$F[x,y]$ AS A DIALGEBRA AND A LEIBNIZ ALGEBRA

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ABSTRACT. We define a new associative dialgebra over a polynomial algebra $F[x,y]$ with two indeterminates x and y. Left derivations, right derivations, derivations and automorphisms of $F[x,y]$ as associative dialgebra are determined. Meanwhile, we also determine all homogeneous derivations of $F[x,y]$ as $\mathbb{Z}$-graded Leibniz algebra, and automorphisms of the Leibniz algebra $F[x,y]$ preserving the standard filtration.
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1. INTRODUCTION

The theory of Leibniz algebras has been actively studied by many mathematicians for years (cf. [1-12]). It is well-known that any associative algebra gives rise to a Lie algebra by $[x,y] = xy - yx$. It is J.-L. Loday who introduced a new notion (cf. [11]), namely, associative dialgebra, which gives (by similar procedure) a Leibniz algebra. In this paper, we introduce an associative dialgebraic structure on a polynomial algebra space $F[x,y]$ over a field F with two variables x and y. Thus $F[x,y]$ also becomes a Leibniz algebra. The purpose of this paper is to study derivation algebras and automorphism groups of $F[x,y]$ as an associative dialgebra and as a Leibniz algebra.

§ 1.1 Basic notations and results.

In this subsection we first recall some basic conceptions and notations which are all standard (cf. [11]).

Let $\mathcal{D}$ be a vector space over a field F equipped with two associative multiplications $\dashv$ and $\vdash$: $\mathcal{D} \times \mathcal{D} \to \mathcal{D}$. If for any $x, y, z \in \mathcal{D}$, the following conditions hold

$$x \dashv (y \dashv z) = x \dashv (y \vdash z), \tag{1.1}$$

$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \tag{1.2}$$

$$(x \vdash y) \vdash z = (x \dashv y) \vdash z, \tag{1.3}$$

then $(\mathcal{D}, \dashv, \vdash)$ is called an associative dialgebra (or dialgebra for short) (where $\dashv$ and $\vdash$ are called left multiplication and right multiplication, respectively).

Obviously, associative dialgebras merge into associative algebras if $a \dashv b = a \vdash b = ab$.

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Let $\mathcal{D}$ be a dialgebra. An element $e \in \mathcal{D}$ is called a bar-unit of $\mathcal{D}$ if e satisfies

$$x \dashv e = x = e \vdash x, \quad \forall\, x \in \mathcal{D}.$$

The set of bar-units of $\mathcal{D}$ is called its halo.

Suppose that $\mathcal{D}_1$ and $\mathcal{D}_2$ are two dialgebras over a field F, and φ is a linear map from $\mathcal{D}_1$ to $\mathcal{D}_2$. If for any $a, b \in \mathcal{D}_1$,

$$\begin{aligned}\varphi(a \dashv b) =& \varphi(a) \dashv \varphi(b),\\ \varphi(a \vdash b) =& \varphi(a) \vdash \varphi(b),\end{aligned}$$

then φ is called a homomorphism of dialgebras from $\mathcal{D}_1$ to $\mathcal{D}_2$.

Isomorphism of dialgebras and automorphism of dialgebra can be defined similarly.

Suppose that I is a subspace of a dialgebra $\mathcal{D}$. If for any $a, b \in I$, we have $a \vdash b \in I$ and $a \dashv b \in I$, then I is called a sub-dialgebra of $\mathcal{D}$.

Suppose that I is a subspace of a dialgebra $\mathcal{D}$. If for any $a \in \mathcal{D}$, $b \in I$, we have $a \vdash b \in I$, $b \vdash a \in I$, $b \dashv a \in I$ and $a \dashv b \in I$, then I is called an ideal of $\mathcal{D}$.

If I is an ideal of a dialgebra $\mathcal{D}$, then on the quotient space $\mathcal{D}/I$ we can define a natural dialgebra structure, the dialgebra $\mathcal{D}/I$ is called the quotient dialgebra determined by I. It is easy to verify that the kernel of any homomorphism of dialgebras is an ideal.

