that $\{f_n(x)\}_{n \in \mathbb{Z}}$ is an orthonormal system, it can be checked that $\{\sqrt{2}f_n(2x)\}_{n \in \mathbb{Z}}$ is also an orthonormal system. Furthermore, The
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fact that \( \{f_n(x)\}_{n \in \mathbb{Z}} \subset \mathcal{K}_1 \) clearly implies that \( \{\sqrt{2}f_n(2x)\}_{n \in \mathbb{Z}} \subset \mathcal{K}_2 \). We will show that the orthonormal system \( \{\sqrt{2}f_n(2x)\}_{n \in \mathbb{Z}} \) is complete. By Lemma 4 of Chapter I, it is equivalent to show that for any \( h \in \mathcal{K}_2 \), we have

\[
||h||_2^2 = \sum_{n \in \mathbb{Z}} |\langle h, \sqrt{2}f_n(2x) \rangle|^2.
\]

Indeed, for any \( h \in \mathcal{K}_2 \), there is a \( f \in \mathcal{K}_1 \) such that \( h(x) = f(2x) \). Now,

\[
||h||_2^2 = \int_{-\infty}^{\infty} |h(x)|^2 \, dx = \int_{-\infty}^{\infty} |f(2x)|^2 \, dx = \frac{1}{2} \int_{-\infty}^{\infty} |f(u)|^2 \, du = \frac{1}{2} ||f||_2^2,
\]

Also

\[
\langle h, \sqrt{2}f_n(2x) \rangle = \int_{-\infty}^{\infty} h(x)\sqrt{2}f_n(2x) \, dx = \int_{-\infty}^{\infty} f(2x)\sqrt{2}f_n(2x) \, dx
\]

\[
= \frac{\sqrt{2}}{2} \int_{-\infty}^{\infty} f(u)f_n(u) \, du = \frac{1}{\sqrt{2}} \langle f, f_n \rangle.
\]

Note that \( \{f_n(x)\}_{n \in \mathbb{Z}} \) is complete in \( \mathcal{K}_1 \), so for such \( f \in \mathcal{K}_1 \), we have

\[
||f||_2^2 = \sum_{n \in \mathbb{Z}} |\langle f, f_n \rangle|^2.
\]

Now we see that

\[
||h||_2^2 = \frac{1}{2} ||f||_2^2 = \frac{1}{2} \sum_{n \in \mathbb{Z}} |\langle f, f_n \rangle|^2 = \sum_{n \in \mathbb{Z}} |\langle h, \sqrt{2}f_n(2x) \rangle|^2.
\]

\((b) \implies (a)\) can be similarly proved. We omit the details. \( \square \)

Now we are ready to find an complete orthonormal system for each subspace \( V_n \) and \( W_n \).

**Proposition 1.** Let \( \{V_n\}_{n \in \mathbb{Z}} \) be a sequence of subspaces of \( L^2(\mathbb{R}) \) such that \( V_n = \{ f \in L^2(\mathbb{R}) \mid \forall j \in \mathbb{Z}, f|_{[j, j+1]} \text{ is constant} \} \). Let \( W_n = V_{n+1} \ominus V_n \) for each \( n \in \mathbb{Z} \). Let \( H \) be the Haar function and

\[
\varphi(x) = \begin{cases} 
1 & 0 \leq x < 1 \\
0 & \text{otherwise}
\end{cases}
\]

then for for each \( n \in \mathbb{Z} \),

\((a)\) \( \{\sqrt{2}^n\varphi(2^nx - l) \mid l \in \mathbb{Z}\} \) is a complete orthonormal system of \( V_n \).

\((b)\) \( \{\sqrt{2}^nH(2^nx - l) \mid l \in \mathbb{Z}\} \) is a complete orthonormal system of \( W_n \).

It should be clear that Proposition 1 is an immediate consequence of the Lemmas and facts developed in this chapter. There is nothing much to prove. Finally, we have
Theorem 1. Let \( \{V_n\}_{n \in \mathbb{Z}} \) be a sequence of subspaces of \( L^2(\mathbb{R}) \) such that \( V_n = \{ f \in L^2(\mathbb{R}) : f|_{[\frac{2n}{2^n}, \frac{2n+1}{2^n}]} \text{ is constant} \} \). Let \( W_n = V_{n+1} \ominus V_n \) for each \( n \in \mathbb{Z} \).

