FARGUES-RAPOPORT CONJECTURE FOR p-ADIC PERIOD DOMAINS IN THE NON-BASIC CASE

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ABSTRACT. We prove the Fargues-Rapoport conjecture for p-adic period domains in the non-basic case with minuscule cocharacter. More precisely, we give a group theoretical criterion for the cases when the admissible locus and weakly admissible locus coincide.

CONTENTS

INTRODUCTION

Let F be a finite extension of $\mathbb{Q}_p$, and let $\breve{F}$ be the p-adic completion of the maximal unramified extension of F with Frobenius σ. We consider the flag variety $\mathcal{F}(G, \mu)$ associated to a pair (G, μ), where G is a reductive group over F, and μ a minuscule cocharacter of G. It's a projective variety over the local reflex field $E = E(G, \{\mu\})$, the field of definition of the geometric conjugacy class $\{\mu\}$ of μ. We still denote by $\mathcal{F}(G, \mu)$ the associated adic space over $\breve{E}$. For any $b \in G(\breve{F})$ satisfying certain conditions with respective to μ (cf. 2.2), we are interested in two open adic subspaces

$$\mathcal{F}(G, \mu, b)^a \subseteq \mathcal{F}(G, \mu, b)^{wa} \subseteq \mathcal{F}(G, \mu)$$

inside $\mathcal{F}(G, \mu)$, where $\mathcal{F}(G, \mu, b)^{wa}$ is the weakly admissible locus, defined by Rapoport and Zink ([30]) by removing a profinite number of Schubert varieties inside $\mathcal{F}(G, \mu)$ which contradicts the weakly admissibility condition defined by Fontaine (cf. section 2.2), and where $\mathcal{F}(G, \mu, b)^a$ is the admissible locus, or called the p-adic period domain. It's much more mysterious. The existence of the admissible locus has been conjectured by Rapoport and Zink. It's characterized by the properties that it has the same classical points as the weakly admissible locus (see [8], in which way weakly admissible locus is also considered as its algebraic approximation) and there exists a local system with G-structures on it which interpret the crystalline representations on all classical points. When the triple (G, μ, b) is of PEL type, the admissible locus $\mathcal{F}(G, \mu, b)^a$ is the image of the p-adic period mapping from the Rapoport-Zink space associated to (G, μ, b) to the flag variety $\mathcal{F}(G, \mu)$. There is also direct construction of $\mathcal{F}(G, \mu, b)^a$ in special cases by Hartl [20] and Faltings [11]. In the most general case, the existence of the admissible locus equipped with the étale local system is known due to the work of Fargues-Fontaine ([16]), Kedlaya-Liu ([21]) and Scholze ([35]). In fact, the admissible locus is defined by semi-stable conditions on the modification of the G-bundle associated to b of type μ on the Fargues-Fontaine curve. Moreover, it can also be considered as the image of the p-adic period mapping from the local Shimura variety associated to (G, μ, b) to the flag variety $\mathcal{F}(G, \mu)$ ([34], [28]).

We want to understand the structure of the p-adic period domains. As the structure of its approximation, the weakly admissible locus, is well known, it's natural to ask when the p-adic period domain coincides with the weakly admissible locus. Hartl classified all the cases

for $G = \mathrm{GL}_n$ in [20]. For general group G, Rapoport and Fargues have conjectured a group theoretic criterion when b is basic, which has now become the following theorem.

Theorem (Fargues-Rapoport conjecture, [6], Theorem 6.1). *Suppose b is basic. The equality $\mathcal{F}(G, \mu, b)^{wa} = \mathcal{F}(G, \mu, b)^a$ holds if and only if (G, μ) is fully Hodge-Newton decomposable.*

Here the fully Hodge-Newton-decomposability condition is purely group theoretic. Görtz, He and Nie classified all the fully Hodge-Newton decomposable pairs and give more equivalent conditions of fully Hodge-Newton decomposability in their article [18]. When G is a general linear group, the triple (G, μ, b) is Hodge-Newton-indecomposable if all the breakpoints of the Newton polygon defined by b do not touch the Hodge polygon defined by μ. Otherwise, the triple is called Hodge-Newton-decomposable. Let $B(G)$ be the set of σ-conjugacy classes of $G(\breve{F})$. For $b \in G(\breve{F})$, let $[b] \in B(G)$ be the σ-conjugacy class of b. An element $[b] \in B(G)$ is called basic if the Newton polygon of b is central. Kottwitz defined a subset $B(G, \mu)$ in $B(G)$. When G is the general linear group, $[b] \in B(G, \mu)$ if and only if the Newton polygon of $[b]$ lies on or above the Hodge polygon of μ and they have the same endpoint. The pair (G, μ) is fully Hodge-Newton decomposable if for any non basic $[b']$ in the Kottwitz set $B(G, \mu)$, the triple (G, μ, b') is Hodge-Newton-decomposable. We refer to section 3 for the details of these notions.

The main result of this article is a generalized version of Fargues-Rapoport conjecture which works for any b. It is inspired by the Fargues-Rapoport conjecture for basic elements and Hartl's result for GL_n. For simplicity, we assume that G is quasi-split in the introduction. There is also a similar description for the non-quasi-split case (Theorem 4.3) in Section 4.

Theorem (Theorem 4.1). *Suppose M is the standard Levi subgroup of G such that $[b] \in B(M, \mu)$ and (M, b, μ) is Hodge-Newton-indecomposable. Then the equality $\mathcal{F}(G, \mu, b)^{wa} = \mathcal{F}(G, \mu, b)^a$ holds if and only if $(M, \{\mu\})$ is fully Hodge-Newton-decomposable and $[b]$ is basic in $B(M)$.*

The key ingredients of the proof of the main theorem are Proposition 3.6 which describes the relation of the (weakly) admissible locus for different groups G and its Levi subgroup M and the following proposition.

Proposition (Proposition. 4.5). *Suppose (G, μ, b) is Hodge-Newton-indecomposable. If b is not basic, then $\mathcal{F}(G, \mu, b)^a \neq \mathcal{F}(G, \mu, b)^{wa}$.*

Indeed, we know that on the flag variety there is a group action of $\tilde{J}_b$. We prove that the group action preserves the admissible locus but not the weakly admissible locus by producing a point which is weakly admissible but not admissible. Such a point exists in the $\tilde{J}_b$-orbit of a non weakly admissible point.

We briefly describe the structure of this article. In section 1, we review the basic facts about the Kottwitz set and G-bundles on the Fargues-Fontaine curve. In section 2, we review the reduction of G-bundles and introduce the weakly admissible locus and admissible locus in the flag variety in term of (weakly) semi-stable condition on the modification of G-bundles. In section 3, we review the Hodge-Newton-decomposability condition and prove Proposition 3.6 which is one of the main ingredients for the proof of the main theorems. In section 4 we prove Proposition 4.5 and the main theorems 4.1 and 4.3 by using the $\tilde{J}_b$-action on the flag variety. In section 5, we discuss the relation between the Newton strata and the weakly admissible locus. We introduce a conjecture predicting which Newton strata contain weakly admissible points in the basic case. We prove this conjecture in a very special case in Proposition 5.3.

NOTATIONS

We use the following notations:

- F is a finite degree extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$ and a uniformizer π_F.
- $\overline{F}$ is an algebraic closure of F and $\Gamma = \mathrm{Gal}(\overline{F}|F)$.
- $\breve{F} = \widehat{F^{un}}$ is the completion of the maximal unramified extension with Frobenius σ.

- G is a connected reductive group over F and H is a quasi-split inner form of G equipped with an inner twisting $G_{\breve{F}} \xrightarrow{\sim} H_{\breve{F}}$.
- $A \subseteq T \subseteq B$, where A is a maximal split torus, $T = Z_H(A)$ is the centralizer of A in T, and B is a Borel subgroup in H.
- $(X^*(T), \Phi, X_*(T), \Phi^\vee)$ is the absolute root datum with positive roots Φ^+ and simple roots Δ with respect to the choice of B.
- $W = N_H(T)/T$ is the absolute Weyl group of T in H, and w_0 is the longest length element in W.
- $(X^*(A), \Phi_0, X_*(A), \Phi_0^\vee)$ is the relative root datum with positive roots Φ_0^+ and simple (reduced) roots Δ_0.
- If M is a standard Levi subgroup in H we denote by Φ_M the corresponding roots or coroots showing up in $\mathrm{Lie}\, M$, and by W_M the Weyl group of M. If P is the standard parabolic subgroup of H with Levi component M, sometimes we also write W_P for W_M.

1. KOTTWITZ SET AND G-BUNDLES ON THE FARGUES-FONTAINE CURVE

In this section, we will recall the some basic facts about the G-bundles on the Fargues-Fontaine curve which will be the main tool for our study of p-adic period domains.

1.1. **The Fargues-Fontaine curve.** Let K be a perfectoid field over $\mathbb{F}_q$ with $\omega_K \in K$ satisfying $0 < |\omega_K| < 1$. Let

$$\mathcal{Y}_K = \mathrm{Spa}(W_{\mathcal{O}_F}(\mathcal{O}_K)) \backslash V(\pi_F[\omega_K])$$

be an adic space over F equipped with an automorphism φ induced from the Frobenius $K|\mathbb{F}_q$. The Fargues-Fontaine curve over F associated to K is the schema

$$X = X_K := \mathrm{Proj}(\bigoplus_{d \geq 0} B_K^{\varphi = \pi_F^d}),$$

where $B_K = H^0(\mathcal{Y}_K, \mathcal{O}_{\mathcal{Y}_K})$. The scheme X is a curve (which means it's a one dimensional noetherian regular scheme) over F([16] thm 6.5.2, 7.3.3).

If we replace K by an affinoid perfectoid space $S = \mathrm{Spa}(R, R^+)$ over $\mathbb{F}_q$, we can similiarly construct $\mathcal{Y}_S$ and X_S over F which is called the relative Fargues-Fontaine curve (cf. [21]).

1.2. G-bundles. From now on, suppose $K = C^\flat$ is the tilt of a complete algebraically closed field C over F. Then the curve X is equipped with a closed point ∞ with residue field $k(\infty) = C$. Let Bun_X be the category of vector bundles on X. The classification of vector bundles on X is well known due to the work of Fargues-Fontaine.

Theorem 1.1 ([16] theorem 8.2.10). *Every vector bundle on X is a direct sum of stable sub-vector bundles, and the isomorphism classes of stable vector bundles on X are parametrized by the slope in $\mathbb{Q}$.*

For $\lambda \in \mathbb{Q}$, let $\mathcal{O}(\lambda)$ be a stable vector bundle on X of slope $\lambda \in \mathbb{Q}$.

By [16], $\{\infty\} = V^+(t)$ with $t \in H^0(X, \mathcal{O}(1))$. Then

$$X \backslash \{\infty\} = \mathrm{Spec}(B_e) \text{ and } \widehat{X_\infty} = \mathrm{Spec}(B_{dR}^+)$$

where $B_e = B_K[\frac{1}{t}]^{\varphi=1}$ is a principal ideal domain, and B_{dR}^+ is a complete discrete valuation ring with residue field C. Let B_{dR} be the fraction field of B_{dR}^+. The following proposition tells us that a vector bundle on X is determined by its restrictions to $X \backslash \{\infty\}$ and $\widehat{X_\infty}$ with a gluing datum.

Proposition 1.2. *[16] Corollary 5.3.2]Let $\mathcal{C}$ be the category of triples (M_e, M_{dR}, u) where*

- *M_e is a free B_e-module of finite rank;*
- *M_{dR} is a free B_{dR}^+-module of finite rank;*
- *$u : M_e \otimes_{B_e} B_{dR} \xrightarrow{\sim} M_{dR} \otimes_{B_{dR}^+} B_{dR}$ is an isomorphism of B_{dR}-modules.*

Then there is an equivalence of categories

$$\mathrm{Bun}_X \xrightarrow{\sim} \mathcal{C}$$

$$\mathcal{E} \mapsto (\Gamma(X \backslash \{\infty\}, \mathcal{E}), \widehat{\mathcal{E}}_\infty, \mathrm{Id}).$$

Let $\mathrm{Isoc}_{\breve{F}|F}$ be the category of isocrystals relative to $\breve{F}|F$. By [16] thm 8.2.10, there is an essentially surjective functor

$$\mathcal{E}(-) : \mathrm{Isoc}_{\breve{F}|F} \to \mathrm{Bun}_X$$

where for any $(D, \varphi) \in \mathrm{Isoc}_{\breve{F}|F}$, the vector bundle $\mathcal{E}(D, \varphi)$ on X is associated to the graded $\bigoplus_{d \geq 0} B_K^{\varphi = \pi_F^d}$-module

$$\bigoplus_{d \geq 0} (D \otimes B_K)^{\varphi \otimes \varphi = \pi_F^d}.$$

In fact, it maps a simple isocrystal of slope λ to $\mathcal{O}(-\lambda)$.

To any $b \in G(\breve{F})$, we can associate an isocrystal with G-structures:

$$\mathcal{F}_b : \mathrm{Rep}\,G \longrightarrow \mathrm{Isoc}_{\breve{F}|F}$$
$$V \mapsto (V_{\breve{F}}, b\sigma).$$

Its isomorphism class only depends on the σ-conjugacy class $[b] \in B(G)$ of b, where $B(G)$ is the set of σ-conjugacy classes in $G(\breve{F})$. In this way, $B(G)$ parametrises the set of isomorphism classes of F-isocrystals with G-structure, cf. [29] Remarks 3.4 (i).

Recall that a G-bundle on X is a G-torsor on X which is locally trivial for the étale topology. Equivalently, a G-bundle on X can also be viewed as an exact functor $\mathrm{Rep}\,G \to \mathrm{Bun}_X$ where $\mathrm{Rep}\,G$ is the category of rational algebraic representations of G. The étale cohomology set $H^1_{\text{ét}}(X, G)$ classifies the isomorphism classes of G-bundles on X.

For $b \in G(\breve{F})$, we can associate to b a G-bundle on X.

$$\mathcal{E}_b = \mathcal{E}(-) \circ \mathcal{F}_b : \mathrm{Rep}\,G \xrightarrow{\mathcal{F}_b} \mathrm{Isoc}_{\breve{F}|F} \xrightarrow{\mathcal{E}(-)} \mathrm{Bun}_X.$$

By [14] theorem 5.1, there is a bijection of sets

$$B(G) \xrightarrow{\sim} H^1_{\text{ét}}(X, G)$$
$$[b] \mapsto [\mathcal{E}_b].$$

In this way, the set $B(G)$ also classifies G-bundles on X.

1.3. **Kottwitz set.** There are two invariants on the set $B(G)$, the Newton map and the Kottwitz map. For any $b \in G(\breve{F})$, there is a composed functor

$$\mathcal{F} : \mathrm{Rep}\,G \xrightarrow{\mathcal{F}_b} \mathrm{Isoc}_{\breve{F}|F} \longrightarrow \mathbb{Q} - \mathrm{grVect}_{\breve{F}}$$

where $\mathbb{Q} - \mathrm{grVect}_{\breve{F}}$ is the category of $\mathbb{Q}$-graded vector spaces over $\breve{F}$ and the second functor is given by the Dieudonné-Manin's classification of isocrystals which decomposes an isocrystal into isocline sub-isocrystals parametrized by $\mathbb{Q}$. We can attach to $\mathcal{F}$ a slope morphism

$$\nu_b : \mathbb{D}_{\breve{F}} = \mathrm{Aut}^{\otimes}(\omega) \to \mathrm{Aut}^{\otimes}(\omega \circ \mathcal{F}) = G_{\breve{F}},$$

where $\omega : \mathbb{Q} - \mathrm{grVect}_{\breve{F}} \to \mathrm{Vect}_{\breve{F}}$ is the natural forgetful functor and $\mathbb{D}$ is the pro-algebraic torus over $\breve{F}$ with $X^*(\mathbb{D}) = \mathbb{Q}$.

The conjugacy class of the slope morphism ν_b is defined over F and it only depends on the σ-conjugacy class of b. We thus obtain the Newton map

$$\nu : B(G) \longrightarrow \mathcal{N}(G)$$
$$[b] \mapsto [\nu_b],$$

where $\mathcal{N}(G) = \mathcal{N}(H) = X_*(A)_{\mathbb{Q}}^+$ is the Newton chamber. The σ-conjugacy class $[b] \in B(G)$ is called basic if ν_b is central. Denote by $B(G)_{basic}$ the subset of basic σ-conjugacy classes in $B(G)$.

The other invariant is the Kottwitz map ([23] 4.9, 7.5, [29] 1.15):

$$\kappa_G : B(G) \longrightarrow \pi_1(G)_{\Gamma}$$
$$[b] \mapsto \kappa_G([b]),$$

where $\pi_1(G) = \pi_1(H) = X_*(T)/\langle \Phi^{\vee} \rangle$ is the algebraic fundamental group of G, and $\pi_1(G)_{\Gamma}$ is the Galois coinvariants. For $G = \mathrm{GL}_n$, $\kappa_{\mathrm{GL}_n}([b]) = v_p(\det(b)) \in \mathbb{Z} = \pi_1(\mathrm{GL}_n)_{\Gamma}$ where v_p denotes the p-adic valuation on $\breve{F}$. For general G, the Kottwitz map is characterized by the unique natural transformation $\kappa_{(-)} : B(-) \to \pi_1(-)_{\Gamma}$ of set valued functors on the category of

connected reductive groups over F such that κ_{GL_n} is defined as above. Let $B(G)_{basic}$ be the subset of basic elements in $B(G)$, then the restriction of κ_G on $B(G)_{basic}$ induces a bijection

$$\kappa_G : B(G)_{basic} \xrightarrow{\sim} \pi_1(G)_\Gamma.$$

Moreover if $[b] \in B(G)_{basic}$ with $\kappa_G([b]) = \mu^\sharp \in \pi_1(G)_\Gamma$ with $\mu \in X_*(T)^+$, then $[\nu_b] = \mathrm{Av}_W \mathrm{Av}_\Gamma \mu$ where Av_W (resp. Av_Γ) denotes the W-average (resp. Γ-average).