Let $\mathcal{G}$ be a vector space over a field F. If $\mathcal{G}$ is equipped with a multiplication $[-,-]$: $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$, and satisfies the Leibniz identity

$$[a, [b, c]] = [[a, b], c] - [[a, c], b], \quad \forall\, a, b, c \in \mathcal{G},$$

then $(\mathcal{G}, [-,-])$ is called a Leibniz algebra.

Suppose that $\mathcal{G}$ is a Leibniz algebra, J is a subspace of $\mathcal{G}$. If $[\mathcal{G}, J] \subseteq J$, then J is called a left ideal. If $[J, \mathcal{G}] \subseteq J$, then J is called a right ideal of $\mathcal{G}$. If J is both a left and right ideal of $\mathcal{G}$, then J is called an (two-sided) ideal of $\mathcal{G}$.

For a Leibniz algebra $\mathcal{G}$, one puts

$$Z^r(\mathcal{G}) = \{a \in \mathcal{G} \mid [x, a] = 0,\ x \in \mathcal{G}\},$$

then $Z^r(\mathcal{G})$ is an ideal of $\mathcal{G}$ and is called the right annihilator of $\mathcal{G}$.

Let $\mathcal{G}$ be a Leibniz algebra. The lower central series of $\mathcal{G}$ is the sequence

$$\mathcal{G}^1 \supseteq \mathcal{G}^2 \supseteq \cdots \supseteq \mathcal{G}^n \supseteq \cdots$$

of ideals of $\mathcal{G}$ defined inductively as follows

$$\mathcal{G}^1 = \mathcal{G},\ \mathcal{G}^{k+1} = [\mathcal{G}^k, \mathcal{G}], \quad k \in \mathbb{N}.$$

$\mathcal{G}$ is said to be nilpotent if there is an integer $s > 1$, such that $\mathcal{G}^s = 0$.

The derived series of a Leibniz algebra $\mathcal{G}$ is the sequence

$$\mathcal{G}^{(0)} \supseteq \mathcal{G}^{(1)} \supseteq \cdots \supseteq \mathcal{G}^{(n)} \supseteq \cdots$$

of ideals of $\mathcal{G}$ defined inductively as follows

$$\mathcal{G}^{(0)} = \mathcal{G},\ \mathcal{G}^{(k+1)} = [\mathcal{G}^{(k)}, \mathcal{G}^{(k)}], \quad k \in \mathbb{Z}_+.$$

$\mathcal{G}$ is said to be solvable if there is an integer $s > 0$, such that $\mathcal{G}^{(s)} = 0$.

Assume that $\mathcal{G}$ is a Leibniz algebra, $d \in \mathrm{End}(\mathcal{G})$. If d satisfies:

$$d[a,b] = [d(a), b] + [a, d(b)], \quad \forall\, a, b \in \mathcal{G},$$

then d is called a derivation of the Leibniz algebra $\mathcal{G}$. We denote $\mathrm{der}(\mathcal{G})$ the set of all derivations of $\mathcal{G}$.

Suppose that $\mathcal{G}$ is a Leibniz algebra, and $x \in \mathcal{G}$. We define a linear transformation $\mathrm{ad}\, x$ of $\mathcal{G}$ as follows

$$\mathrm{ad}\, x(y) = [y, x], \quad \forall\, y \in \mathcal{G}.$$

Then it is easy to verify that $\mathrm{ad}\, x$ is a derivation of $\mathcal{G}$, and is called an inner derivation of $\mathcal{G}$. We denote $\mathrm{ad}(\mathcal{G})$ the set of all inner derivations of $\mathcal{G}$.