Let \( H \) be the Haar function and
\[
\varphi(x) = \begin{cases} 
1 & 0 \leq x < 1 \\
0 & \text{otherwise}
\end{cases}
\]

Let \( H_{j,l}(x) = 2^j H(2^j x - l) \) and \( \varphi_{j,l}(x) = 2^j \varphi(2^j x - l) \) for each \( j, l \in \mathbb{Z} \). Then for any \( h \in L^2(\mathbb{R}) \) and for each \( j \in \mathbb{Z} \),
\[
P_{V_{j+1}} h = P_{V_j} h + P_{W_j} h.
\]

Specifically,
\[
\sum_{l \in \mathbb{Z}} \langle h, \varphi_{j+1,l} \rangle \varphi_{j+1,l} = \sum_{l \in \mathbb{Z}} \langle h, \varphi_{j,l} \rangle \varphi_{j,l} + \sum_{l \in \mathbb{Z}} \langle h, H_{j,l} \rangle \varphi_{j,l}.
\]

Proof. Note that \( V_{j+1} = V_j \oplus W_j \) for any \( j \in \mathbb{Z} \), so for any \( h \in L^2(\mathbb{R}) \), we can consider the orthogonal projection of \( P_{V_{j+1}} h \in V_{j+1} \) onto subspaces \( V_j \) and \( W_j \).

Clearly by Proposition 4 of Chapter 5, we have
\[
P_{V_{j+1}} h = P_{V_j} (P_{V_{j+1}} h) + P_{W_j} (P_{V_{j+1}} h) = P_{V_j} h + P_{W_j} h.
\]

Now use the complete orthonormal systems of \( V_n \)’s and \( W_n \)’s obtained in Proposition 1 above and Proposition 3 of Chapter 5, we can easily obtain the explicit form of \( P_{V_{j+1}} h = P_{V_j} h + P_{W_j} h \). □

Note that
\[
\varphi_{j+1,k} = \begin{cases} 
2^{\frac{1}{2j+1}} & x \in \left[ \frac{k}{2j+1}, \frac{k+1}{2j+1} \right) \\
0 & \text{otherwise}
\end{cases}
\]

\[
\varphi_{j,l} = \begin{cases} 
2^{\frac{l}{2j}} & x \in \left[ \frac{2l}{2j+1}, \frac{2l+2}{2j+1} \right) \\
0 & \text{otherwise}
\end{cases}
\]

and
\[
H_{j,l} = \begin{cases} 
2^{\frac{l}{2j}} & x \in \left[ \frac{2l}{2j+1}, \frac{2l+1}{2j+1} \right) \\
-2^{\frac{l}{2j}} & x \in \left[ \frac{2l+1}{2j+1}, \frac{2l+2}{2j+1} \right) \\
0 & \text{otherwise}
\end{cases}
\]

So
\[
H_{j,l} = \frac{\varphi_{j+1,2l} - \varphi_{j+1,2l+1}}{\sqrt{2}}.
\]
Thus
\[ \langle h, H_{j,l} \rangle = \frac{\langle h, \varphi_{j+1,2l} \rangle - \langle h, \varphi_{j+1,2l+1} \rangle}{\sqrt{2}}. \]

On the other hand, we also have
\[ \langle h, \phi_{j,l} \rangle = \frac{\langle h, \varphi_{j+1,2l} \rangle + \langle h, \varphi_{j+1,2l+1} \rangle}{\sqrt{2}}. \]

Finally, we obtain these so called **Haar Decomposition and Reconstruction Formulas.** These formulas allow one to efficiently find the orthogonal projections of functions onto subspaces \( V_n \) and \( W_n \). Once we take the orthogonal projection of any function \( f \) onto certain subspace \( V_j \) (for convenience, let us denote \( P_{V_j} f \) as \( f_j \)), then by using Decomposition formulas, it is very easy to further find its orthogonal projection onto \( V_{j-1} \) and \( W_{j-1} \). Denote \( P_{W_j} f \) as \( W_j \), then \( f_j = f_{j-1} + w_{j-1} \). And this process can be repeated so that we can find orthogonal projections of \( f_{j-1} \) onto \( V_{j-2} \) and \( W_{j-2} \), hence \( f_j = f_{j-2} + w_{j-1} + w_{j-1} \). Then this process can be repeated once again to further decompose the original function. On the other hand, if we have the orthogonal projections of a function onto various \( V_n \) and \( W_n \), using Reconstruction formulas we can recover the function very fast. This actually is one of the reasons why wavelets are favored by people working on signal processing. We will explain in class how this works using some signal processing example.