The elements in $B(G)$ are determined by the Newton map and Kottwitz map. Namely, the map

(1.3.1) $$(\nu, \kappa_G) : B(G) \longrightarrow \mathcal{N}(G) \times \pi_1(G)_\Gamma$$

is injective ([23] 4.13).

Definition 1.3.   (1) *Let* $[b] \in B(G)$ *and* $\mu \in X_*(T)^+$, *we define the Kottwitz set*

$$B(G, \mu) := \{[b] \in B(G) | [\nu_b] \leq \mu^\diamond, \kappa_G(b) = \mu^\sharp\}$$

which is a finite subset in $B(G)$, *where*
- $\mu^\diamond := \mathrm{Av}_\Gamma(\mu) \in \mathcal{N}(G)$ *is the Galois average of* μ,
- $\mu^\sharp \in \pi_1(G)_\Gamma$ *is the image of* μ *via the natural quotient map* $X_*(T) \to \pi_1(G)_\Gamma$,
- *the order* $\leq$ *on* $\mathcal{N}(G)$ *is the usual order:* $\nu_1 \leq \nu_2$ *if and only if* $\nu_2 - \nu_1 \in \mathbb{Q}_{\geq 0} \Phi_0^+$.

(2) *We define a partial order on* $B(G, \mu)$: *For* $[b_1], [b_2] \in B(G, \mu)$, *we say* $[b_1] \leq [b_2]$ *if* $[\nu_{b_1}] \leq [\nu_{b_2}]$.

We will also need the following generalized Kottwitz set defined in [6].

Definition 1.4. *For* $\epsilon \in \pi_1(G)_\Gamma$ *and* $\delta \in X_*(A)_\mathbb{Q}^+$ *we set*

$$B(G, \epsilon, \delta) = \{[b] \in B(G) \mid \kappa_G(b) = \epsilon \text{ and } [\nu_b] \leq \delta\}.$$

Remark 1.5.   (1) *The partial order* $\leq$ *on* $B(G, \mu)$ *extends naturally to a partial order (which we still denote by* $\leq$ *) on* $B(G, \delta, \epsilon)$.
(2) $B(G, \mu) = B(G, \mu^\sharp, \mu^\diamond)$.

We have the following characterization of the generalized Kottwitz set.

Lemma 1.6. *Suppose* G *is quasi-split. Let* $[b] \in B(G)$ *and* $M = \mathrm{Cent}_G([\nu_b])$. *After choosing a suitable representative* $b \in [b]$, *we may assume* $b \in M(\breve{F})$. *Then*

(1) $[b] \in B(G, \mu)$ *if and only if* $\kappa_M(b) \preceq_M \mu^\sharp$ *in* $\pi_1(M)_\Gamma$ *where* $\mu^\sharp$ *denotes the image of* μ *in* $\pi_1(M)_\Gamma$ *by abuse of notation and the partial order in* $\pi_1(M)_\Gamma$ *is defined as follows: for* $y_1, y_2 \in \pi_1(M)_\Gamma$, *we write* $y_1 \preceq_M y_2$ *if and only* $y_2 - y_1$ *is a non-negative integral linear combination of images in* $\pi_1(M)_\Gamma$ *of coroots corresponding to the simple roots of* T *in* N *with* N *the unipotent radical of the standard parabolic subgroup of* G *with Levi component* M.

(2) $[b] \in B(G, \epsilon, \delta)$ *if and only if* $\kappa_G(b) = \epsilon$ *and* $\kappa_M(b) \preceq_M \delta^\sharp$ *in* $\pi_1(M)_{\Gamma,\mathbb{Q}}$, *where* $\delta^\sharp$ *denotes for the image of* δ *via the natural map* $X_*(A)_\mathbb{Q} \to \pi_1(M)_{\Gamma,\mathbb{Q}}$ *by abuse of notation and the partial order* $\preceq_M$ *is defined in a similar way.*

Proof. The proof is the same as [24] Proposition 4.10. We repeat the proof of (2) here, while the proof of (1) is similar. The necessity is obvious. For the sufficiency, the inequality $\kappa_M(b) \preceq_M \delta^\sharp$ in $\pi_1(M)_{\Gamma,\mathbb{Q}}$ implies

$$[\nu_b] = \mathrm{Av}_M \mathrm{Av}_\Gamma(\widetilde{\kappa_M(b)}) \leq \mathrm{Av}_M \mathrm{Av}_\Gamma \delta \leq \delta,$$

where $\widetilde{\kappa_M(b)}$ denotes a preimage of $\kappa_M(b)$ via the natural map $X_*(T) \to \pi_1(M)_\Gamma$ and the second inequality is due to the fact that δ is dominant. $\square$

The following lemma will be used in the proof of the main theorem.

Lemma 1.7. *Suppose* G *is quasi-split. Let* M *be a standard Levi subgroup of* G. *Let* $b \in M(\breve{F})$ *such that* $[\nu_b]$ *is* G-*dominant.*

(1) *Then the natural map* $\pi_1(M)_{\Gamma,tor} \to \pi_1(G)_{\Gamma,tor}$ *is injective.*
(2) *Suppose* $[b] \in B(G, \mu)$ *with* $\mu \in X_*(T)^+$, *then there exists* $\mu' \in X_*(T)$ *such that* $[b]_M \in B(M, \mu')$ *and* $(\mu')_{G-dom} = \mu$.

(3) *Suppose $[b] \in B(G, \mu^\sharp + \epsilon, \mu^\diamond)$ with $\epsilon \in \pi_1(G)_{\Gamma, tor}$ and $\mu \in X_*(T)^+$, then*

$$\epsilon \in \mathrm{Im}(\pi_1(M)_{\Gamma, tor} \to \pi_1(G)_{\Gamma, tor})$$

and there exists $w \in W$ such that $[b] \in B(M, (w\mu)^{\sharp, M} + \epsilon, (w\mu)^{\diamond, M})$, here we view ϵ as an element in $\pi_1(M)_{\Gamma, tor}$.

Proof. For (1), the map $\pi_1(M)_{\Gamma, tor} \to \pi_1(G)_{\Gamma, tor}$ can be identified with the map $H^1(F, M) \to H^1(F, G)$ which is injective(cf. [36] Exercise 1 in III §2.1).

For (2), the lemma is proved when G is unramified by [31] Lemma 8.1 (2) when G is unramified. For the general case, we want to reduce to the unramified case. Without loss of generality, we may assume G is adjoint and simple by [23] 6.5. Moreover, after replacing M by a smaller Levi subgroup, we may assume $[b]_M \in B(M)_{basic}$. Then by Lemma 1.6,

$$[b] \in B(G, \mu) \Leftrightarrow \kappa_M(b) \preceq_M \mu^\sharp \text{ in } \pi_1(M)_\Gamma.$$

We want to show that there exists $\mu' \in X_*(T)$ which is conjugate to μ such that $\kappa_M(b) = (\mu')^\sharp$ in $\pi_1(M)_\Gamma$.

Let $Q := \mathrm{Ker}(\pi_1(M) \to \pi_1(G))$.
Claim: $Q_\Gamma = \mathrm{Ker}(\pi_1(M)_\Gamma \to \pi_1(G)_\Gamma)$.

Let $A := \mathrm{Ker}(\pi_1(M)_\Gamma \to \pi_1(G)_\Gamma)$ which is torsion free by the proof of [6] Lemma 4.11. As the functor $(-)_\Gamma$ is right exact, there exists a natural surjection $Q_\Gamma \to A$. We need to show it's injective. Therefore it suffices to show $\mathrm{rank}_\mathbb{Q} A_\mathbb{Q} = \mathrm{rank}_\mathbb{Q} Q_{\Gamma, \mathbb{Q}}$. Since the functor $(-)_{\Gamma, \mathbb{Q}}$ is canonically isomorphic to the functor $(-)_\mathbb{Q}^\Gamma$, we have

$$A_\mathbb{Q} \simeq \mathrm{Ker}(\pi_1(M)_\mathbb{Q}^\Gamma \to \pi_1(G)_\mathbb{Q}^\Gamma) = Q_\mathbb{Q}^\Gamma.$$

The Claim follows.

By the Claim, $\mu - \kappa_M(b) \in Q_\Gamma$. We need to show there exists $\mu' \in X_*(T)$ which is conjugate to μ such that

$$\mu - \kappa_M(b) = \mu - \mu' \text{ in } Q_\Gamma.$$

It's a question only about root system with Galois action. Indeed, by Galois decent, we can construct an adjoint unramified group $\tilde{G}$ over F with $\tilde{T} \subset \tilde{B} \subset \tilde{G}$ over F where $\tilde{T}$ is a maximal torus and $\tilde{B}$ is a Borel subgroup such that

- $X^*(T) \simeq X^*(\tilde{T})$ and via this identification $\Delta_G = \Delta_{\tilde{G}}$;
- Δ_G and $\Delta_{\tilde{G}}$ have the same Galois orbits.

Then the absolute Weyl group of (G, T) and $(\tilde{G}, \tilde{T})$ are isomorphic. Let $\tilde{M}$ be the standard Levi subgroup of $\tilde{G}$ such that $\Delta_{\tilde{M}} = \Delta_M$. The ismorphism between the character groups induces an identification $\pi_1(\tilde{M})_\Gamma = \pi_1(M)_\Gamma$. Let $\tilde{\mu} \in X_*(\tilde{T})$ be the cocharacter corresponding to μ via the idendtification $X_*(T) \simeq X_*(\tilde{T})$. Let $[\tilde{b}] \in B(\tilde{M})_{basic}$ such that $\kappa_{\tilde{M}}(\tilde{b}) \in \pi_1(\tilde{M})_\Gamma$ maps to $\kappa_M(b) \in \pi_1(M)_\Gamma$ via the identication $\pi_1(\tilde{M})_\Gamma = \pi_1(M)_\Gamma$. As $\tilde{G}$ is unramified, we can find $\tilde{\mu}'$ for $[\tilde{b}] \in B(\tilde{G}, \tilde{\mu})$. Then μ' is the cocharcter of G corresponding to $\tilde{\mu}'$.

For (3), as before, we may assume $[b] \in B(M)_{basic}$. Then $[\nu_b] \leq \mu^\diamond$ implies that

$$\kappa_M(b) \preceq_M \mu^{\sharp, M} \text{ in } \pi_1(M)_{\Gamma, \mathbb{Q}}.$$

Hence there exists $\epsilon' \in \pi_1(M)_{\Gamma, tor}$ such that

$$\kappa_M(b) - \epsilon' \preceq_M \mu^{\sharp, M} \text{ in } \pi_1(M)_\Gamma.$$

Let $[b'] \in B(M)_{basic}$ such that $\kappa_M(b') = \kappa_M(b) - \epsilon'$. Then $[b'] \in B(G, \mu)$. By (2), there exists $w \in W$ such that $[b'] \in B(M, w\mu)$. Therefore $[\nu_b]_M = [\nu_{b'}]_M = (w\mu)^{\diamond, M}$ and

$$\mu^\sharp = \kappa_G(b') = \kappa_G(b) - \epsilon' = \mu^\sharp + \epsilon - \epsilon' \text{ in } \pi_1(G)_\Gamma,$$

where ϵ' is considered to be an element in $\pi_1(G)_{\Gamma, tor}$ via the natural map in (1). Hence $\epsilon = \epsilon'$ and

$$\kappa_M(b) = \kappa_M(b') + \epsilon = (w\mu)^\sharp + \epsilon \text{ in } \pi_1(M)_\Gamma.$$

$\square$

1.4. Classification of G-bundles in terms of φ-modules over $\bar{B}$. Let

$$B^{b,+} := W_{\mathcal{O}_F}(\mathcal{O}_K)[\frac{1}{\pi_F}],$$

$$\bar{B} := (B^{b,+}/[\omega_K])_{red}.$$

Here $\bar{B}$ is a local F-algebra with residue field $W_{\mathcal{O}_F}(k_K)_{\mathbb{Q}}$.

The Frobenius on $\mathcal{O}_K$ induces an automorphism φ on $B^{b,+}$ and on $\bar{B}$.

Let $\varphi-\mathrm{Mod}_{\bar{B}}$ (resp. $\varphi-\mathrm{Mod}_{W_{\mathcal{O}_F}(k_K)_{\mathbb{Q}}}$) be the category of free $\bar{B}$-modules (resp. $W_{\mathcal{O}_F}(k_K)_{\mathbb{Q}}$-vector spaces) of finite rank equipped with a semi-linear isomorphism.

Theorem 1.8 ([16] Theorem 11.1.7 and 11.1.9). *There is an equivalence of additive tensor categories:*

$$\mathrm{Bun}_X \xrightarrow{\sim} \varphi - \mathrm{Mod}_{\bar{B}}.$$

For $(M, \varphi) \in \varphi-\mathrm{Mod}_{\bar{B}}$, the Harder-Narasimham filtration of the corresponding vector bundle gives a $\mathbb{Q}$-filtration $(M^{\geq\lambda})_{\lambda\in\mathbb{Q}}$ of M which is called the Harder-Narasimham filtration of M (cf. [14] 5.4.1).

For any $\beta \in G(\bar{B})$, we define

$$\mathcal{E}_\beta : \mathrm{Rep}G \longrightarrow \varphi - \mathrm{Mod}_{\bar{B}} \xrightarrow{\sim} \mathrm{Bun}_X$$

$$(V, \rho) \mapsto (V \otimes_F \bar{B}, \rho(\beta)\varphi)$$

Proposition 1.9 ([14] Proposition 5.11). *The functor $\beta \mapsto \mathcal{E}_\beta$ induces a bijection between the set of φ-conjugacy classes in $G(\bar{B})$ and the set of isomorphism classes of G-bundles on X.*

We also define a functor $\mathrm{red}_{\bar{B},\breve{F}}$ as composition of two functors:

$$\mathrm{red}_{\bar{B},\breve{F}} : \varphi - \mathrm{Mod}_{\bar{B}} \xrightarrow{\otimes_{\bar{B}} W_{\mathcal{O}_F}(k_K)_{\mathbb{Q}}} \varphi - \mathrm{Mod}_{W_{\mathcal{O}_F}(k_K)_{\mathbb{Q}}} \xrightarrow{\sim} \mathrm{Isoc}_{\breve{F}|F}$$

where the second functor is a quasi-inverse of the functor

$$(-) \otimes_{\breve{F}} W_{\mathcal{O}_F}(k_K)_{\mathbb{Q}} : \mathrm{Isoc}_{\breve{F}|F} \xrightarrow{\sim} \varphi - \mathrm{Mod}_{W_{\mathcal{O}_F}(k_K)_{\mathbb{Q}}}$$

which is an equivalence of categories due to Dieudonné-Manin's theorem of classification of isocrystals.

1.5. The automorphism group $\tilde{J}_b$.

For $[b] \in B(G)$, let $\tilde{J}_b = \underline{\mathrm{Aut}}(\mathcal{E}_b)$ be the pro-étale sheaf of automorphisms of $\mathcal{E}_b$ on the category of affinoid perfectoid spaces $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ over $\overline{\mathbb{F}}_q$. More precisely, for any affinoid perfectoid space S over $\overline{\mathbb{F}}_q$, one has $\tilde{J}_b(S) = \mathrm{Aut}(\mathcal{E}_{b|X_S})$.

In this subsection, we review the structure of the group $\tilde{J}_b(K)$ studied in [14] section 5.4.2. Suppose $\mathcal{E}_b$ corresponds to the φ-conjugay class of $\beta \in G(\bar{B})$ as in Proposition 1.9. Then

$$\tilde{J}_b(K) \simeq \{g \in G(\bar{B}) | g\beta = \beta\varphi(g)\}.$$

We will identify these two groups via this isomorphism. In order to study the structure of $\tilde{J}_b(K)$, we need to use a parabolic subgroup of $G \otimes \bar{B}$ that contains $\tilde{J}_b(K)$.

Consider the functor

$$\mathrm{Rep}G \longrightarrow \varphi - \mathrm{Mod}_{\bar{B}} \longrightarrow \mathbb{Q} - \text{filtered } \bar{B} - \text{modules}$$

$$(V, \rho) \mapsto (V \otimes_F \bar{B}, \rho(\beta)\varphi)$$

where the second functor is given by the Harder-Narasimham filtration. By [39] theorem 4.40, this functor corresponds to a parabolic subgroup $\mathcal{P} \subset G \otimes_F \bar{B}$ with $\tilde{J}_b(K) \subset \mathcal{P}(\bar{B})$. The structure of $\mathcal{P}$ is well understood. Let

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$$

be the adjoint representation with $\mathfrak{g} := \mathrm{Lie}G$. Then $(\mathfrak{g} \otimes_F \bar{B}, \mathrm{Ad}(\beta)\varphi)$ is the φ-module over $\bar{B}$ corresponds to the vector bundle $\mathrm{Ad}(\mathcal{E}_\beta) := \mathcal{E}_\beta \times_{G,\mathrm{Ad}} \mathfrak{g}$. Hence it has the Harder-Narasimham filtration $(\mathfrak{g}_{\bar{B}}^{\geq\lambda})_{\lambda\in\mathbb{Q}}$. In particular, for $\lambda \neq 0$, the dimension of $\mathrm{gr}^\lambda\mathfrak{g}_{\bar{B}}$ equals to the number of roots $\alpha \in \Phi$ such that $\langle\alpha,\nu_b\rangle = \lambda$. Then

$$\mathcal{P} = \{g \in G_{\bar{B}} | \mathrm{Ad}(g)(\mathfrak{g}_{\bar{B}}^{\geq\bullet}) = \mathfrak{g}_{\bar{B}}^{\geq\bullet}\}$$

$$\mathrm{Lie}\mathcal{P} = \mathfrak{g}_{\bar{B}}^{\geq 0}$$

Moreover, the parabolic subgroup $\mathcal{P}$ is filtered by $(\mathcal{P}^{\geq \lambda})_{\lambda \in \mathbb{Q}_{\geq 0}}$ such that

$$\mathcal{P}^{>0} = R_u \mathcal{P};$$

$$\forall \lambda > 0, \mathcal{P}^{\geq \lambda}/\mathcal{P}^{>\lambda} \xrightarrow{\sim} \mathrm{gr}^{\lambda} \mathfrak{g}_{\bar{B}} \otimes \mathbb{G}_a;$$

$$\mathcal{P}^{\geq \lambda} = \{g \in G_{\bar{B}} | (\mathrm{Ad}(g) - \mathrm{Id})(\mathfrak{g}_{\bar{B}}^{> \bullet}) = \mathfrak{g}_{\bar{B}}^{> \bullet + \lambda}\}.$$

Let $\tilde{J}_b^{\geq \lambda}(K) = \tilde{J}_b(K) \cap \mathcal{P}^{\geq \lambda}(\bar{B})$ for all $\lambda \in \mathbb{Q}_{\geq 0}$, then we can understand the graded pieces:

$$\tilde{J}_b(K)/\tilde{J}_b^{>0}(K) \simeq J_b = \{g \in G(\breve{F}) | b\sigma(g) = gb\};$$

$$\forall \lambda > 0, \tilde{J}_b^{\geq \lambda}(K)/\tilde{J}_b^{>\lambda}(K) \simeq (\mathrm{gr}^{\lambda} \mathfrak{g}_{\bar{B}})^{\mathrm{Ad}(\beta)\varphi = \mathrm{Id}}$$

where $(\mathrm{gr}^{\lambda} \mathfrak{g}_{\bar{B}})^{\mathrm{Ad}(\beta)\varphi = \mathrm{Id}}$ is $\frac{\dim \mathrm{gr}^{\lambda} \mathfrak{g}_{\bar{B}}}{h}$ copies of $H^0(X, \mathcal{O}_X(\lambda))$ if $\lambda = \frac{d}{h}$ with $(d, h) = 1$. In particular $\tilde{J}_b^{\geq \lambda}(K) \not\supseteq \tilde{J}_b^{>\lambda}(K)$ if there exists $\alpha \in \Phi$ such that $\langle \alpha, \nu_b \rangle = \lambda > 0$.