Suppose that $(\mathcal{D}, \dashv, \vdash)$ is an associative dialgebra over F, we can define a new multiplication $[-,-]$ on $\mathcal{D}$:

$$[a, b] = a \dashv b - b \vdash a.$$

Then for any $a, b, c \in \mathcal{D}$, we have

$$[a, [b, c]] = [[a, b], c] - [[a, c], b].$$

According to the definition of Leibniz algebra, $(\mathcal{D}, [-,-])$ is a Leibniz algebra.

§ 1.2 Dialgebra and Leibniz algebra $F[x,y]$.

Suppose that $F[x,y]$ is a polynomial algebra over F with commuting indeterminates x and y, where F is any field of characteristic 0. In this subsection, we introduce an associative dialgebraic structure over the vector space $F[x,y]$ and discuss some fundamental properties of $F[x,y]$ as a dialgebra and a Leibniz algebra.

We define two multiplications $\dashv$ and $\vdash$ on $F[x,y]$ as follows

$$\begin{aligned} f(x,y) \dashv g(x,y) =& f(x,y)g(y,y), \\ f(x,y) \vdash g(x,y) =& f(x,x)g(x,y). \end{aligned} \tag{1.4}$$

Then we have

Theorem 1.2.1. $(F[x,y], \dashv, \vdash)$ *is an associative dialgebra, where the left multiplication* $\dashv$ *and the right multiplication* $\vdash$ *are defined as (1.4).*

Proof. Since $\{x^m y^n \mid m, n \in \mathbb{Z}_+\}$ is a basis of $F[x,y]$ it is sufficient to prove (1.1)-(1.3) on basis elements. By (1.4), for any $x^m y^n$, $x^s y^t$, $x^i y^j \in F[x,y]$, we have

$$\begin{aligned} x^m y^n \dashv (x^s y^t \dashv x^i y^j) &= x^m y^n \dashv x^s y^{t+i+j} = x^m y^{n+s+t+i+j}, \\ (x^m y^n \dashv x^s y^t) \dashv x^i y^j &= x^m y^{n+s+t} \dashv x^i y^j = x^m y^{n+s+t+i+j}. \end{aligned}$$

Thus

$$x^m y^n \dashv (x^s y^t \dashv x^i y^j) = (x^m y^n \dashv x^s y^t) \dashv x^i y^j.$$

Since

$$x^m y^n \vdash (x^s y^t \vdash x^i y^j) = x^m y^n \vdash x^{s+t+i} y^j = x^{m+n+s+t+i} y^j,$$
$$(x^m y^n \vdash x^s y^t) \vdash x^i y^j = x^{m+n+s} y^t \vdash x^i y^j = x^{m+n+s+t+i} y^j,$$

we have

$$x^m y^n \vdash (x^s y^t \vdash x^i y^j) = (x^m y^n \vdash x^s y^t) \vdash x^i y^j.$$

Thus both $\dashv$ and $\vdash$ satisfy the associative law. On the other hand

$$x^m y^n \dashv (x^s y^t \dashv x^i y^j) = x^m y^n \dashv x^s y^{t+i+j} = x^m y^{n+s+t+i+j},$$
$$x^m y^n \dashv (x^s y^t \vdash x^i y^j) = x^m y^n \dashv x^{s+t+i} y^j = x^m y^{n+s+t+i+j},$$

thus (1.1) holds. But

$$(x^m y^n \vdash x^s y^t) \dashv x^i y^j = x^{m+n+s} y^t \dashv x^i y^j = x^{m+n+s} y^{t+i+j},$$
$$x^m y^n \vdash (x^s y^t \dashv x^i y^j) = x^m y^n \vdash x^s y^{t+i+j} = x^{m+n+s} y^{t+i+j},$$

then (1.2) holds. Furthermore

$$(x^m y^n \vdash x^s y^t) \vdash x^i y^j = x^{m+n+s} y^t \vdash x^i y^j = x^{m+n+s+t+i} y^j,$$
$$(x^m y^n \dashv x^s y^t) \vdash x^i y^j = x^m y^{n+s+t} \vdash x^i y^j = x^{m+n+s+t+i} y^j,$$

thus (1.3) holds. Therefore $(F[x,y], \dashv, \vdash)$ is an associative dialgebra. □

Since the left (and right) multiplication of the associative dialgebra $F[x,y]$ satisfies associative law, $(F[x,y], \dashv)$ and $(F[x,y], \vdash)$ are all associative algebras.

From the above subsection, we know that any associative dialgebra can be turned naturally into a Leibniz algebra. Thus $F[x,y]$ can be considered as a Leibniz algebra, its Leibniz bracket is given by

$$[f(x,y), g(x,y)] = f(x,y)\big(g(y,y) - g(x,x)\big), \tag{1.5}$$

or on its basis, we have

$$[x^m y^n, x^s y^t] = x^m y^n (y^{s+t} - x^{s+t}).$$

Lemma 1.2.2. *Let $F[x,y]$ be the Leibniz algebra defined as above, then $F[x,y]^{(1)} = (y-x)F[x,y]$.*

Proof. Obviously, $(y-x) \mid g(y,y) - g(x,x)$, for any $g(x,y) \in F[x,y]$, thus by (1.5),

$$F[x,y]^{(1)} = \big[F[x,y], F[x,y]\big] \subseteq (y-x)F[x,y].$$

On the other hand, for any $f(x,y) \in F[x,y]$, $(y-x)f(x,y) = [f(x,y), x] \in F[x,y]^{(1)}$, so $(y-x)F[x,y] \subseteq F[x,y]^{(1)}$. Hence, $F[x,y]^{(1)} = (y-x)F[x,y]$. □

Theorem 1.2.3. *$F[x,y]$ is a solvable Leibniz algebra, but not a nilpotent Leibniz algebra.*

Proof. By Lemma 1.2.2, we have $F[x,y]^{(1)} = (y-x)F[x,y]$. Since

$$[(y-x)f(x,y), (y-x)g(x,y)] = 0, \quad \forall\, f(x,y), g(x,y) \in F[x,y],$$

thus $F[x,y]^{(2)} = 0$. Hence $F[x,y]$ is solvable.

However, for any integer $n > 0$,

$$[\cdots[[1, \underbrace{x], x], \cdots, x}_{n}] = (y-x)^n.$$

Thus, $F[x,y]^n \neq 0$, for any $n > 0$. So $F[x,y]$ is not nilpotent. □

Lemma 1.2.4. *Let $Z^r(F[x,y])$ be the right annihilator of the Leibniz algebra $F[x,y]$, then*

$$Z^r(F[x,y]) = \{a + (y-x)h(x,y) \mid a \in F,\ h(x,y) \in F[x,y]\}.$$

Proof. For any $a \in F$, and $f(x,y), h(x,y) \in F[x,y]$, we have

$$[f(x,y), a + (y-x)h(x,y)] = [f(x,y), a] + [f(x,y), (y-x)h(x,y)] = 0,$$

thus $a + (y-x)h(x,y) \in Z^r(F[x,y])$.

Conversely, if $g(x,y) \in Z^r(F[x,y])$, then $0 = [1, g(x,y)] = g(y,y) - g(x,x)$. Hence $g(x,x) = a \in F$. $(y-x) \mid g(x,y) - g(x,x)$ implies that there is $h(x,y) \in F[x,y]$, such that $g(x,y) - g(x,x) = (y-x)h(x,y)$, so $g(x,y) = g(x,x) + \big(g(x,y) - g(x,x)\big) = a + (y-x)h(x,y)$. This completes the proof. □

We can also get the following result.

Lemma 1.2.5. *The halo of the associative dialgebra $F[x,y]$ is*

$$\{1 + (y-x)g(x,y) \mid g(x,y) \in F[x,y]\}.$$

In particular, 1 is a bar-unit of $F[x,y]$.