1.6. Modifications of a G-bundle on X.

Definition 1.10. *Let $\mathcal{E}$ be a G-bundle on X. A modification of G-bundles of $\mathcal{E}$ (on ∞) is a pair $(\mathcal{E}', u)$, where $\mathcal{E}'$ is a G-bundles on X and*

$$u : \mathcal{E}|_{X \setminus \{\infty\}} \xrightarrow{\sim} \mathcal{E}'|_{X \setminus \{\infty\}}$$

is an isomorphisme of G-bundles on $X \setminus \{\infty\}$. Two modifications $(\mathcal{E}', u)$ and $(\tilde{\mathcal{E}}', \tilde{u})$ of $\mathcal{E}$ are said to be equivalent if there exists an isomorphism $f : \mathcal{E}' \xrightarrow{\sim} \tilde{\mathcal{E}}'$ such that $\tilde{u} = f|_{X \setminus \{\infty\}} \circ u$.

Consider the B_{dR}-affine Grassmannian $\mathrm{Gr}_G^{B_{dR}}$ attached to G (cf. [34]). We only need its C-points

$$\mathrm{Gr}_G^{dR}(C) := G(B_{dR})/G(B_{dR}^+).$$

For any $b \in B(G)$, let $\mathcal{E}_b$ be the associated G-bundle on X. For any $x \in \mathrm{Gr}_G^{dR}(C)$, we can construct a modification $\mathcal{E}_{b,x}$ of $\mathcal{E}_b$ à la Beauville-Laszlo given by gluing $\mathcal{E}_{b|X \setminus \{\infty\}}$ and the trivial bundle on $\mathrm{Spec}(B_{dR}^+)$ via the gluing datum given by x (cf. [5] Theorem 3.4.5 and [13] 4.2, [12] Proposition 3.20). Moreover, by [12] Proposition 3.20, there is a bijection

$$(1.6.1) \qquad \mathrm{Gr}_G^{dR}(C) \xrightarrow{\sim} \{\text{equivalent classes of modifications of } \mathcal{E}_b\}$$
$$x \longmapsto \text{equivalent class of } (\mathcal{E}_{b,x}, \mathrm{Id})$$

For $\mu \in X_*(T)^+$, the corresponding affine Schubert cell is

$$\mathrm{Gr}_{G,\mu}^{B_{dR}}(C) = G(B_{dR}^+)\mu(t)^{-1}G(B_{dR}^+)/G(B_{dR}^+) \subset \mathrm{Gr}_G^{B_{dR}}(C).$$

Definition 1.11. *A modification of $\mathcal{E}_b$ is of type μ if its equivalent class falls in the affine Schubert cell $\mathrm{Gr}_{G,\mu}^{B_{dR}}(C)$ via (1.6.1).*

The natural action of $\tilde{J}_b(K) = \mathrm{Aut}(\mathcal{E}_b)$ on the set of modifications of $\mathcal{E}_b$ induces via (1.6.1) an action of $\tilde{J}_b(K)$ on $\mathrm{Gr}_G^{B_{dR}}(C)$.

Let $\widehat{\mathcal{E}}_{b\infty}$ be the local completion of $\mathcal{E}_b$ at ∞. It is canonically trivialized. Hence there is a natural morphism

$$\alpha_{b,G} : \tilde{J}_b(K) = \mathrm{Aut}(\mathcal{E}_b) \to \mathrm{Aut}(\widehat{\mathcal{E}}_{b\infty}) = G(\mathrm{B}_{dR}^+)$$

The action of $\tilde{J}_b(K)$ on $\mathrm{Gr}_G^{B_{dR}}(C)$ is given by left multiplication via $\alpha_{b,G}$.

Lemma 1.12. *Let $\gamma \in \tilde{J}_b(K)$. For any $x \in \mathrm{Gr}_G^{B_{dR}}(C)$, the automorphism $\gamma : \mathcal{E}_b \xrightarrow{\sim} \mathcal{E}_b$ induces an automorphism*

$$\tilde{\gamma} : \mathcal{E}_{b,x} \xrightarrow{\sim} \mathcal{E}_{b,\gamma(x)}$$

such that the following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{E}_{b|X \setminus \{\infty\}} & \xrightarrow{\gamma} & \mathcal{E}_{b|X \setminus \{\infty\}} \\
\mathrm{Id} \downarrow & & \downarrow \mathrm{Id} \\
\mathcal{E}_{b,x|X \setminus \{\infty\}} & \xrightarrow{\tilde{\gamma}} & \mathcal{E}_{b,\gamma(x)|X \setminus \{\infty\}}
\end{array}$$

Proof. By Tannakian formalism, it suffices to deal with the case when $G = \mathrm{GL}_n$. Then G-bundles on X is the same thing as vector bundles of rank n. Suppose $\mathcal{E}_b$ corresponds to the triple (M_e, M_{dR}, u) as in Proposition 1.2, then $x \in \mathrm{Gr}_G^{B_{dR}}(C)$ corresponds to a B_{dR}^+-lattice M_x in $M_{dR} \otimes_{B_{dR}^+} B_{dR}$ and $\mathcal{E}_{b,x}$ corresponds to the triple (M_e, M_x, u). The automorphism $\gamma : \mathcal{E}_b \xrightarrow{\sim} \mathcal{E}_b$ corresponds to a pair (γ_e, γ_{dR}) of automorphisms compatible with u, where

$$\gamma_e : M_e \xrightarrow{\sim} M_e, \gamma_{dR} : M_{dR} \xrightarrow{\sim} M_{dR}.$$

Then $\mathcal{E}_{b,\gamma(x)}$ corresponds to the triple $(M_e, \gamma_{dR}(M_x), u)$ where $\gamma_{dR}(M_x)$ is the image of M_x via $\gamma_{dR} \otimes_{B_{dR}^+} B_{dR} : M_{dR} \otimes_{B_{dR}^+} B_{dR} \xrightarrow{\sim} M_{dR} \otimes_{B_{dR}^+} B_{dR}$. We define $\tilde{\gamma} : \mathcal{E}_{b,x} \xrightarrow{\sim} \mathcal{E}_{b,\gamma(x)}$ to be the automorphism corresponds to

$$\gamma_e : M_e \xrightarrow{\sim} M_e, \gamma_{dR} : M_x \xrightarrow{\sim} \gamma_{dR}(M_x).$$

The commutativity of the diagram can be verified directly. $\qquad\square$

We have the *Bialynicki-Birula map* (cf. [5] Propostion 3.4.3)

$$\pi_{G,\mu} : \mathrm{Gr}_{G,\mu}^{B_{dR}}(C) \longrightarrow \mathcal{F}(G,\mu)(C).$$

From now on, suppose μ is minuscule. In this case, the Bialynicki-Birula map $\pi_{G,\mu}$ is an isomorphism. For $x \in \mathcal{F}(G,\mu)(C)$, we denote by $\mathcal{E}_{b,x}$ the modifcation $\mathcal{E}_{b,\pi_{G,\mu}^{-1}(x)}$ of $\mathcal{E}_b$ of type μ.

When $[b] \in B(G)$ is basic, the isomorphism classes of the modifications of $\mathcal{E}_b$ can be classified as follows.

Proposition 1.13 ([28] A.10, [6] Prop. 5.2)**.** *Let $[b] \in B(G)$ be basic. Let*

$$B(G, \kappa_G(b) - \mu^\sharp, \nu_b \mu^{-1}) := B(G, \kappa_G(b) - \mu^\sharp, \nu_b(w_0\mu^{-1})^\diamond) \subseteq B(G).$$

The map $[b'] \mapsto [\mathcal{E}_{b'}]$ gives a bijection

$$B(G, \kappa_G(b) - \mu^\sharp, \nu_b \mu^{-1}) \simeq \{\mathcal{E}_{b,x} |\ x \in \mathcal{F}(G,\mu)(C)\}/ \sim .$$

The action of $\tilde{J}_b(K)$ on $\mathrm{Gr}_{G,\mu}^{B_{dR}}(C)$ defined by the multiplication on the left via the morphism $\alpha_{b,G}$ induces an action of $\tilde{J}_b(K)$ on $\mathcal{F}(G,\mu)(C)$ via the Bialynicki-Birula map $\pi_{G,\mu}$. For any $\gamma \in \tilde{J}_b(K)$, we have an automorphism (still denoted by)

$$\tilde{\gamma} : \mathcal{E}_{b,x} \xrightarrow{\sim} \mathcal{E}_{b,\gamma(x)}$$

of G-bundles for any $x \in \mathcal{F}(G,\mu)(C)$.

2. Admissible locus and weakly admissible locus

2.1. Reductions of G-bundles.

Definition 2.1.   (1) *Let $H \subseteq G$ be a closed subgroup of G. Suppose $\mathcal{E}$ is a G-bundle on X. A reduction of $\mathcal{E}$ to H is a pair $(\mathcal{E}_H, \iota)$ where H-bundle and $\iota : \mathcal{E}_H \times_H G \xrightarrow{\sim} \mathcal{E}$ is an isomorphism of G-bundles. We will also write $\mathcal{E}_H$ for such a reduction if we don't need to emphasis ι.*
     (2) *Two reductions $(\mathcal{E}_H, \iota)$ and $(\mathcal{E}'_H, \iota')$ of $\mathcal{E}$ to H are called equivalent if there exists an isomorphism $u : \mathcal{E}_H \xrightarrow{\sim} \mathcal{E}'_H$ such that $\iota = \iota' \circ (u \times_H G)$.*

Remark 2.2. *The equivalence classes of redcutions of $\mathcal{E}$ to H is in bijection to the sections of the fibration $H\backslash\mathcal{E} \to X$.*

We will assume G quasi-split in the rest of this subsection.

Definition 2.3. *Let $b \in G(\breve{F})$. For a Levi subgroup M of G, a reduction of b to M is a pair (b_M, g) with $b_M \in M(\breve{F})$ and $g \in G(\breve{F})$ such that $b = g b_M \sigma(g)^{-1}$. We also write b_M for such a reduction if we don't need to emphasize g. Two reductions (b_M, g) and (b'_M, g') of b to M are equivalent if and only if there exists $h \in M(\breve{F})$ such that $(b'_M, g') = (h b_M \sigma(h)^{-1}, gh^{-1})$. Similarly, we can define the same notion for parabolic subgroups.*

There is a natural injective map

{equivalent classes of reductions of b to M} $\longrightarrow$ {equivalent classes of reductions of $\mathcal{E}_b$ to M}.

This map is in general not surjective.

Example 2.4. *Let $G = \mathrm{GL}_5$ with Levi subgroup $M = \mathrm{GL}_3 \times \mathrm{GL}_2$. Let $b \in G(\breve{F})$ with New-ton slopes $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2})$. Then there exists a unique equivalent class of reductions of b to M. However, as the decomposition of $\mathcal{E}_b = \mathcal{O}(-\frac{1}{3}) \oplus \mathcal{O}(-\frac{1}{2})$ on semi-stable vector bundles is not canonical, there exist infinite equivalent classes of reductions of $\mathcal{E}_b$ to M.*

The following lemma will be used frequently in the sequel.

Lemma 2.5. *[6] Lemma 2.4] Let $\mathcal{E}$ and $\mathcal{E}'$ be two G-bundles on X with a modification $\mathcal{E}|_{X \setminus \{\infty\}} \xrightarrow{\sim} \mathcal{E}'|_{X \setminus \{\infty\}}$. Then for any parabolic subgroup P of G, we have a bijection*

$$\{Reductions \ of \ \mathcal{E} \ to \ P\} \longrightarrow \{Reductions \ of \ \mathcal{E}' \ to \ P\}.$$

Let $\mathcal{E}$ be a G-bundle on X, by [14] 5.1, there exists the canonical reduction $\mathcal{E}_P$ of $\mathcal{E}$ to a unique standard parabolic subgroup P of G such that

- the assoicated M-bundle $\mathcal{E}_P \times_P M$ is semi-stable, where M is the Levi component of P,
- for any $\chi \in X^*(P/Z_G) \setminus \{0\} \cap \mathbb{N}\Delta_G$, we have $\deg_{\chi_*}\mathcal{E}_P > 0$.

Using the Harder-Narasimham reduction $\mathcal{E}_P$, we can define the slope

$$\nu_{\mathcal{E}} \in X_*(A)_{\mathbb{Q}}^+$$

of $\mathcal{E}$ by the Galois invariant morphism $X^*(P) \to \mathbb{Z}$ which maps $\chi \in X^*(P)$ to $\deg_{\chi_*}\mathcal{E}_P$ combined with the inclusion

$$\mathrm{Hom}_{\mathbb{Z}}(X^*(P), \mathbb{Z}) = X_*(M^{ab}) \subset X_*(A_M)_{\mathbb{Q}} \subseteq X_*(A)_{\mathbb{Q}},$$

where M^{ab} is the cocenter of M and $A_M \subseteq A$ is a maximal split central torus of M.

Proposition 2.6 ([16] Proposition 6.6, [5] Lemma 3.5.5). *Let $[b] \in B(G)$ and $x \in \mathrm{Gr}_{G,\mu}^{B_{dR}}(C)$. Then*

(1) *we have an equality in the positive Weyl chamber*

$$\nu_{\mathcal{E}_b} = -w_0[\nu_b],$$

where w_0 is the element of longest length in the Weyl group W;

(2)

$$c_1^G(\mathcal{E}_{b,x}) = \mu^{\sharp} - \kappa_G(b) \in \pi_1(G)_{\Gamma}.$$

In particular, $c_1^G(\mathcal{E}_b) = -\kappa_G(b)$.

Recall the following fact:

Theorem 2.7 ([32] theo. 4.5.1). *Let $\mathcal{E}$ be a G-bundle on X.*

(1) *Suppose $\mathcal{E}_Q$ is a reduction of $\mathcal{E}$ to the standard parabolic subgroup Q. Consider the vector*

$$v : X^*(Q) \longrightarrow \mathbb{Z}$$
$$\chi \longmapsto \deg \chi_* \mathcal{E}_Q$$

seen as an element of $X_*(A)_{\mathbb{Q}}$. Then one has $v \leq \nu_{\mathcal{E}}$. Moreover, if this inequality is an equality, then $Q \subset P$ and $\mathcal{E}_P \simeq \mathcal{E}_Q \times_Q P$, where $\mathcal{E}_P$ is the canonical reduction of $\mathcal{E}$.

(2) *The vector $\nu_{\mathcal{E}}$ can be defined as being the supremum of all such vectors v associated to all possible reductions $\mathcal{E}_Q$ in the poset $X_*(A)_{\mathbb{Q}}$.*

Remark 2.8. *If we view v as an element in $X_*(A)_{\mathbb{Q}}$, then $v = \mathrm{Av}_{M_Q}(\nu_{\mathcal{E}_{M_Q}})$, where M_Q is the Levi component of Q and $\mathcal{E}_{M_Q} = \mathcal{E}_Q \times_Q M$.*

Corollary 2.9. *Suppose $\mathcal{E}$ is a G-bundle with $\mathcal{E}_P$ a reduction to P. Let $\mathcal{E}' = (\mathcal{E}_P \times_P M) \times_M G$. Then $\nu_{\mathcal{E}} \preceq \nu_{\mathcal{E}'}$. In particular, if $\mathcal{E}_P \times_P M$ is a trivial M-bundle, then $\mathcal{E}$ is a trivial G-bundle.*

Proof. Suppose $\mathcal{E}_Q$ is a reduction of $\mathcal{E}$ to a standard parabolic subgroup Q. Suppose

$$s_P : X \to P \backslash \mathcal{E}, \ s_Q : X \to Q \backslash \mathcal{E}$$

are the corresponding sections for $\mathcal{E}_P$ and $\mathcal{E}_Q$ respectively. Then the relative position map:

$$Q \backslash G \times P \backslash G \quad \to \quad Q \backslash G / P = W_Q \backslash W / W_P$$
$$(Qg_1, Pg_2) \quad \mapsto \quad Qg_1 g_2^{-1} P$$

gives

$$X \xrightarrow{(s_Q, s_P)} Q \backslash \mathcal{E} \times P \backslash \mathcal{E} \to W_Q \backslash W / W_P.$$

Let $W_Q \dot{w} W_P$ be the image of the generic point of X, where $\dot{w}$ is the minimal length representative of the double coset $W_Q \dot{w} W_P$. Let $X' \subseteq X$ be the preimage of $W_Q \dot{w} W_P$. It's an open subscheme of X.