Assume that $F[x,y]_n$ is the subspace consisting of all homogeneous polynomials of degree n in $F[x,y]$, $n = 0, 1, 2, \cdots$, then $F[x,y] = \bigoplus_{n=0}^{\infty} F[x,y]_n$ is a $\mathbb{Z}$-graded dialgebra and a $\mathbb{Z}$-graded Leibniz algebra. We can also define an increasing filtration on $F[x,y]$ by setting $F[x,y]_{(n)} = \bigoplus_{i=0}^{n} F[x,y]_i$, then $F[x,y]$ is also a filtered dialgebra and a filtered Leibniz algebra. The above gradation and filtration of $F[x,y]$ are said to be *standard*.

2. Derivation algebra and automorphism group of the dialgebra $F[x,y]$

In this section we consider $F[x,y]$ as an associative dialgebra. We first determine left derivations $\mathrm{LDer}(F[x,y])$ and right derivations $\mathrm{RDer}(F[x,y])$ of the associative dialgebra $F[x,y]$. Then derivation algebra $\mathrm{Der}(F[x,y])$ and automorphism group $\mathrm{Aut}(F[x,y])$ are also determined.

§ 2.1 Left derivations of the associative dialgebra $F[x,y]$.

Definition 2.1.1. *Assume that $(\mathcal{D}, \dashv, \vdash)$ is a dialgebra, $d \in \mathrm{End}(F[x,y])$. If d satisfies:*

$$d(a \dashv b) = d(a) \dashv b + a \dashv d(b), \quad \forall\, a, b \in \mathcal{D},$$

then d is called a left derivation of the dialgebra $(\mathcal{D}, \dashv, \vdash)$. Denote the set of all left derivations of $\mathcal{D}$ by $\mathrm{LDer}(\mathcal{D})$. Thus $\mathrm{LDer}(F[x,y])$ is the set of left derivations of the associative dialgebra $F[x,y]$.

Suppose that $(\mathcal{D}, \dashv, \vdash)$ is an associative dialgebra over F. For $z \in \mathcal{D}$, define $\widehat{\mathrm{ad}}z(a) = z \dashv a - a \dashv z$, $\forall\, a \in \mathcal{D}$, then $\widehat{\mathrm{ad}}z$ is a left derivation of $\mathcal{D}$, and call it a *left inner derivation* of the associative dialgebra $(\mathcal{D}, \dashv, \vdash)$.

Since $x^m y^n = x^m \dashv x^n = x^m \dashv \underbrace{x \dashv x \dashv \cdots \dashv x}_{n}$, the associative algebra $(F[x,y], \dashv)$ is generated by $\{x^n\}_{n\in\mathbb{Z}_+}$ (note that we set $x^0 = 1$, and 1 is a right unit of $(F[x,y], \dashv)$, but not a unit element).

Lemma 2.1.2. *Given any $f(x,y) \in F[x,y]$, we define a linear map $\widehat{f}(x,y)$: $F[x,y] \to F[x,y]$ as follows:*

$$g(x,y) \mapsto f(x,y)g(x,y), \quad \forall\, g(x,y) \in F[x,y].$$

Then $\widehat{f}(x,y)$ is a left derivation of the associative dialgebra $F[x,y]$ if and only if $f(y,y)=0$, or equivalently, $(x-y) \mid f(x,y)$.

Proof. We prove the necessity first. If $\widehat{f}(x,y)$ is a left derivation of the dialgebra $F[x,y]$, then we have

$$\widehat{f}(x,y)(1 \dashv 1) = \widehat{f}(x,y)(1) \dashv 1 + 1 \dashv \widehat{f}(x,y)(1),$$

so $1 \dashv \widehat{f}(x,y)(1) = 0$. Thus $f(y,y)=0$, that is, $(x-y) \mid f(x,y)$.

For sufficiency, we have to prove that if $f(y,y)=0$, then $\widehat{f}(x,y)$ is a left derivation of $F[x,y]$. It is enough to verify it on the basis $\{x^s y^t \mid s,t \in \mathbb{Z}_+\}$.

$$\widehat{f}(x,y)(x^m y^n \dashv x^s y^t) = \widehat{f}(x,y)(x^m y^{n+s+t}) = x^m y^{n+s+t} f(x,y),$$

$$\begin{aligned}\widehat{f}(x,y)(x^m y^n) \dashv x^s y^t + x^m y^n \dashv \widehat{f}(x,y)(x^s y^t) &= f(x,y) x^m y^n y^{s+t} + 0 \\ &= x^m y^{n+s+t} f(x,y).\end{aligned}$$

Thus,

$$\widehat{f}(x,y)(x^m y^n \dashv x^s y^t) = \widehat{f}(x,y)(x^m y^n) \dashv x^s y^t + x^m y^n \dashv \widehat{f}(x,y)(x^s y^t).$$

Hence, $\widehat{f}(x,y)$ is a left derivation of $F[x,y]$. This proves the result. □

In general, for any $d \in \mathrm{LDer}(F[x,y])$, we have

$$d(1) = d(1 \dashv 1) = d(1) \dashv 1 + 1 \dashv d(1).$$

Thus $d(1) = d(1) + 1 \dashv d(1)$. So $1 \dashv d(1) = 0$. Hence $(x-y)|d(1)$.

Theorem 2.1.3. *Suppose that $d \in \mathrm{LDer}(F[x,y])$. Then*

$$d(x^m y^n) = (my^{m+n-1} + nx^m y^{n-1})g(y) + (x-y)y^n f_m(x,y), \tag{2.1}$$

where $f_m(x,y) \in F[x,y]$, $m = 0,1,2,3,\cdots$, $g(y) \in F[y]$. Conversely, for any $f_m(x,y) \in F[x,y]$, $m=0,1,2,3,\cdots$, $g(y) \in F[y]$, the map $d \in \mathrm{End}(F[x,y])$ defined by formula (2.1) is an element in $\mathrm{LDer}(F[x,y])$.

Proof. If $d \in \mathrm{LDer}(F[x,y])$, we set $d(x^n) = h_n(x,y) \in F[x,y]$, $n = 0,1,2,\cdots$, then

$$1 \dashv d(x^n) = 1 \dashv d(x) \dashv x \dashv \cdots \dashv x + \cdots + 1 \dashv x \dashv x \dashv \cdots \dashv d(x).$$

That is,

$$h_n(y,y) = ny^{n-1} h_1(y,y).$$

Since $(x-y)|h_n(x,y) - h_n(y,y)$, there exist $f_n(x,y) \in F[x,y]$, $n \in \mathbb{Z}_+$, such that

$$h_n(x,y) = h_n(y,y) + (x-y)f_n(x,y) = ny^{n-1}h_1(y,y) + (x-y)f_n(x,y).$$

From the definition of left derivation and the fact $x^m y^n = x^m \dashv x^n$ we get

$$\begin{aligned} d(x^m y^n) &= y^n d(x^m) + x^m h_n(y,y) \\ &= (my^{m+n-1} + nx^m y^{n-1}) h_1(y,y) + (x-y) y^n f_m(x,y). \end{aligned}$$

Conversely, given $f_m(x,y) \in F[x,y]$, $m = 0,1,2,3,\cdots$, $g(y) \in F[y]$, we have to prove that the linear transformation d determined by formula (2.1) is an element in $\mathrm{LDer}(F[x,y])$, that is, we have to check: for any $x^i y^j, x^m y^n \in F[x,y]$, the following identity holds:

$$d(x^i y^j \dashv x^m y^n) = d(x^i y^j) \dashv x^m y^n + x^i y^j \dashv d(x^m y^n), \quad \forall\, i,j,m,n \geqslant 0.$$