By Theorem 2.7(2), the corollary follows from the following Claim.

Claim: $\mathcal{E}_Q$ induces a reduction $\mathcal{E}'_Q$ of $\mathcal{E}'$ to the parabolic subgroup Q such that $\mathcal{E}_Q$ and $\mathcal{E}'_Q$ has the same vector v as defined in Theorem 2.7.

Now it remains to prove the Claim. The composition of morphisms

$$\mathcal{E}_P \to \mathcal{E} \overset{\times \dot{w}}{\to} \mathcal{E}$$

induces a monomorphism

$$P \cap \dot{w}^{-1} Q \dot{w} \backslash \mathcal{E}_P \to Q \backslash \mathcal{E}.$$

The pullback by the section $s_Q : X \to Q \backslash \mathcal{E}$ of this morphism gives a section of

$$P \cap \dot{w}^{-1} Q \dot{w} \backslash (\mathcal{E}_P)_{|X'} \to X'.$$

Combined with the natural morphism

$$P \cap \dot{w}^{-1} Q \dot{w} \backslash \mathcal{E}_P \to P'_M \backslash \mathcal{E}'_M$$

induced by the projection to the Levi quotient, where $\mathcal{E}'_M = \mathcal{E}_P \times_P M$ and $P'_M = M \cap \dot{w}^{-1} Q \dot{w}$, we get a reduction $(\mathcal{E}'_{M, P'_M})_{|X'}$ of $(\mathcal{E}'_M)_{|X'}$ to its standard parabolic subgroup P'_M. The composition of morphisms

$$\mathcal{E}'_M \to \mathcal{E}' \overset{\times \dot{w}}{\to} \mathcal{E}'$$

induces a morphism

$$(P'_M \backslash \mathcal{E}'_M)_{|X'} \to (Q \backslash \mathcal{E}')_{|X'},$$

and we get a reduction $\mathcal{E}'_{|X', Q}$ of $\mathcal{E}'_{|X'}$ to Q induced from $\mathcal{E}'_{M, P'_M}$ via this morphism. Then the desired reduction $\mathcal{E}'_Q$ of the Claim is obtained by the application of the valuative criterion of properness to $Q \backslash \mathcal{E}' \to X$ as $Q \backslash G$ is proper and X is a Dedekind scheme.

Now it remains to show that $\mathcal{E}_Q$ and $\mathcal{E}'_Q$ has the same vector v as defined in Theorem 2.7. By the construction of $\mathcal{E}'_Q$, this results from the commutativity of the following diagram for any $\chi \in X^*(Q)$

(2.1.1)
$$
\begin{array}{ccccc}
P \cap \dot{w}^{-1} Q \dot{w} & \xrightarrow{\mathrm{ad}\dot{w}} & Q & \xrightarrow{\chi} & \mathbb{G}_m \\
{\scriptstyle \mathrm{pr}_M} \downarrow & & & & \| \\
P'_M & \xrightarrow{\mathrm{ad}\dot{w}} & Q & \xrightarrow{\chi} & \mathbb{G}_m
\end{array}
$$

where pr_M is restriction to $P \cap w^{-1} Q w$ of the projection of P to its Levi component M. $\square$

Remark 2.10. *When $G = \mathrm{GL}_n$, the above corollary gives a necessary condition of whether a vector bundle over Fargues-Fontaine curve X is an extension of two given vector bundles over X.*

The converse side of the corollary is the following conjecture.

Conjecture 2.11. *Let P be a standard parabolic subgroup of G with Levi component M. Let $\mathcal{E}$ be a G-bundle and $\mathcal{E}'_M$ be a M-bundle with $\nu_{\mathcal{E}} \preceq \nu_{\mathcal{E}'}$ where $\mathcal{E}' = \mathcal{E}'_M \times_M G$. Suppose $\nu_{\mathcal{E}'_M}$ is G-anti-dominant. Then $\mathcal{E}$ has a reduction $\mathcal{E}_P$ to P such that $\mathcal{E}_P \times_P M \simeq \mathcal{E}'_M$.*

Remark 2.12. *When $G = \mathrm{GL}_n$ and $\mathcal{E}'_M$ is semi-stable, the conjecture is proved in [1]. The G-anti-dominant assumption for $\nu_{\mathcal{E}'_M}$ is necessary since $H^1(X, \mathcal{O}(\lambda)) = 0$ for $\lambda \geq 0$.*

Let b_M be a reduction of b to M, where M is a standard Levi subgroup of G. Let P be the standard parabolic subgroup with Levi component M. Recall that for any $w \in W$, there is an affine fibration

$$\mathrm{pr}_w : \mathcal{F}(G, \mu)(C)^w := P(C) w P_\mu(C) / P_\mu(C) \to \mathcal{F}(M, w\mu)(C)$$

by projection to the Levi quotient. We have the following fact.

Lemma 2.13 ([6] Lemma 2.6). *For $x \in P(C) w P_\mu(C) / P_\mu(C)$ there is an isomorphism*

$$(\mathcal{E}_{b,x})_P \times_P M \simeq \mathcal{E}_{b_M, \mathrm{pr}_w(x)}.$$

where b_M is a reduction of b to M and $(\mathcal{E}_{b,x})_P$ is the reduction of $\mathcal{E}_{b,x}$ to P induced by the reduction $\mathcal{E}_{b_M} \times_M P$ of $\mathcal{E}_b$ to P.

2.2. **Weakly admissible locus.** Recall that $\{\mu\}$ is a geometric conjugacy class of a minuscule cocharacter $\mu : \mathbb{G}_m \to G_{\bar{F}}$. After choosing a suitable representative in $\{\mu\}$, we may assume $\mu \in X_*(T)^+$ via inner twisting, where $^+$ stands for the dominant cocharacters. We consider the adic space $\mathcal{F}(G, \mu)$ associated to the flag variety over $\mathrm{Spa}(\breve{E})$. For $b \in G(\breve{F})$, Rapoport and Zink has defined a weakly admissible locus

$$\mathcal{F}(G, \mu, b)^{wa} \subseteq \mathcal{F}(G, \mu)$$

associated to (G, μ, b). Now we recall its definition.

Let $L|\breve{F}$ be a complete field extension. For any $x \in \mathcal{F}(G, \mu)(L)$, we can associate a cocharacter $\mu_x \in \{\mu\}$ defined over L. Let $\varphi - \mathrm{FilMod}_{L|\breve{F}}$ be the category of filtered isocrystals over $L|\breve{F}$. There is a functor

$$\mathcal{I}_{b,x} : \mathrm{Rep}G \longrightarrow \varphi - \mathrm{FilMod}_{L/\breve{F}}$$
$$(V, \rho) \mapsto (V_{\breve{F}}, \rho(b)\sigma, \mathrm{Fil}^\bullet_{\rho \circ \mu_x} V_L)$$

The pair (b, x) is called weakly admissible if for any $(V, \rho) \in \mathrm{Rep}G$, the filtered isocrystal $\mathcal{I}_{b,x}(V, \rho)$ is weakly admissible in the sense of Fontaine. Let

$$\mathcal{F}(G, \mu, b)^{wa}(L) := \{x \in \mathcal{F}(G, \mu)(L) | (b, x) \text{is weakly admissible}\}.$$

This defines the weakly admissible locus $\mathcal{F}(G, \mu, b)^{wa}$ which is a partially proper open subspace inside $\mathcal{F}(G, \mu)$ by [30] Proposition 1.36.

Remark 2.14. (1) Let $b, b' \in G(\breve{F})$ with $[b] = [b'] \in B(G)$, then $\mathcal{F}(G, \mu, b)^{wa} \simeq \mathcal{F}(G, \mu, b)^{wa}$.
(2) By [31] Proposition 3.1, $\mathcal{F}(G, \mu, b)^{wa}$ is non-empty if and only if

$$[b] \in A(G, \mu) := \{[b] \in B(G) | [\nu_b] \leq \mu^\diamond\}.$$

(3) Suppose the Frobenius maps on $H(\breve{F})$ maps to $g\sigma(-)g^{-1}$ via the inner twisting $H_{\breve{F}} \overset{\sim}{\to} G_{\breve{F}}$ with $g \in G(\breve{F})$. We have a bijection $B(G) \overset{\sim}{\to} B(H)$ which maps $[b]$ to $[b^H]$ where b^H maps to $bg^{-1} \in G(\breve{F})$ via the inner twisting. By [9] Proposition 9.5.3, there is an identification

$$\mathcal{F}(G, b, \mu)^{wa} = \mathcal{F}(H, b^H, \mu)^{wa}.$$

Therefore for the study of weakly admissible locus, it usually suffices to reduce to the quasi-split case.

In the following proposition, we will use the modification of G-bundles on the curve X to give an equivalent definition of the weakly admissibility of a pair (b, x) when G is quasi-split.

Proposition 2.15. [6] Proposition 2.7] Assume that G is quasi-split. Let $[b] \in A(G, \mu)$ and $x \in \mathcal{F}(G, \mu)(L)$. Then the pair (b, x) is weakly admissible if and only if for any standard parabolic P with Levi component M, any reduction b_M of b to M, and any $\chi \in X^*(P/Z_G)^+$ where Z_G is the center of G, we have

$$\deg \chi_*(\mathcal{E}_{b,x})_P \leq 0,$$

where $(\mathcal{E}_{b,x})_P$ is the reduction to P of $\mathcal{E}_{b,x}$ induced by the reduction $\mathcal{E}_{b_M} \times_M P$ of $\mathcal{E}_b$ by Lemma 2.5.

Remark 2.16. The condition

$$\deg \chi_*(\mathcal{E}_{b,x})_P \leq 0, \ \forall \chi \in X^*(P/Z_G)^+$$

is equivalent to $v \leq 0$ in $X_*(A)_{\mathbb{Q}}$, where

$$v : X^*(P) \longrightarrow \mathbb{Z}$$
$$\chi \longmapsto \deg \chi_*(\mathcal{E}_{b,x})_P.$$

2.3. **Admissible locus.** Rapoport and Zink has conjectured in [30] the existence of an open subspace

$$\mathcal{F}(G, \mu, b)^a \subseteq \mathcal{F}(G, \mu, b)^{wa}$$

with a étale-G-local system $\mathcal{L}$ on $\mathcal{F}(G, \mu, b)^a$ such that these two spaces have the same classical points and the G-local system $\mathcal{L}$ interpolates a family of crystalline representations with value in $G(F)$.

When the local Shimura datum (G, μ, b) corresponds to a Rapoport-Zink space $\mathcal{M}(G, \mu, b)$ over $\breve{E}$, then the admissible locus $\mathcal{F}(G, \mu, b)^a$ is the image of the p-adic period mapping ([30] Chapter 5)

$$\breve{\pi} : \mathcal{M}(G, \mu, b) \to \mathcal{F}(G, \mu),$$

and the G-local system $\mathcal{L}$ corresponds to the Tate module of the universal p-divisble group with G-structures by descent via the p-adic period mapping.

For the general local Shimura datum (G, μ, b), the existence of the admissible locus is due to the work of Fargues-Fontaine [16], Kedlaya-Liu [21] and Scholze [35].

Definition 2.17. *Let $\mathcal{F}(G, \mu, b)^a$ be a subspace of $\mathcal{F}(G, \mu)$ stable under generalization with C-points defined as follows:*

$$\mathcal{F}(G, \mu, b)^a(C) = \{x \in \mathcal{F}(G, \mu)(C) | \nu_{\mathcal{E}_{b,x}} \text{ is trivial }\}$$

for any complete algebraically closed field C over F.

Remark 2.18.   (1) *$\mathcal{F}(G, \mu, b)^a$ is a partially proper open subset of $\mathcal{F}(G, \mu)$ ([21]), and by definition*

$$\mathcal{F}(G, \mu, b)^a \subset \mathcal{F}(G, \mu, b)^{wa}.$$

   Moreover,

$$\mathcal{F}(G, \mu, b)^a(K) = \mathcal{F}(G, \mu, b)^{wa}(K)$$

   for any finite extension K over $\breve{E}$ ([28] A.5, [8]). In particular, $\mathcal{F}(G, \mu, b)^a \neq \emptyset$ if and only if $[b] \in A(G, \mu)$.
   (2) *For $[b] \in A(G, \mu)$, the admissible locus $\mathcal{F}(G, \mu, b)^a$ coincides with the image of the p-adic period mapping from the local Shimura variety attached to (G, μ, b) to the flag variety $\mathcal{F}(G, \mu)$ ([34], [33]). It also coincides with the construction of admissible locus of Hartl [20] and Faltings [11] when (G, μ, b) is a Hodge type local Shimura datum.*
   (3) *Via the bijection $B(G) \xrightarrow{\sim} B(H)$ by inner twisting which maps $[b]$ to $[b^H]$, there is an identification*

$$\mathcal{F}(G, \mu, b)^a = \mathcal{F}(H, \mu, b^H)^a.$$

   Therefore for the study of admissible locus, we can also reduce to the quasi-split case.

Consider the Newton stratification:

$$\mathcal{F}(G, \mu) = \coprod_{[b'] \in B(G)} \mathcal{F}(G, \mu, b)^{[b']}$$

where $\mathcal{F}(G, \mu, b)^{[b']}$ is a subspace of $\mathcal{F}(G, \mu)$ stable under generalization with C-points defined by

$$\mathcal{F}^{[b']}(C) = \{x \in \mathcal{F}(C) | \mathcal{E}_{b,x} \simeq \mathcal{E}_{b'}\}$$

for any complete algebraically closed field C over F. Each stratum is locally closed by Kedlaya-Liu [21]. It's clear that

$$\mathcal{F}(G, \mu, b)^{[1]} = \mathcal{F}(G, \mu, b)^a.$$

Remark 2.19. *The $\tilde{J}_b(K)$-action on $\mathcal{F}(G, \mu)(C)$ induces an action on each stratum $\mathcal{F}(G, \mu, b)^{[b']}(C)$. In particular, $\tilde{J}_b(K)$ acts on $\mathcal{F}(G, \mu, b)^a(C)$.*

3. Hodge-Newton-decomposability

Let $M_b = \text{Cent}_H([\nu_b])$ be the centralizer of $[\nu_b]$.

Definition 3.1.   (1) *A triple (G, μ, b) (resp. (G, δ, b) with $\delta \in X_*(A)^+_{\mathbb{Q}}$) is called Hodge-Newton-decomposable (or HN-decomposable for short) if $[b] \in A(G, \mu)$ (resp. $[b] \in B(G, \epsilon, \delta)$ with $\epsilon = \kappa_G(b) \in \pi_1(G)_\Gamma$) and there exists a strict standard Levi subgroup M of the quasi-split inner form H of G containing M_b, such that $\mu^\diamond - [\nu_b] \in \langle \Phi^\vee_{0,M} \rangle_{\mathbb{Q}}$ (resp. $\delta - [\nu_b] \in \langle \Phi^\vee_{0,M} \rangle_{\mathbb{Q}}$). Otherwise, the triple (G, μ, b) (resp. (G, δ, b)) is called Hodge-Newton-indecomposable (or HN-indecomposable for short).*
   (2) *A pair (G, μ) is called fully Hodge-Newton decomposable (or fully HN-decomposable for short) if for any non-basic $[b] \in B(G, \mu)$, the triple (G, μ, b) is HN-decomposable.*

Remark 3.2. *In [18], Görtz, He and Nie give equivalent conditions for a pair (G, μ) to be fully HN-decomposable. They also classified all such pairs.*

In the quasi-split case, we have the following equivalent definition for the HN-decomposability.

Lemma 3.3. *[comp. [6] Lemma 4.11]Suppose G is quasi-split. Let $[b] \in B(G, \mu)$. Then the following four conditions are equivalent:*

(1) *the triple (G, μ, b) is HN-decomposable,*
(2) *there exists a strict standard Levi subgroup M containing M_b such that $\mu^{\sharp} = \kappa_M(b_M) \in \pi_1(M)_\Gamma$,where b_M is the reduction of b to M deduced from its canonical reduction to M_b combined with the inclusion $M_b \subseteq M$,*
(3) *there exists a strict standard Levi subgroup M containing M_b such that $[b_M] \in B(M, \mu)$,*
(4) *there exists a strict standard Levi subgroup M containing $w_0 M_b w_0^{-1}$ such that $[\tilde{b}_M] \in B(M, w_0\mu)$, where $\tilde{b}_M$ is the reduction of b to M deduced from its canonical reduction $w_0 b_{M_b} w_0^{-1}$ to $w_0 M_b w_0^{-1}$ combined with the inclusion $w_0 M_b w_0^{-1} \subseteq M$.*

Proof. The proof of the equivalence of (1)-(3) is the same as that of [6] Lemma 4.11. It is deduced from 1.7(1). The equivalence between (3) and (4) is due to the fact that there is a bijection between $B(M, \mu)$ and $B(w_0 M w_0^{-1}, \tilde{w}_0\mu)$ induced by the conjugation by w_0, where $\tilde{w}_0\mu = (w_0\mu)_{M-dom}$ is the M-dominant representative in $W_M w_0 \mu$. $\qquad\square$

Corollary 3.4. *Suppose G is quasi-split. Let $[b] \in B(G, \mu)$. Then the following three conditions are equivalent:*

(1) *the triple (G, μ, b) is HN-indecomposable,*
(2) *$\mu^{\sharp} - \kappa_{M_b}(b_M) \in \pi_1(M_b)_\Gamma$ is a linear combination of*

$$\{\alpha^{\vee,\sharp} \in \pi_1(M_b)_\Gamma | \alpha \in \Delta_0, \langle \alpha, [\nu_b] \rangle > 0\}$$

   with coefficients of positive integers,
(3) *for all $\alpha \in \Delta_0$, $\langle \alpha, [\nu_b] \rangle > 0$ implies the coefficient of $\alpha^{\vee}$ in $\mu^{\diamond} - [\nu_b]$ is positive.*

Corollary 3.5. *Suppose (G, μ, b) is HN-decomposable with G quasi-split. Then there exists a strict standard Levi subgroup M and a reduction $\tilde{b}_M$ such that $[\tilde{b}_M] \in B(M, \tilde{w}_0\mu)$ and $(M, \tilde{w}_0\mu, \tilde{b}_M)$ is HN-indecomposable, where $\tilde{w}_0\mu = (w_0\mu)_{M-dom}$.*

The following proposition is a key ingredient to the proof of the main result.

Proposition 3.6. *Suppose (G, μ, b) is HN-decomposable with G quasi-split. Let $M \subseteq G$ be the strict standard Levi subgroup such that $(M, w_0\mu, \tilde{b}_M)$ is HN-indecomposable as in Corollary 3.5. Then*

$$\mathrm{pr}_{\tilde{w}_0}^{-1}(\mathcal{F}(M, \tilde{w}_0\mu, \tilde{b}_M)^a(C)) = \mathcal{F}(G, \mu, b)^a(C),$$

$$\mathrm{pr}_{\tilde{w}_0}^{-1}(\mathcal{F}(M, \tilde{w}_0\mu, \tilde{b}_M)^{wa}(C)) = \mathcal{F}(G, \mu, b)^{wa}(C).$$

Remark 3.7. *The appeearence of $\tilde{w}_0$ in the statement is due to the fact that $\nu_{\mathcal{E}_b} = -w_0[\nu_b]$ by Proposition 2.6.*

The remaining of this section is devoted to the proof of this proposition. Suppose (G, μ, b) is HN-decomposable with G quasi-split. Suppose M is the Levi subgroup of G such that $(M, \tilde{w}_0\mu, \tilde{b}_M)$ is HN-indecomposable as in Corollary 3.5. Let P be the standard parabolic subgroup of G with M as Levi component.

Lemma 3.8. $\mathcal{F}(G, \mu, b)^{wa}(C) \subseteq P(C)w_0 P_\mu(C)/P_\mu(C)$.

Proof. Suppose $\mathcal{F}(G, \mu, b)^{wa}(C) \cap P(C)w P_\mu(C)/P_\mu(C) \neq \emptyset$ for some $w \in W$. We want to show $w_0 \in W_P w W_{P_\mu}$. For any $x \in \mathcal{F}(G, \mu, b)^{wa}(C) \cap P(C)w P_\mu(C)/P_\mu(C)$, we have

$$(\mathcal{E}_{b,x})_P \simeq \mathcal{E}_{\tilde{b}_M, \mathrm{pr}_w(x)}$$

by Lemma 2.13. The weak semi-stability of $\mathcal{E}_{b,x}$ implies that $\deg\chi_*(\mathcal{E}_{\tilde{b}_M, \mathrm{pr}_w(x)}) \leq 0$ for any $\chi \in X^*(M/Z_G)^+$. On the other side, using the fact that $[\tilde{b}_M] \in B(M, \tilde{w}_0\mu)$, we have

$$c_1^M(\mathcal{E}_{\tilde{b}_M, \mathrm{pr}_w(x)}) = (w\mu)^{\sharp} - \kappa_M(\tilde{b}_M) = (w\mu)^{\sharp} - (w_0\mu)^{\sharp} \text{ in } \pi_1(M)_\Gamma,$$

and $\deg\chi_*(\mathcal{E}_{\tilde{b}_M, \mathrm{pr}_w(x)}) = \langle w\mu - w_0\mu, \chi \rangle \geq 0$ for any $\chi \in X^*(M/Z_G)^+$ by Proposition 2.6. Therefore the equality holds for any χ and $(w\mu)^{\sharp} = (w_0\mu)^{\sharp}$ in $\pi_1(M)_\Gamma$. On the other hand, as $w\mu \geq w_0\mu$, $w\mu$ and $w_0\mu$ has the same image in $\pi_1(M)$. The result follows. $\qquad\square$

Consider the following commutative diagram (cf. [26] [37]) of de Rham period maps for different groups from local Shimura varieties at infinite level to flag varieties.

$$\mathcal{M}(P, \tilde{w}_0\mu, \tilde{b}_P)_\infty$$

$$\xi_M \qquad \qquad \xi_G$$

$$\mathcal{M}(M, \tilde{w}_0\mu, \tilde{b}_M)_\infty \qquad \pi_{dR}^P \qquad \mathcal{M}(G, \mu, b)_\infty$$

$$\pi_{dR}^M \qquad \mathcal{F}(P, \tilde{w}_0\mu, \tilde{b}_P)^a \qquad \pi_{dR}^G$$

$$\tilde{\xi}_M \qquad \qquad \tilde{\xi}_G$$

$$\mathcal{F}(M, \tilde{w}_0\mu, \tilde{b}_M)^a \qquad \qquad \mathcal{F}(G, \mu, b)^a$$

where

- $\tilde{b}_P$ is the reduction of b to P induced by the reduction $\tilde{b}_M$ of b to M combined with the inclusion $M \subseteq P$.
- $\mathcal{M}(G, \mu, b)_\infty$ (resp. $\mathcal{M}(M, \tilde{w}_0\mu, \tilde{b}_M)$, resp. $\mathcal{M}(P, \tilde{w}_0\mu, \tilde{b}_P)$) classifies modifications of type μ (resp. $\tilde{w}_0\mu$, resp. $\tilde{w}_0\mu$) between $\mathcal{E}_b^G$ (resp. $\mathcal{E}_b^M$, resp. $\mathcal{E}_b^P$) and $\mathcal{E}_1^G$ (resp. $\mathcal{E}_1^M$, resp. $\mathcal{E}_1^P$).
- π_{dR}^G, π_{dR}^M, π_{dR}^P are the de Rham period maps. More precisely, for a modification in $\mathcal{M}(G, \mu, b)_\infty$ its image by π_{dR}^G is x if $\mathcal{E}_{b,x}^G = \mathcal{E}_1^G$. Similarly for π_{dR}^M and π_{dR}^P. We define $\mathcal{F}(P, \tilde{w}_0\mu, \tilde{b}_P)^a$ to be the image of π_{dR}^P.
- ξ_G (resp. ξ_M) is the induced modification via the natural morphism $P \to G$ (resp. the projection to the Levi quotient $P \to M$).
- $\tilde{\xi}_G$ is induced from

$$(3.0.1) \qquad \mathcal{F}(P, \tilde{w}_0\mu) = P/P_{\tilde{w}_0\mu} \cap P \quad \to \quad P\tilde{w}_0 P_\mu/P_\mu \subseteq G/P_\mu = \mathcal{F}(G, \mu)$$
$$aP_{\tilde{w}_0\mu} \cap P \quad \mapsto \quad a(P_{\tilde{w}_0\mu} \cap P)\tilde{w}_0 P_\mu = a\tilde{w}_0 P_\mu$$

- $\tilde{\xi}_M$ is induced from the natural projection $P \to M$ to the Levi component

$$\mathcal{F}(P, \tilde{w}_0\mu) = P/P_{\tilde{w}_0\mu} \to M/M_{\tilde{w}_0\mu} = \mathcal{F}(M, \tilde{w}_0\mu)$$

By the proof of [6] Lemma 6.3, we have the following fact.

Lemma 3.9. *Let $\mathcal{E}$ be a G-bundle and let $P' \subseteq P$ be a standard parabolic subgroup of G contained in P. There is a bijection between*

- *reductions $\mathcal{E}_{P'}$ of $\mathcal{E}$ to P',*
- *reductions $\mathcal{E}_P$ to P together with a reduction $(\mathcal{E}_P \times_P M)_{M \cap P'}$ of $\mathcal{E}_P \times_P M$ of $M \cap P'$.*

Moreover, this bijection indentifies $\mathcal{E}_{P'} \times_{P'} M'$ and $(\mathcal{E}_P \times_P M)_{M \cap P'} \times_{M \cap P'} M'$, where M' is the Levi component of P'.

Now we can prove the following result.

Lemma 3.10. *In the above diagram, we have $\tilde{\xi}_G$ is an isomorphism of adic spaces and*

$$\mathcal{M}(G, \mu, b)_\infty \simeq \coprod_{G(F)/P(F)} \mathcal{M}(P, \tilde{w}_0\mu, b)_\infty.$$

Proof. For the first assertion, as $\mathcal{F}(P, \tilde{w}_0\mu) \to \mathcal{F}(G, \mu)$ is an open immersion, it suffices to show

$$\mathcal{F}(P, \tilde{w}_0\mu, \tilde{b}_P)^a(C) \to \mathcal{F}(G, \mu, b)^a(C)$$

is surjective for any complete algebraic closed field C. For any $x \in \mathcal{F}(G, \mu, b)^a(C)$, by Lemma 3.8, $x \in P(C)\omega_0 P_\mu(C)/P_\mu(C)$. Let $(\mathcal{E}_{b,x})_P$ be the reduction of $\mathcal{E}_{b,x}$ to P induced by the reduction $\mathcal{E}_{\tilde{b}_P}$ of $\mathcal{E}_b$ to P. Write $(\mathcal{E}_{b,x})_P = \mathcal{E}_{\tilde{b}_P,y}$ where y is the preimage of x via (3.0.1). We want to show $y \in \mathcal{F}(P, \tilde{w}_0\mu, \tilde{b}_P)^a(C)$. It's equivalent to show $(\mathcal{E}_{b,x})_P = \mathcal{E}_{\tilde{b}_P,y}$ is a trivial G-bundle. The isomorphism of G-bundles $\mathcal{E}_{b,x} \simeq \mathcal{E}_1$ induces a reduction $(\mathcal{E}_1)_P$ of $\mathcal{E}_1$ to P and an isomorphism of P-bundles. $(\mathcal{E}_{b,x})_P \simeq (\mathcal{E}_1)_P$. We want to show that $(\mathcal{E}_1)_P$ is a trivial P-bundle. By corollary 2.9, it suffices to show that $\mathcal{E}_M := (\mathcal{E}_1)_P \times_P M$ is a trivial M-bundle. We first show that

$$(3.0.2) \qquad c_1^M((\mathcal{E}_M) = 0.$$

We have the following equalities

$$
\begin{aligned}
c_1^M((\mathcal{E}_M) & = & c_1^M((\mathcal{E}_{b,x})_P \times_P M) \\
& \overset{\text{Lemma}2.13}{=} & c_1^M(\mathcal{E}_{\tilde{b}_M, \mathrm{pr}_{w_0}(x)}) \\
& = & (w_0\mu)^\sharp - \kappa_M(\tilde{b}_M) \\
& = & 0 \in \pi_1(M)_\Gamma,
\end{aligned}
$$

where the last equality holds because $[\tilde{b}_M] \in B(M, \tilde{w}_0\mu)$. Now it remains to show the slope $\nu_{\mathcal{E}_M} = 0$. Let $(\mathcal{E}_M)_{P'_M}$ be the canonical reduction of $\mathcal{E}_M$ to a standard parabolic subgroup P'_M of M. Write $P' = P'_M \cdot R_u(P)$, where $R_u(P)$ denotes the unipotent radical of P. Note that P' is a standard parabolic subgroup of G with $P' \cap M = P'_M$. Let M' be the Levi component of P'_M. By [14] Proposition 5.16,

$$
\nu_{(\mathcal{E}_M)_{P'_M} \times_{P'_M} M'} = \nu_{\mathcal{E}_M} \in X_*(A)_\mathbb{Q}.
$$

By Lemma 3.9, $(\mathcal{E}_M)_{P'_M}$ corresponds to a reduction of $\mathcal{E}_1$ to P'. Hence the semi-stability of $\mathcal{E}_1$ implies that

$$
\langle \chi, \nu_{\mathcal{E}_M} \rangle \leq 0, \ \forall \chi \in X^*(M'/Z_G)^+.
$$

On the other hand, the equality (3.0.2) implies that $\nu_{\mathcal{E}_M}$ is a non-negative linear combination of simple coroots in M. Hence

$$
\langle \chi, \nu_{\mathcal{E}_M} \rangle \geq 0, \ \forall \chi \in X^*(M'/Z_G)^+.
$$

It follows that $\nu_{\mathcal{E}_M} = 0$.

The second assertion is obtained from the first assertion and the fact that $\mathcal{M}(G, \mu, b)_\infty$ (resp. $\mathcal{M}(P, \tilde{w}_0\mu, \tilde{b}_P)_\infty$) is the $\underline{G(F)}$-torsor (resp. $\underline{P(F)}$-torsor) over $\mathcal{F}(G, \mu, b)^a$ (resp. $\mathcal{F}(P, \tilde{w}_0\mu, \tilde{b}_P)$). $\qquad\square$

Remark 3.11. *This lemma was proved for unramified local Shimura datum of PEL type by Mantovan* [26] *section 8.2 and Shen* [37] *corollary 6.4.*

In order to prove Proposition 3.6, we also need the following Lemmas.

Lemma 3.12. *Suppose G is quasi-split. We consider $\mathcal{F}(M, w_0\mu)$ as a subspace of $\mathcal{F}(G, \mu)$ via the natural injective morphism $\mathcal{F}(M, \tilde{w}_0\mu) \to \mathcal{F}(G, \mu)$. Then we have*

$$
\begin{aligned}
\mathcal{F}(M, \tilde{w}_0\mu, \tilde{b}_M)^a & \subseteq & \mathcal{F}(G, \mu, b)^a, \\
\mathcal{F}(M, \tilde{w}_0\mu, \tilde{b}_M)^{wa} & \subseteq & \mathcal{F}(G, \mu, b)^{wa}.
\end{aligned}
$$

Proof. The first assertion is clear. For the second one, let $x \in \mathcal{F}(M, \tilde{w}_0\mu, \tilde{b}_M)^{wa}(C)$ and let x_G be its image via the natural morphism $\mathcal{F}(M, \tilde{w}_0\mu) \to \mathcal{F}(G, \mu)$. Suppose Q is a standard parabolic subgroup with Levi component M_Q. Let b_{M_Q} be a reduction of b to M_Q. Let $w \in W^\Gamma$ and $b_1 \in M_1(\breve{F})$ with $M_1 = M \cap w^{-1}M_Q w$ as in Lemma 3.13. Let Q_1 be the standard parabolic subgroup of M with Levi component M_1. Since

$$
(\mathcal{E}_{b_1} \times_{M_1} Q_1) \times_{Q_1, \mathrm{ad}(w)} Q \simeq \mathcal{E}_{b_{M_Q}} \times_{M_Q} Q
$$

by Lemma 3.13, we have

$$
(\mathcal{E}_{b,x_G})_Q \simeq (\mathcal{E}_{\tilde{b}_M,x})_{Q_1} \times_{Q_1} Q,
$$

where $Q_1 \to Q$ is induced by the $\mathrm{ad}\dot{w}$ and where $(\mathcal{E}_{b,x_G})_Q$ (resp.$(\mathcal{E}_{\tilde{b}_M,x})_{Q_1}$) is the reduction of $\mathcal{E}_{b,x_G}$ (resp. $\mathcal{E}_{\tilde{b}_M,x}$) to Q (resp. Q_1) induced by the reduction $\mathcal{E}_{b_{M_Q}} \times_{M_Q} Q$ (resp. $\mathcal{E}_{b_1} \times_{M_1} Q_1$) of $\mathcal{E}_b$ (resp. $\mathcal{E}_{\tilde{b}_M}$) to Q (resp. Q_1). For any $\chi \in X^*(Q/Z_G)^+$, we have $\deg\chi_*(\mathcal{E}_{b,x_G})_Q = \deg\chi'_*(\mathcal{E}_{\tilde{b}_M,x})_{Q_1}$ with $\chi' = \chi \circ \mathrm{ad}\dot{w} \in X^*(Q_1)$. Write $\chi' = \chi'_1 + \chi'_2$ with $\chi'_1 = \mathrm{Av}_{W_M}(\chi')$ the W_M average of χ'. Up to replacing χ by a multiple $m\chi$ for $m \in \mathbb{N}$ big enough, we may assume $\chi'_1 \in (\Phi_M^\vee)^\perp = X^*(M)$, $\chi'_2 \in X^*(M_1/Z_M)$. Moreover, the choice of w implies $w\beta \in \Phi_G^+$ for any $\beta \in \Phi_M^+$. Hence $\chi'_2 \in X^*(M_1/Z_M)^+$, here $+$ stands for M-dominant. Then

$$
\begin{aligned}
\deg\chi_*(\mathcal{E}_{b,x})_Q & = & \deg\chi'_*(\mathcal{E}_{\tilde{b}_M,x})_{Q_1} \\
& = & \underbrace{\deg\chi'_{1*}(\mathcal{E}_{\tilde{b}_M,x})}_{=0} + \underbrace{\deg\chi'_{2*}(\mathcal{E}_{\tilde{b}_M,x})_{Q_1}}_{\leq 0} \leq 0
\end{aligned}
$$

since $[\tilde{b}_M] \in B(M, \tilde{w}_0\mu)$ combined with the weak admissibility of x. $\qquad\square$

Lemma 3.13. *Suppose G is quasi-split. Let b_{M_Q} be a reduction of b to M_Q, where M_Q is the Levi component of a standard parabolic subgroup Q. Then there exist $w \in W^\Gamma$, $g_1 \in M(\breve{F})$, $g_1' \in w^{-1}M_Q w(\breve{F})$ and $b_1 \in M_1(\breve{F})$ with $M_1 = w^{-1}M_Q w \cap M$ such that*

   (1) w is the minimal length element in $W_{M_Q}wW_M$;
   (2) (b_1, g_1) is a reduction of $\tilde{b}_M$ to M_1;
   (3) (b_1, g_1') is a reduction of $\dot{w}^{-1}b_{M_Q}\sigma(\dot{w})$ to M_1 with $\dot{w} \in N(T)(\breve{F})$ a representative of w.

Moreover, $W_{M_Q}wW_M$ is the generic relative position between the reduction $\mathcal{E}_{\tilde{b}_M} \times_M P$ of $\mathcal{E}_b$ to P and the reduction $\mathcal{E}_{b_{M_Q}} \times_{M_Q} Q$ of $\mathcal{E}_b$ to Q.

Proof. The last assertion is implied by the conditions (2) and (3). We first show that there exists $\tilde{w} \in W^\Gamma$, $\tilde{g}_1 \in M(\breve{F})$, $\tilde{g}_1' \in \tilde{w}^{-1}M_Q\tilde{w}(\breve{F})$ and $\tilde{b}_1 \in \tilde{M}_1(L)$ with $\tilde{M}_1 = \tilde{w}^{-1}M_Q\tilde{w} \cap M$ such that

   (2') $(\tilde{b}_1, \tilde{g}_1)$ is a reduction of $\tilde{b}_M$ to $\tilde{M}_1$;
   (3') $(\tilde{b}_1, \tilde{g}_1')$ is a reduction of $\dot{\tilde{w}}^{-1}b_{M_Q}\sigma(\dot{\tilde{w}})$ to $\tilde{M}_1$ with $\dot{\tilde{w}} \in N(T)(\breve{F})$ a representative of $\tilde{w}$.

Note that a representative $\dot{\tilde{w}}$ of $\tilde{w}$ can be chosen in $N(T)(\breve{F})$ by Steinberg's theorem ([36] III 2.3).

By definition, $\tilde{b}_M$ is induced from $w_0 b_{M_b} w_0^{-1}$ via the natural inclusion $w_0 M_b w_0^{-1} \subseteq M$. Therefore, without loss of generality, we may assume $M = w_0 M_b w_0^{-1}$. Since b_{M_Q} is a reduction of b to M_Q, Up to σ-conjugation, we may assume $\nu_{b_{M_Q}} : \mathbb{D} \to M_Q$ has image in the split maximal torus A and is M_Q-anti-dominant. Choose $\tilde{w} \in W^\Gamma$ such that $\tilde{w}^{-1}\nu_{b_{M_Q}} = w_0[\nu_b]$ is G-antidominant. Then $\mathrm{Cent}_{\tilde{w}^{-1}M_Q\tilde{w}}(\nu_{\dot{\tilde{w}}^{-1}b_{M_Q}\sigma(\dot{\tilde{w}})}) = \tilde{w}^{-1}M_Q\tilde{w} \cap M = \tilde{M}_1$ and $\dot{\tilde{w}}^{-1}b_{M_Q}\sigma(\dot{\tilde{w}})$ has a canonical reduction $(\tilde{b}_1, \tilde{g}_1')$ to $\tilde{M}_1$. Then (3') follows.

Let $\tilde{b}_1'$ be the image of $\tilde{b}_1$ via the inclusion $\tilde{M}_1 \subseteq M$. For (2'), it suffices to show $[\tilde{b}_1'] = [\tilde{b}_M]$ in $B(M)$. Clearly,

$$[\nu_{\tilde{b}_1'}]_M = [\nu_{\tilde{b}_M}]_M = (w_0[\nu_b])_{M-dom} \in \mathcal{N}(M).$$

By the injectivity of the map (1.3.1)

$$(\nu, \kappa_M) : B(M) \to \mathcal{N}(M) \times \pi_1(M)_\Gamma,$$

it suffices to show $\kappa_M(\tilde{b}_1') = \kappa_M(\tilde{b}_M) \in \pi_1(M)_\Gamma$. Since

$$\kappa_M(a) = \nu_a \text{ in } \pi_1(M)_\Gamma \otimes_\mathbb{Z} \mathbb{Q}, \quad \forall a \in M(\breve{F}),$$

we have $\kappa_M(\tilde{b}_1') - \kappa_M(\tilde{b}_M) \in \pi_1(M)_{\Gamma,tor}$. As $\tilde{b}_1$ and $\tilde{b}_M$ are both reductions of b, we have

$$\kappa_G(\tilde{b}_1') = \kappa_G(\tilde{b}_1) = \kappa_G(\tilde{b}_M) \text{ in } \pi_1(G)_\Gamma.$$

The result follows by the injectivity of the map $\pi_1(M)_{\Gamma,tor} \to \pi_1(G)_{\Gamma,tor}$ (cf. proof of Lemma 3.3).

Let $w = w_1\tilde{w}w_2$ be the unique minimal length element in $W_{M_Q}\tilde{w}W_M$ with $w_1 \in W_{M_Q}^{\mathrm{Gal}(\bar{\breve{F}}|\breve{F})}$ and $w_2 \in W_M^{\mathrm{Gal}(\bar{\breve{F}}|\breve{F})}$ (Indeed, we work in the finite Weyl group $W^{\mathrm{Gal}(\bar{\breve{F}}|\breve{F})}$ associated to a maximal $\breve{F}$-split torus of T over F). Then $w \in W^\Gamma$ and we can choose representatives $\dot{w}_2 \in M(\breve{F})$, $\dot{w} \in G(\breve{F})$ of w_2 and w respectively again by Steinberg's theorem. Let

$$
\begin{aligned}
b_1 &:= \dot{w}_2^{-1}\tilde{b}_1\sigma(\dot{w}_2) \in M_1(\breve{F}), \\
g_1 &:= \tilde{g}_1\dot{w}_2 \in M(\breve{F}), \\
g_1' &:= \dot{w}^{-1}\dot{\tilde{w}}\tilde{g}_1'\dot{w}_2 \in w^{-1}M_Q w(\breve{F}).
\end{aligned}
$$

They satisfy the desired properties (1)-(3). $\qquad\square$

Now we are ready to prove the main result of this section.

Proof of proposition 3.6. By Lemma 3.8 and Lemma 3.10, we have the equality of morphisms: $\mathrm{pr}_{\tilde{w}_0} = \tilde{\xi}_M \circ \tilde{\xi}_G^{-1}$ and

$$\mathcal{F}(G, \mu, b)^a \subseteq \mathrm{pr}_{w_0}^{-1}(\mathcal{F}(M, w_0\mu, \tilde{b}_M)^a).$$

Now it remains to show that $\mathcal{E}_{b,x}$ is a trivial G-bundle for any $x \in \mathrm{pr}_{\tilde{w}_0}^{-1}(\mathcal{F}(M, \tilde{w}_0\mu, b)^a(C))$. By Lemma 2.13, $(\mathcal{E}_{b,x})_P \times_P M \simeq \mathcal{E}_{\tilde{b}_M, \mathrm{pr}_{\tilde{w}_0}(x)}$ which is a trivial M-bundle. The result then follows by Corollary 2.9.

The proof for the weak admissibility is similar. Suppose $x \in \mathcal{F}(G, \mu, b)^{wa}(C)$. Then $x \in P(C)w_0 P_\mu(C)/P_\mu(C)$ by Lemma 3.8. We show that $\mathrm{pr}_{\tilde{w}_0}(x) \in \mathcal{F}(M, \tilde{w}_0\mu, \tilde{b}_M)^{wa}(C)$. Suppose $M' \subset M$ is a standard Levi subgroup of M with $b' \in M'(L)$ a reduction of $\tilde{b}_M$ to M'. Let

$P'_M \subseteq M$ (resp. $P' \subseteq G$) be the standard parabolic subgroup of M (resp. G) with Levi component M'. We need to show that

$$\deg\chi_*(\mathcal{E}_{\tilde{b}_M, \mathrm{pr}_{\tilde{w}_0}(x)})_{P'_M} \le 0, \forall \chi \in X^*(P'_M/Z_M)^+$$

where $(\mathcal{E}_{\tilde{b}_M, \mathrm{pr}_{\tilde{w}_0}(x)})_{P'_M}$ is the reduction to P'_M of $\mathcal{E}^M_{b'_M, \mathrm{pr}_{\tilde{w}_0}(x)}$ induced by the reduction $\mathcal{E}_{b'} \times_{M'} P'_M$ of $\mathcal{E}_{\tilde{b}_M}$. By Remark 2.8 and Remark 2.16, it's equivalent to show

$$(3.0.3) \qquad\qquad \mathrm{Av}_{M'}(\nu_{\mathcal{E}_{M'}}) \le_M 0 \text{ in } X_*(A)_{\mathbb{Q}},$$

where $\mathcal{E}_{M'} := (\mathcal{E}_{\tilde{b}_M, \mathrm{pr}_{w_0}(x)})_{P'_M} \times_{P'_M} M'$. By Theorem 2.7,

$$(3.0.4) \qquad\qquad \mathrm{Av}_{M'}(\nu_{\mathcal{E}_{M'}}) \le_M \nu_{\mathcal{E}_M} \text{ in } X_*(A)_{\mathbb{Q}},$$

where $\mathcal{E}_M := \mathcal{E}_{\tilde{b}_M, \mathrm{pr}_{w_0}(x)}$. On the other hand, by Lemma 2.13 and Lemma 3.9, we have

$$\mathcal{E}_{M'} = (\mathcal{E}_{\tilde{b}_M, \mathrm{pr}_{w_0}(x)})_{P'_M} \times_{P'_M} M' \simeq ((\mathcal{E}_{b,x})_P \times_P M)_{P'_M} \times_{P'_M} M' \simeq (\mathcal{E}_{b,x})_{P'} \times_{P'} M',$$

where $(\mathcal{E}_{b,x})_{P'}$ is the reduction to P' of $\mathcal{E}_{b,x}$ induced by the reduction $\mathcal{E}_{b'} \times_{M'} P'$ of $\mathcal{E}_b$. The weak semi-stability of $\mathcal{E}_{b,x}$ implies that

$$(3.0.5) \qquad\qquad \mathrm{Av}_{M'}(\nu_{\mathcal{E}_{M'}}) \le_G 0 \text{ in } X_*(A)_{\mathbb{Q}}.$$

Hence the inequality (3.0.3) follows from (3.0.4) and (3.0.5) combined with the fact that

$$\mathrm{Av}_M(\nu_{\mathcal{E}_M}) = 0$$

since $[\tilde{b}_M] \in B(M, \tilde{w}_0\mu)$.

For the other side, suppose $x \in P(C)w_0 P_\mu(C)/P_\mu(C)$ with $\mathrm{pr}_{\tilde{w}_0}(x) \in \mathcal{F}(M, \tilde{w}_0\mu, \tilde{b}_M)^{wa}(C)$. We want to show $x \in \mathcal{F}(G, \mu, b)^{wa}(C)$. Suppose Q is a standard parabolic subgroup of G with Levi-component M_Q. Let b_{M_Q} be a reduction of b to M_Q. We need to show

$$(3.0.6) \qquad\qquad \deg\chi_*(\mathcal{E}_{b,x})_Q \le 0, \forall \chi \in X^*(Q/Z_G)^+,$$

where $(\mathcal{E}_{b,x})_Q$ is the reduction to Q of $\mathcal{E}_{b,x}$ induced by the reduction $\mathcal{E}_{b_{M_Q}} \times_{M_Q} Q$ to Q of $\mathcal{E}_b$. By the proof of Corollary 2.9, the reduction $(\mathcal{E}_{b,x})_Q$ to Q of $\mathcal{E}_{b,x}$ induces a reduction

$$(\mathcal{E}_{\tilde{b}_M, \mathrm{pr}_{w_0}(x)} \times_M G)_Q$$

of $\mathcal{E}_{\tilde{b}_M, \mathrm{pr}_{w_0}(x)} \times_M G$ to Q. Moreover, we have

$$\deg\chi_*(\mathcal{E}_{b,x})_Q = \deg\chi_*(\mathcal{E}_{\tilde{b}_M, \mathrm{pr}_{w_0}(x)} \times_M G)_Q.$$

Hence the inequality (3.0.6) follows by the weak admissibility of $\mathrm{pr}_{w_0}(x)$ and Lemma 3.12. $\square$

4. The action of $\tilde{J}_b$ on the modifications of G-bundles

Now we state the main results of this article. We first state the main result in the quasi-split case.

Theorem 4.1. *Suppose G is quasi-split and μ is minuscule. Let M be the standard Levi subgroup of G such that $[b] \in B(M, \mu)$ and (M, b, μ) is Hodge-Newton-indecomposable. Then the equality $\mathcal{F}(G, \mu, b)^{wa} = \mathcal{F}(G, \mu, b)^a$ holds if and only if $(M, \{\mu\})$ is fully Hodge-Newton-decomposable and $[b]$ is basic in $B(M)$.*

Remark 4.2. *For GL_n, this theorem is proved by Hartl [20] Theorem 9.3.*

We have a similar description when the group G is non-quasi-split .

Theorem 4.3. *Suppose μ is minuscule. Let $[b^H] \in B(H)$ be the image of $[b]$ via the natural morphism $B(G) \simeq B(H)$. Let M^H be the standard Levi subgroup of H with a reduction b_M^H of b^H to M^H such that $[b_M^H] \in A(M^H, \mu)$ and (M^H, b_M^H, μ) is HN-indecomposable. Then*

   (1) *the Levi-subgroup M^H of H corresponds to a Levi-subgroup M of G via the inner twisting,*
   (2) *the equality $\mathcal{F}(G, \mu, b)^{wa} = \mathcal{F}(G, \mu, b)^a$ holds if and only if the set $B(M, \{\mu\})$ is fully HN-indecomposable and $[b_M]$ is basic in $B(M)$, where $[b_M]$ corresponds to $[b_M^H]$ via $B(M) \simeq B(M^H)$.*

In order to prove the main theorems, we need some preparations.

Lemma 4.4. *Suppose M is a standard Levi subgroup of H defined over F. Let $\dot{w}_0 \in G(\breve{F})$ be a representative of w_0. Then the map*

$$M(\breve{F}) \simeq (w_0 M w_0^{-1})(\breve{F})$$

$$g \mapsto \dot{w}_0 g \sigma(\dot{w}_0)^{-1}$$

induces the bijections

$$B(M) \simeq B(w_0 M w_0^{-1}),$$

$$B(M)_{basic} \simeq B(w_0 M w_0^{-1})_{basic},$$

$$B(M, \mu) \simeq B(w_0 M w_0^{-1}, \tilde{w}_0 \mu),$$

where $\tilde{w}_0 \mu := (w_0 \mu)_{w_0 M w_0^{-1} - dom}$ is the $w_0 M w_0^{-1}$-domominant representative of $w_0 \mu$ in its $W_{w_0 M w_0^{-1}}$-orbit. Moreover (M, μ) is fully HN-decomposable if and only if so is $(w_0 M w_0^{-1}, \tilde{w}_0 \mu)$.

Proof. We may assume that $\tilde{w}_0$ is the minimal length representative in $W_{w_0 M w_0^{-1}} w_0$. Note that for any $g \in M(\breve{F})$,

$$[\nu_{\dot{w}_0 g \sigma(\dot{w}_0)^{-1}}] = \tilde{w}_0 [\nu_g] \in \mathcal{N}(w_0 M w_0^{-1}).$$

The other assertions can be checked easily. $\square$

We also need the following proposition which is a key ingredient of the proof of the main result.

Proposition 4.5. *Suppose (G, μ, b) is HN-indecomposable with G quasi-split. If b is not basic, then $\mathcal{F}(G, \mu, b)^a \neq \mathcal{F}(G, \mu, b)^{wa}$.*

Before proving this proposition, we first show how to use it combined with Proposition 3.6 to prove the main theorem.

Proof of theorem 4.1. For the sufficiency, suppose $(M, \{\mu\})$ is fully HN-decomposable and $[b_M]$ is basic in $B(M)$. Then $(\tilde{M}, \{w_0 \mu\})$ is fully HN-decomposable and $[\tilde{b}_M]$ is basic in $B(\tilde{M})$ with $\tilde{M} = w_0 M w_0^{-1}$ and $\tilde{b}_M = w_0 b_M \sigma(w_0)^{-1}$ by Lemma 4.4. After applying [6] Theorem 6.1 to the triple $(\tilde{M}, \tilde{w}_0 \mu, \tilde{b}_M)$, we get that $\mathcal{F}(\tilde{M}, \tilde{w}_0 \mu, \tilde{b}_M)^a = \mathcal{F}(\tilde{M}, \tilde{w}_0 \mu, \tilde{b}_M)^{wa}$. The result follows by Proposition 3.6.

For the necessity, if b_M is not basic in M or (M, μ) is not fully HN-indecomposable, it's equivalent to say that $\tilde{b}_M$ is not basic in $\tilde{M}$ or $(\tilde{M}, w_0 \mu)$ is not fully HN-indecomposable, then after applying Proposition 4.5 (in the first case) or [6] Theorem 6.1 to the triple $(\tilde{M}, \tilde{w}_0 \mu, \tilde{b}_M)$, we get $\mathcal{F}(M, \tilde{w}_0 \mu, \tilde{b}_M)^a \neq \mathcal{F}(M, \tilde{w}_0 \mu, \tilde{b}_M)^{wa}$. And $\mathcal{F}(G, \mu, b)^{wa} = \mathcal{F}(G, \mu, b)^a$ follows again by Proposition 3.6. $\square$

Proof of theorem 4.3. The reduction to the quasi-split case is similar to the proof of [6] Theorem 6.1. We may assume that G is adjoint, then $G = J_{b^*}$ is an extended pure inner form of H, where $[b^*] \in B(H)_{basic}$. We have

$$\mathcal{F}(G, \mu, b)^a = \mathcal{F}(H, \mu, b^H)^a,$$

$$\mathcal{F}(G, \mu, b)^{wa} = \mathcal{F}(H, \mu, b^H)^{wa},$$

and $[b^H] \in B(H, \mu^\sharp + \kappa(b^*), \mu^\diamond)$. By Lemma 1.7,

$$\kappa(b^*) \in \operatorname{Im}(\pi_1(M^H)_{\Gamma, tor} \to \pi_1(H)_{\Gamma, tor})$$

and if we view $\kappa(b^*)$ as an element in $\pi_1(M^H)_{\Gamma, tor}$, we have

$$[b_M^H] \in B(M^H, \mu^\sharp + \kappa(b^*), \mu^\diamond).$$

Let $[b_M^*]_{M^H} \in B(M^H)_{basic}$ with $\kappa_{M^H}(b_M^*) = \kappa(b^*) \in \pi_1(M^H)_\Gamma$. Then $[b^*] = [b_M^*] \in B(H)$ and we may assume $b^* = b_M^* \in M^H(\breve{F})$. It follows that $M = J_{b_M^*}$ is a pure inner form of M^H which is also a Levi subgroup of G. We can check that all the arguments in the proof of Theorem 4.1 still holds when we work with $B(M^H, \mu^\sharp + \kappa(b_M^*), \mu^\diamond)$ and $B(M^H, \kappa(b_M^*), w_{M^H, 0} \mu^{-1, \diamond})$ instead of $B(M^H, \mu)$ and $B(M^H, 0, w_{M^H, 0} \mu^{-1, \diamond})$. Moreover, by [6] Corollary 4.15, $B(M, \mu)$ is fully HN-decomposable if and only if $B(M^H, \kappa(b_M^*), w_{M^H, 0} \mu^{-1, \diamond})$. $\square$

The rest of the section is devoted to prove Proposition 4.5. Suppose G is quasi-split. For any $\beta \in \Delta_G$, let M_β be the stardard Levi subgroup of G such that $\Delta_{M_\beta} = \Delta_G \backslash \Gamma \beta$. For any $\alpha \in \Delta_{G,0}$, let $M_\alpha := M_\beta$ for any $\beta \in \Delta_G$ such that $\beta|_A = \alpha$.

Lemma 4.6. *Suppose (G, μ, b) is HN-indecomposable and $[b] \in B(G)$ is not basic. Then there exist $\tilde{\alpha} \in \Delta_0$, $w \in W$ and $x \in \mathcal{F}(M_{\tilde{\alpha}}, w\mu)(C)$ satisfying the following conditions:*

(1) $\langle \tilde{\alpha}, [\nu_b] \rangle > 0$,

(2) $w\mu$ *is $M_{\tilde{\alpha}}$-dominant,*

(3) $\mathcal{E}_{b_{M_{\tilde{\alpha}}}, x} \simeq \mathcal{E}_{b'_{M_{\tilde{\alpha}}}}$, *where $b_{M_{\tilde{\alpha}}}$ is the reduction of b to $M_{\tilde{\alpha}}$ deduced from its canonical reduction b_{M_b} to M_b combined with the inclusion $M_b \subseteq M_{\tilde{\alpha}}$, and $b'_{M_{\tilde{\alpha}}} \in B(M_{\tilde{\alpha}})_{basic}$ such that $\kappa_{M_{\tilde{\alpha}}}(b'_{M_{\tilde{\alpha}}}) = -(\tilde{\beta}^{\vee})^{\sharp}$ with $\tilde{\beta} \in \Delta$ and $\tilde{\beta}_{|A} = \tilde{\alpha}$.*

Proof. In order to distinguish the roots for different groups, we will write Δ_G and $\Delta_{G,0}$ for Δ and Δ_0 respectively. Suppose $[b_{M_b}] \in B(M_b, w_1\mu)$ for some $w_1 \in W$ by Lemma 1.7. For any $\alpha \in \Delta_{G,0}$ such that $\langle \alpha, [\nu_b] \rangle > 0$, let $[b_{M_\alpha}]$ be the image of $[b_{M_b}]$ via the natural map $B(M_b) \to B(M_\alpha)$. Then $[b_{M_\alpha}] \in B(M_\alpha, (w_1\mu)_{M_\alpha - dom})$. As (b, μ) is not HN-decomposable, we have $(w_1\mu)_{M_\alpha - dom} \neq \mu$. Therefore there exists $\beta \in \Delta_G$ such that $\beta_{|A} = \alpha$ and $\langle \beta, (w_1\mu)_{M_\alpha - dom} \rangle < 0$. Let

$$R_\alpha := \{(s_\beta(w_1\mu)_{M_\alpha - dom})_{M_\alpha - dom} \in W\mu | \beta \in \Delta_G, \beta_{|A} = \alpha, \langle \beta, (w_1\mu)_{M_\alpha - dom} \rangle < 0\},$$

where $s_\beta \in W$ is the reflection corresponding to β. Let $w\mu$ be a maximal element in the subset

$$\bigcup_{\alpha \in \Delta_{G,0}, \langle \alpha, [\nu_b] \rangle > 0} R_\alpha \subset W\mu.$$

Suppose $w\mu \in R_\alpha$ for any $\alpha \in \Delta_{G,0}$, then $(\alpha, w\mu)$ satisfies (1) and (2). It remains to find $x \in \mathcal{F}(M_{\tilde{\alpha}}, w\mu)(C)$ satisfying condition (3) for some $\tilde{\alpha}$ with $w\mu \in R_{\tilde{\alpha}}$.

It's equivalent to find $\tilde{\alpha} \in \Delta_{G,0}$, $y \in \mathcal{F}(M_{\tilde{\alpha}}, (w\mu)^{-1})(C)$ such that $\mathcal{E}_{b_{M_{\tilde{\alpha}}}} \simeq \mathcal{E}_{b'_{M_{\tilde{\alpha}}}, y}$ and $w\mu \in R_{\tilde{\alpha}}$.

By Proposition 1.13, we need to find $\tilde{\alpha} \in \Delta_{G,0}$ such that $w\mu \in R_{\tilde{\alpha}}$ and

$$[b_{M_{\tilde{\alpha}}}] \in B(M_{\tilde{\alpha}}, \kappa_{M_{\tilde{\alpha}}}(b'_{M_{\tilde{\alpha}}}) + (w\mu)^{\sharp}, \nu_{b'_{M_{\tilde{\alpha}}}}(w\mu)^{\diamond}).$$

By Lemma 1.6, the later is equivalent to the conditions

(4.0.1) $\qquad \kappa_{M_{\tilde{\alpha}}}(b_{M_{\tilde{\alpha}}}) \quad = \quad (w\mu - \tilde{\beta}^{\vee})^{\sharp}$ in $\pi_1(M_{\tilde{\alpha}})_\Gamma$,

(4.0.2) $\qquad \kappa_{M_b}(b_{M_b}) \quad \preceq_{M_b} \quad -\text{Av}_{W_{M_{\tilde{\alpha}}}}\text{Av}_\Gamma(\tilde{\beta}^{\vee}) + \text{Av}_\Gamma(w\mu)$ in $\pi_1(M_b)_{\Gamma, \mathbb{Q}}$.

Let $(\beta_j)_{j \in J}$ be a set of representatives of Galois orbits in $\Delta_G \backslash \Delta_{M_b}$. Let

$$w\mu - w_1\mu = \sum_{j \in J} n_j \beta_j^{\vee} \text{ in } \pi_1(M_b)_\Gamma$$

with $n_j \in \mathbb{N}$ for all $j \in J$.

Claim 1: $n_j \geq 1, \forall j \in J$.

Suppose $n_{j_0} = 0$ for some $j_0 \in J$. Let $\alpha_0 = \beta_{j_0}|_A$. Then $(w\mu)_{M_{\alpha_0} - dom} = (w_1\mu)_{M_{\alpha_0} - dom}$. Again by HN-indecomposability, $(w\mu)_{M_{\alpha_0} - dom} \neq \mu$. Then there exists $\beta \in \Gamma\beta_{j_0}$ such that $\langle \beta, (w\mu)_{M_{\alpha_0} - dom} \rangle < 0$. It follows that

$$(s_\beta(w_1\mu)_{M_{\alpha_0} - dom})_{M_{\alpha_0} - dom} \succsneq w\mu.$$

This contradicts with the maximality of $w\mu$. Hence Claim 1 follows.

By definition, suppose $w\mu \in R_\alpha$ for some α and $w\mu = (s_\beta(w_1\mu)_{M_\alpha - dom})_{M_\alpha - dom}$, where $\alpha \in \Delta_{G,0}$ with $\langle \alpha, [\nu_b] \rangle > 0$ and $\beta \in \Delta_G$ with $\beta_{|A} = \alpha$. Suppose $\beta \in \Gamma\beta_{j_0}$ for some $j_0 \in J$, then $n_{j_0} = 1$ by the definition of $w\mu$. If the subset

$$J_1 := \{j \in J | n_j = 1\}$$

of J consists of the single element of j_0. Let $\tilde{\alpha} := \alpha$, $\tilde{\beta} := \beta$. We will verify that $\tilde{\alpha}$ satisfies the desired properties. Since $[b_{M_b}] \in B(M_b, w_1\mu)$, we have

$$\kappa_{M_{\tilde{\alpha}}}(b_{M_{\tilde{\alpha}}}) = (w_1\mu)^{\sharp} = (w\mu - \tilde{\beta}^{\vee})^{\sharp} \text{ in } \pi_1(M_{\tilde{\alpha}})_\Gamma.$$

The equality (4.0.1) follows. For the inequality (4.0.2), we have

$$\text{Av}_\Gamma(w\mu) - \kappa_{M_b}(b_{M_b}) - \text{Av}_{W_{M_{\tilde{\alpha}}}}\text{Av}_\Gamma(\tilde{\beta}^{\vee})$$

$$= \quad (w\mu - w_1\mu) - \text{Av}_{W_{M_{\tilde{\alpha}}}}(\tilde{\beta}^{\vee})$$

$$\succeq_{M_b} \quad 0$$

in $\pi_1(M_b)_{\Gamma,\mathbb{Q}}$, where the last inequality follows from Lemma 4.7 (1) (because E_8 case does not occur as there is only trivial minuscule cocharacters in that case) combined with the fact that $n_j \geq 2$ for any $j \in J\backslash\{j_0\}$.

It remains to deal with the case when J_1 has at least two elements. By Claim 1, for any $j \in J$, up to replacing β_j by some other representative in the same Galois orbit, we may assume β_j appears in the linear combination of $w\mu - w_1\mu$.

Claim 2: $w\mu = \mu = (s_{\beta_{j'}}(w_1\mu)_{M_{\beta_{j'}}-dom})_{M_{\beta_{j'}}-dom}$ *for any* $j' \in J_1$.

We want to show $w\mu$ is G-dominant. Suppose $j_0' \in J_1$ with $j_0 \neq j_0'$. Let $w\mu = w_3 s_{\beta_{j_0'}} w_2 w_1 \mu$ with $w_2, w_3 \in W_{M_{\beta_{j_0'}}}$. Then we have

$$(s_{\beta_{j_0'}}(w_1\mu)_{M_{\beta_{j_0'}}-dom})_{M_{\beta_{j_0'}}-dom} \succcurlyeq s_{\beta_{j_0'}} w_2 w_1 \mu.$$

Since both sides are in $W\mu$ with difference a linear combination of coroots in $M_{\beta_{j_0'}}$, we have

$$\bigcup_\alpha R_\alpha \ni (s_{\beta_{j_0'}}(w_1\mu)_{M_{\beta_{j_0'}}-dom})_{M_{\beta_{j_0'}}-dom} = (s_{\beta_{j_0'}} w_2 w_1 \mu)_{M_{\beta_{j_0'}}-dom} \succcurlyeq w\mu.$$

By the maximality of $w\mu$, we deduce that $w\mu$ is both $M_{\beta_{j_0}}$-dominant and $M_{\beta_{j_0'}}$-dominant. Therefore

$$w\mu = (s_{\beta_{j_0'}} w_2 w_1 \mu)_{M_{\beta_{j_0}}-dom}$$

is G-dominant and Claim 2 follows.

Let $(I_i)_{0 \leq i \leq r}$ be the increasing sequence of subsets in Δ_G as in Lemma 4.7. Suppose i_0 is the smallest integer such that $\{\beta_j | j \in J_1\} \cap I_{i_0}$ is not an empty set. Choose $\tilde{\beta} \in \{\beta_j | j \in J_1\} \cap I_{i_0}$. Let $\tilde{\alpha} := \tilde{\beta}|_A$. By the same arguments as before we can verify that $\tilde{\alpha}$ satisfies the condition (4.0.1). For the condition (4.0.2), let

$$\mathrm{Av}_{M_{\tilde{\alpha}}}(\tilde{\beta}^\vee) := \sum_{j \in J} m_j \beta_j^\vee \text{ in } \pi_1(M_b)_\Gamma,$$

with $m_j \in \mathbb{Q}$ for all $j \in J$. By Lemma 4.7, $0 \leq m_j \leq 2$ (as E_8 case will not occur) for all $j \in J$, and $0 \leq m_j \leq 1$ for all $j \in J_1$. Then

$$
\begin{aligned}
& \mathrm{Av}_\Gamma(w\mu) - \kappa_{M_b}(b_{M_b}) - \mathrm{Av}_{W_{M_{\tilde{\alpha}}}} \mathrm{Av}_\Gamma(\tilde{\beta}^\vee) \\
=\quad & (w\mu - w_1\mu) - \mathrm{Av}_{W_{M_{\tilde{\alpha}}}}(\tilde{\beta}^\vee) \\
=\quad & \sum_{j \in J}(n_j - m_j)\beta_j^\vee \\
\succcurlyeq_{M_b}\quad & \sum_{j \in J_1}(1 - m_j)\beta_j^\vee + \sum_{j \in J\backslash J_1}(2 - m_j)\beta_j^\vee \\
\succcurlyeq_{M_b}\quad & 0.
\end{aligned}
$$

$\square$

Lemma 4.7. *Suppose G is quasi-split. Let $\beta \in \Delta_G$. Suppose*

$$\mathrm{Av}_{M_\beta}\beta^\vee = \beta^\vee + \sum_{\gamma \in (\Delta_{M_\beta})_\Gamma} n_{\beta,\gamma}\gamma^\vee \text{ in } X_*(T)_{\Gamma,\mathbb{Q}}.$$

Then

(1) $0 \leq n_{\beta,\gamma} \leq 3$ *for all* $\gamma \in (\Delta_G)_\Gamma$. *Moreover, if all the connected components of the Dynkin diagram of G is not of type E_8, then $0 \leq n_{\beta,\gamma} \leq 2$ for all $\gamma \in (\Delta_G)_\Gamma$.*

(2) *There exist $r \in \mathbb{N}_{\geq 1}$ and an increasing sequence of Γ-invariant subsets in Δ_G*

$$\emptyset = I_0 \subset I_1 \subset \cdots \subset I_{r-1} \subset I_r = \Delta_G$$

such that if $\beta \in I_i$ for some i, then $n_{\beta,\gamma} \leq 1$ for all $\gamma \notin (I_{i-1})_\Gamma$. In particular, if $r = 1$, then $n_{\beta,\gamma} \leq 1$ for all $\beta \in \Delta_G$, $\gamma \in (\Delta_G)_\Gamma$.

Proof. This lemma only depends on the absolute root system of G with Galois action. After considering separately each connected component of the Dynkin diagram of G, we may assume the Dynkin diagram of G is connected. The first assertion can be checked directly case by case. For the second assertion, we list the increasing sequence of Γ-invariant subsets in Δ_G case by case according to the type of Dynkin diagram of G. We can check directly that this increasing

sequence of subsets satisfies the desired property. We left the details of the verification to the readers.

Case A_n: In the ^1A_n case or ^2A_n case with n even, take $r = 1$. Otherwise, we are in the ^2A_n case with n odd, then take $r = 2$ and $I_1 = \Delta_G\backslash\{\beta\}$ where β is the unique Γ-invariant root in Δ_G.

*Case B_n:* take $r = 2$ and I_1 is the subset of long roots in Δ_G.

Case C_n: take $r = 1$.

Case D_n: In the ^1D_n case with $n > 4$, take $r = 2$ and $\Delta_G\backslash I_1$ consists of two roots which are the end points of the Dynkin diagram and are neighbors to the same simple root.

In the ^2D_n case or $^1D_4$ case, take $r = 1$.

In the $^3D_4$ case, take $r = 2$ and $I_1 = \Delta_G\backslash\{\beta\}$ where β is the unique Γ-invariant root in Δ_G.

Case E_n: Suppose the Dynkin diagram of E_8 is as follows:
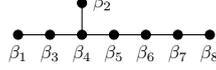
In the E_7 (resp. E_6) case, we remove β_8 (resp. β_7 and β_8) in the diagram.

In the $^1E_6$ case, take $r = 3$, $I_1 = \{\beta_4\}$, $I_2 = \{\beta_2, \beta_3, \beta_4, \beta_5\}$.

In the $^2E_6$ case, take $r = 4$, $I_1 = \{\beta_3, \beta_5\}$, $I_2 = I_1 \cup \{\beta_1, \beta_6\}$, $I_3 = I_2 \cup \{\beta_4\}$.

In the E_7 case, take $r = 4$, $I_1 = \{\beta_4\}$, $I_2 = \{\beta_3, \beta_4, \beta_5\}$, $I_3 = \Delta_G\backslash\{\beta_7\}$.

In the E_8 case, take $r = 5$, $I_1 = \{\beta_4\}$, $I_2 = \{\beta_4, \beta_5\}$, $I_3 := I_2 \cup \{\beta_3, \beta_6\}$, $I_4 = \Delta_G\backslash\{\beta_8\}$.

Case F_4: Suppose the Dynkin diagram is as follows:
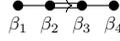
Take $r = 3$, $I_1 = \{\beta_2\}$, $I_2 = \{\beta_1, \beta_2, \beta_3\}$.

Case G_2: Take $r = 2$, I_1 is the set of unique long root in Δ_G. $\qquad\square$

Now we can prove Proposition 4.5.

Proof of Proposition 4.5. As b is not basic, there exist $\alpha \in \Delta_0$, $w \in W$ and $x \in \mathcal{F}(M_{\tilde\alpha}, w\mu)(C)$ satisfies the properties (1)-(3) in Lemma 4.6 which follows. Let $b_{M_{\tilde\alpha}}$ and $b'_{M_{\tilde\alpha}}$ as in loc. cit.

Let $x_G \in \mathcal{F}(G, \mu)(C)$ be the image of x via the natural morphism:

$$
\begin{array}{ccc}
\mathcal{F}(M_{\tilde\alpha}, w\mu)(C) & \to & \mathcal{F}(G, \mu)(C) \\
a(M_{\tilde\alpha} \cap wP_\mu w^{-1}) & \mapsto & awP_\mu
\end{array}
$$

Then $\mathcal{E}_{b,x_G} \simeq \mathcal{E}_{b'_{M_{\tilde\alpha}}} \times_{M_{\tilde\alpha}} G \simeq \mathcal{E}_{b'}$ where $[b'] \in B(G)$ is the image of $[b'_{M_{\tilde\alpha}}] \in B(M_{\tilde\alpha})$ via the natural map $B(M_{\tilde\alpha}) \to B(G)$. Hence $x_G \notin \mathcal{F}^a(C)$. Moreover, $x_G \notin \mathcal{F}^{wa}(C)$. Indeed, the canonical reduction $(\mathcal{E}_{b'})_{P_{\tilde\alpha}}$ of $\mathcal{E}_{b'}$ to the standard parabolic subgroup $P_{\tilde\alpha}$ corresponding to $M_{\tilde\alpha}$ induces the reduction $\mathcal{E}_{b_{M_{\tilde\alpha}}} \times_{M_{\tilde\alpha}} P_{\tilde\alpha}$ of $\mathcal{E}_b$ to $P_{\tilde\alpha}$. Take $\chi \in X^*(P_{\tilde\alpha}/Z_G)^+$, then $\deg\chi_*(\mathcal{E}_{b'})_{P_{\tilde\alpha}} > 0$.

For any element $\gamma \in \tilde{J}_b(K)$, we have $\gamma(x_G) \notin \mathcal{F}^a(C)$. Choose $\gamma \in \tilde{J}_b^{\geq \lambda_{max}}(K)\backslash\{1\}$ with $\lambda_{max} = \max_{\gamma \in \Phi}\langle \nu_b, \gamma\rangle$, it remains to show that $\gamma(x_G) \in \mathcal{F}^{wa}(C)$.

Suppose the pair $(b, \gamma(x_G))$ is not weakly admissible. There exist a standard maximal parabolic subgroup Q, a reduction b_{M_Q} of b to the Levi component M_Q of Q and $\chi \in X^*(Q/Z_G)^+$ such that

$$\deg\chi_*(\mathcal{E}_{b,\gamma(x_G)})_Q > 0,$$

where $(\mathcal{E}_{b,\gamma(x_G)})_Q$ is the reduction of $\mathcal{E}_{b,\gamma(x_G)}$ to Q induced by a reduction $\mathcal{E}_{b,Q}^\gamma$ of $\mathcal{E}_b$ to Q, where $\mathcal{E}_{b,Q}^\gamma := \mathcal{E}_{\tilde{b}_{M_Q}} \times_{M_Q} Q$ is induced by a reduction $\tilde{b}_{M_Q}$ of b to M_Q (and hence to Q). The isomorphism $\gamma : \mathcal{E}_b \xrightarrow{\sim} \mathcal{E}_b$ induces an isomorphism $\mathcal{E}_b/Q \xrightarrow{\sim} \mathcal{E}_b/Q$, hence by Remark 2.2, $\mathcal{E}_{b,Q}$ induces a reduction $\mathcal{E}_{b,Q}^\gamma$ of $\mathcal{E}_b$ to Q and an isomorphism $\mathcal{E}_{b,Q} \xrightarrow{\sim} \mathcal{E}_{b,Q}^\gamma$ satisfying the commutative diagram

$$
\begin{array}{ccc}
\mathcal{E}_b & \xrightarrow[\sim]{\gamma} & \mathcal{E}_b \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\
\mathcal{E}_{b,Q} \times_Q G & \xrightarrow{\sim} & \mathcal{E}_{b,Q}^\gamma \times_Q G
\end{array}
$$

Suppose the reduction $(\mathcal{E}_{b,x_G})_Q$ to Q of $\mathcal{E}_{b,x_G}$ is induced by the reduction $(\mathcal{E}_{b,\gamma(x_G)})_Q$ to Q of $\mathcal{E}_{b,\gamma(x_G)}$ by Lemma 1.12 and Remark 2.2. We get a cubic commutative diagram

where the vertices of the front face are the G-bundles and vertices of the back face are the corresponding reductions of G-bundles to Q, the vertical waved arrows denote the modification of G or Q-bundles. It follows that $\deg\chi_*(\mathcal{E}_{b,x_G})_Q = \deg\chi_*(\mathcal{E}_{b,\gamma(x_G)})_Q > 0$.

According to theorem 2.7 the vector

$$v : X^*(Q/Z_G) \longrightarrow \mathbb{Z}$$
$$\chi \longmapsto \deg\chi_*(\mathcal{E}_{b,x_G})_Q$$

seen as an element of $X_*(A)_\mathbb{Q}$ satisfies $v \leq \nu_{\mathcal{E}_{b,x_G}} = \nu_{\mathcal{E}_{b'}}$ with $v \neq 0$. As $\nu_{\mathcal{E}_{b'}} \in \mathcal{N}_G\backslash\{0\}$ is minimal, one deduces that $v = \nu_{b,x_G}$, $Q = P_{\tilde{\alpha}}$ and $(\mathcal{E}_{b,x_G})_Q$ is the HN canonical reduction of $\mathcal{E}_{b,x_G}$. Therefore $\mathcal{E}_{b,Q} = \mathcal{E}_{b_{M_{\tilde{\alpha}}}} \times_{M_{\tilde{\alpha}}} P_{\tilde{\alpha}}$ to Q of $\mathcal{E}_b$. Pushing forward via the natural projection $P_{\tilde{\alpha}} \to M_{\tilde{\alpha}}$, the isomorphism of $P_{\tilde{\alpha}}$-bundles $\mathcal{E}_{b,P_{\tilde{\alpha}}} \simeq \mathcal{E}^\gamma_{b,P_{\tilde{\alpha}}}$ induces an isomorphism of $M_{\tilde{\alpha}}$-bundles, hence we have $[b_{M_{\tilde{\alpha}}}] = [\tilde{b}_{M_Q}] \in B(M_{\tilde{\alpha}})$. According to the following lemma 4.8, the two reductions $b_{M_{\tilde{\alpha}}}$ and $\tilde{b}_{M_{\tilde{\alpha}}} := \tilde{b}_{M_Q}$ of b to M_Q are equivalent. In particular, the reductions $\mathcal{E}_{b,Q}$ and $\mathcal{E}^\gamma_{b,Q}$ of $\mathcal{E}_b$ to Q are equivalent. Hence they give the same sub-vector bundle

$$\mathcal{E}_{b,Q} \times_{Q,\mathrm{Ad}} \mathrm{Lie}Q = \mathcal{E}^\gamma_{b,Q} \times_{Q,\mathrm{Ad}} \mathrm{Lie}Q$$

over X of $\mathrm{Ad}(\mathcal{E}_b) = \mathcal{E}_b \times_{G,\mathrm{Ad}} \mathfrak{g}$. By theorem 1.8, this sub-vector bundle corresponds to a sub-$\bar{B}$-module $\mathfrak{q}$ in $\mathfrak{g}_{\bar{B}}$ which is stable under the action of $\mathrm{Ad}(\gamma)$, where we identify $\tilde{J}_b(K)$ with a subgroup of $G(\bar{B})$ (cf. section 1.5). Recall that in loc. cit., we have also defined a filtration $(\mathfrak{g}_{\bar{B}}^{\geq\lambda})_{\lambda\in\mathbb{Q}}$ on $\mathfrak{g}_{\bar{B}}$. As $\nu_{b_{M_{\tilde{\alpha}}}} = \nu_b$, the non-zero elements of $\mathfrak{q}$ are in the $-\langle\gamma,\nu_b\rangle$ graded pieces of $\mathfrak{g}_{\bar{B}}$ for some absolute root γ in $Q = P_{\tilde{\alpha}}$. In particular $\mathfrak{q} \cap \mathfrak{g}_{\bar{B}}^{\geq\lambda_{max}} = 0$. On the other side, since $\gamma \neq 1$, we can always choose an element $y \in \mathfrak{q} \cap \mathfrak{g}_{\bar{B}}^{\geq 0}$ such that $\mathrm{Ad}(\gamma)(y) \neq y$. Note that

$$0 \neq \mathrm{Ad}(\gamma)(y) - y \in \mathfrak{g}_{\bar{B}}^{\geq\lambda_{max}},$$

which implies $\mathfrak{q} \cap \mathfrak{g}_{\bar{B}}^{\geq\lambda_{max}} \neq 0$. We get a contradiction. $\square$

Lemma 4.8. *Suppose* $b \in G(\breve{F})$. *Suppose* $\tilde{\alpha} \in \Delta_0$ *such that* $\langle\tilde{\alpha},\nu_b\rangle > 0$. *Let* $(b_{M_{\tilde{\alpha}}},g)$ *and* $(\tilde{b}_{M_{\tilde{\alpha}}},\tilde{g})$ *be two reductions of* b *to* $M_{\tilde{\alpha}}$ *with* $[b_{M_{\tilde{\alpha}}}] = [\tilde{b}_{M_{\tilde{\alpha}}}] \in B(M_{\tilde{\alpha}})$ *and* $\nu_{b_{M_{\tilde{\alpha}}}} = \nu_b$. *Then the two reductions* $(b_{M_{\tilde{\alpha}}},g)$ *and* $(\tilde{b}_{M_{\tilde{\alpha}}},\tilde{g})$ *are equivalent.*

Proof. As $[b_{M_{\tilde{\alpha}}}] = [\tilde{b}_{M_{\tilde{\alpha}}}] \in B(M_{\tilde{\alpha}})$, we may assume $b_{M_{\tilde{\alpha}}} = \tilde{b}_{M_{\tilde{\alpha}}}$. Then

$$\tilde{g}g^{-1} \in J_b = \{h \in G(\breve{F})|b\sigma(h) = hb\}.$$

Since $\langle\tilde{\alpha},\nu_b\rangle > 0$, $J_b \subseteq M_{\tilde{\alpha}}(\breve{F})$. It follows that $(b_{M_{\tilde{\alpha}}},g)$ and $(\tilde{b}_{M_{\tilde{\alpha}}},\tilde{g})$ are equivalent. $\square$

5. Newton stratification and weakly admissible locus

In this section, we suppose G is quasi-split and $[b] \in A(G,\mu)$ is basic. Under this condition, the proof of [6] Theorem 6.1 in fact shows the following finer result.

Theorem 5.1 ([6]). *Let* $[b'] \in B(G,\kappa_G(b) - \mu^\sharp, \nu_b\mu^{-1})$.
(1) *If* $(G,\nu_b(w_0\mu^{-1})^\diamond,b')$ *is HN-decomposable, then* $\mathcal{F}(G,b,\mu)^{[b']} \cap \mathcal{F}(G,b,\mu)^{wa} = \emptyset$;
(2) *If* $(G,\nu_b(w_0\mu^{-1})^\diamond,b')$ *is HN-indecomposable and* $[b']$ *a minimal element in the set* $B(G,\kappa_G(b)-\mu^\sharp, \nu_b\mu^{-1})\backslash[1]$ *for the Bruhat order, then* $\mathcal{F}(G,b,\mu)^{[b']} \cap \mathcal{F}(G,b,\mu)^{wa} \neq \emptyset$.

Inspired by this theorem, we have the following conjecture.

Conjecture 5.2. *Suppose* $[b'] \in B(G, \kappa_G(b) - \mu^\sharp, \nu_b \mu^{-1})$ *with* $(G, \nu_b(w_0\mu^{-1})^\diamond, b')$ *HN-indecomposable, then*

$$\mathcal{F}(G, b, \mu)^{[b']} \cap \mathcal{F}(G, b, \mu)^{wa} \neq \emptyset.$$

In the rest of the section, we will prove this conjecture for the linear algebraic groups for special μ.

For $r, s \in \mathbb{Z}$ with $r > 0$, let $\mathcal{O}([\frac{s}{r}]) := \mathcal{O}(\frac{s}{r})^d$ if $d = (s, r)$. Then $\deg\mathcal{O}([\frac{s}{r}]) = s$ and $\mathrm{rank}\mathcal{O}([\frac{s}{r}]) = r$.

Proposition 5.3. *Let* $G = \mathrm{GL}_n$. *Suppose* $[b'] \in B(G, \kappa_G(b) - \mu^\sharp, \nu_b\mu^{-1})$ *with* $(G, \nu_b(w_0\mu^{-1})^\diamond, b')$ *HN-indecomposable. If* $\mathcal{E}_{b'} \simeq \mathcal{O}([\frac{2}{r_1}]) \oplus \mathcal{O}([\frac{-2}{r_2}])$ *or* $\mathcal{O}([\frac{1}{r_1}]) \oplus \mathcal{O}([\frac{0}{r_2}]) \oplus \mathcal{O}([\frac{-1}{r_3}])$ *for some* $r_1, r_2, r_3 > 0$, *then*

$$\mathcal{F}(G, b, \mu)^{[b']} \cap \mathcal{F}(G, b, \mu)^{wa} \neq \emptyset.$$

In particular, the conjecture 5.2 holds when $\mu = (1^{(r)}, 0^{(n-r)})$ *with* $r(n - r) \leq 2n$.

Proof. We claim that there exists an exact sequence of vector bundles

$$(5.0.1) \hspace{3cm} 0 \to \mathcal{E}' \to \mathcal{E}_b \to \mathcal{E}'' \to 0$$

satisfying a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{E}_b & \longrightarrow & \mathcal{E}'' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \tilde{\mathcal{E}}' & \longrightarrow & \mathcal{E}_{b'} & \longrightarrow & \tilde{\mathcal{E}}'' & \longrightarrow & 0
\end{array}
$$

where $\mathcal{E}'$ and $\mathcal{E}''$ are semi-stable vector bundles and the vertical arrows are the modifications of minuscule type. Indeed, suppose $\mathcal{E}_b = \mathcal{O}([\frac{s}{r}])$. If $\mathcal{E}_{b'} \simeq \mathcal{O}([\frac{2}{r_1}]) \oplus \mathcal{O}([\frac{-2}{r_2}])$, then $r = r_1 + r_2$. Let $\tilde{\mathcal{E}}' = \mathcal{O}([\frac{2}{r_1}])$, $\tilde{\mathcal{E}}'' = \mathcal{O}([\frac{-2}{r_2}])$. If $s \leq r_2$, then let $\mathcal{E}' = \mathcal{O}([\frac{2}{r_1}])$ and $\mathcal{E}'' = \mathcal{O}([\frac{s-2}{r_2}])$. Otherwise, let $\mathcal{E}' = \mathcal{O}([\frac{s+2-r_2}{r_1}])$ and $\mathcal{E}'' = \mathcal{O}([\frac{r_2-2}{r_2}])$. If $\mathcal{E}'_b \simeq \mathcal{O}([\frac{1}{r_1}]) \oplus \mathcal{O}([\frac{0}{r_2}]) \oplus \mathcal{O}([\frac{-1}{r_3}])$, then $r_1 + r_2 + r_3 = r$. We can easily check that one of the following two equalities holds:

$$\frac{s-1}{r-r_1} \leq \frac{r_3-1}{r_3}, \quad \frac{s-r_3+1}{r-r_3} \geq \frac{1}{r_1}.$$

(Otherwise, the equalities give upper and lower bounds for s. The comparison of the two bounds leads to a contradiction $r < r_1 + r_3$). In the former case, let

$$
\begin{array}{llll}
\mathcal{E}' & = \mathcal{O}([\frac{1}{r_1}]), & \mathcal{E}'' & = \mathcal{O}([\frac{s-1}{r-r_1}]), \\
\tilde{\mathcal{E}}' & = \mathcal{O}([\frac{1}{r_1}]), & \tilde{\mathcal{E}}'' & = \mathcal{O}([\frac{0}{r_2}]) \oplus \mathcal{O}([\frac{-1}{r_3}])
\end{array}
$$

In the later case, let

$$
\begin{array}{llll}
\mathcal{E}' & = \mathcal{O}([\frac{s-r_3+1}{r-r_3}]), & \mathcal{E}'' & = \mathcal{O}([\frac{r_3-1}{r_3}]), \\
\tilde{\mathcal{E}}' & = \mathcal{O}([\frac{1}{r_1}]) \oplus \mathcal{O}([\frac{0}{r_2}]), & \tilde{\mathcal{E}}'' & = \mathcal{O}([\frac{-1}{r_3}])
\end{array}
$$

The existence of the extension 5.0.1 is due to [1], and the existence of the modifications given by the left and right vertical arrows are given by Proposition 1.13. The vertical arrow in the middle of the commutative diagram gives a point in $x \in \mathcal{F}(G, b, \mu)^{[b']}(C)$. It suffices to show $x \in \mathcal{F}^{wa}(C)$. Suppose $\mathcal{G} \subseteq \mathcal{E}_{b'}$ is any sub-vector bundle of $\mathcal{E}_{b'}$ corresponding to a reduction $(\mathcal{E}_{b,x})_P$ of $\mathcal{E}_{b'}$ to a maximal parabolic subgroup P such that $\deg\mathcal{G} > 0$. Let $\mathcal{G}' \subseteq \mathcal{E}_b$ be the corresponding sub-vector bundle of $\mathcal{E}_b$ corresponding to the reduction $(\mathcal{E}_b)_P$ induced by $(\mathcal{E}_{b,x})_P$. We want to show that $\mathcal{G}'$ does not come from subisocrystals.

Suppose $\mathcal{G}'$ comes from subisocrystals. The fact that $\deg\mathcal{G} > 0$ combined with the particular choice of b' implies that $\mathcal{G} \supseteq \tilde{\mathcal{E}}'$ or $\mathcal{G} \subseteq \tilde{\mathcal{E}}'$. Then $\mathcal{G}' \supseteq \mathcal{E}'$ or $\mathcal{G}' \subseteq \mathcal{E}'$. The later case is obviously impossible. For the former one, $\mathcal{E}'' = \mathcal{E}_b/\mathcal{E}'$ must have a direct summand of slope $\frac{s}{r}$ which is also impossible as $\mathcal{E}''$ is semi-stable. $\qquad\square$

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