But the left-hand side of the above equation is

$$d(x^i y^{j+m+n}) = (iy^{i+j+m+n-1} + (j+m+n) x^i y^{j+m+n-1}) g(y) + (x-y) y^{j+m+n} f_i(x,y),$$

and the right-hand side is

$$\begin{aligned} & y^{m+n}\big((iy^{i+j-1} + jx^i y^{j-1}) g(y) + (x-y) y^j f_i(x,y)\big) + x^i y^j (m+n) y^{m+n-1} g(y) \\ = {} & (iy^{i+j+m+n-1} + (j+m+n) x^i y^{j+m+n-1}) g(y) + (x-y) y^{j+m+n} f_i(x,y). \end{aligned}$$

Thus the equation holds, and $d \in \mathrm{LDer}(F[x,y])$. □

Corollary 2.1.4. *In Theorem 2.1.3, there exists an element $f(x,y) \in F[x,y]$ such that $d = \widehat{f}(x,y) \in \mathrm{LDer}(F[x,y])$ if and only if $g(y) = 0$ and $f_m(x,y) = x^m f_0(x,y)$.*

Corollary 2.1.5. *Suppose that $d \in \mathrm{LDer}(F[x,y])$ and $g(y) \in F[y]$. Then $g(y)d \in F[x,y]$, where $g(y)d$ is defined by*

$$g(y)d: \ f(x,y) \mapsto g(y) d(f(x,y)), \quad \forall\, f(x,y) \in F[x,y].$$

For left inner derivations, we have the following result.

Theorem 2.1.6. *If d is a left inner derivation of the dialgebra $F[x,y]$, then there exist $f(x,y) \in F[x,y]$ and $f_m(x,y) \in F[x,y]$, $m = 0,1,2,\cdots$, satisfying*

$$(x-y) f_m(x,y) = y^m f(x,y) - x^m f(y,y)$$

such that (2.1) hold with $g(y) = 0$. Conversely, given any $f(x,y) \in F[x,y]$ and $f_m(x,y) \in F[x,y]$, $m = 0,1,2,\cdots$, satisfying

$$(x-y) f_m(x,y) = y^m f(x,y) - x^m f(y,y),$$

the linear transformation d of $F[x,y]$ defined by (2.1), with $g(y) = 0$, is a left inner derivation of the dialgebra $F[x,y]$.

Proof. Let $d = \widehat{\mathrm{ad}} f(x,y)$ be a left inner derivation of $F[x,y]$, for some $f(x,y) \in F[x,y]$, then

$$\begin{aligned} \widehat{\mathrm{ad}} f(x,y)(x^m y^n) &= f(x,y) \dashv x^m y^n - x^m y^n \dashv f(x,y) \\ &= y^{m+n} f(x,y) - x^m y^n f(y,y) \\ &= y^n (y^m f(x,y) - x^m f(y,y)). \end{aligned}$$

Assume that $f(x,y)=\sum_{i,j} a_{i,j}x^iy^j$, where all $a_{i,j}\in F$. Then

$$\begin{aligned}
y^m f(x,y)-x^m f(y,y) &= \sum_{i,j} a_{i,j}(x^iy^{m+j}-x^my^{i+j}) \\
&= \sum_{i,j} a_{i,j}\big(x^iy^{m+j}-x^{i+j+m}+x^{i+j+m}-x^my^{i+j}\big) \\
&= \sum_{i,j} a_{i,j}\big(x^m(x^{i+j}-y^{i+j})-x^i(x^{j+m}-y^{j+m})\big) \\
&= (x-y)\sum_{i,j} a_{i,j}\big(x^m(x^{i+j-1}+x^{i+j-2}y+\cdots+y^{i+j-1}) \\
&\quad -x^i(x^{j+m-1}+x^{j+m-2}y+\cdots+y^{j+m-1})\big).
\end{aligned}$$

Thus there are $f_m(x,y)\in F[x,y]$, such that $(x-y)f_m(x,y)=y^mf(x,y)-x^mf(y,y)$, $m=0,1,2,\cdots$. Hence $d(x^my^n)=(x-y)y^nf_m(x,y)$. Take $g(y)=0$, then (2.1) holds.

Conversely, given any $f(x,y)\in F[x,y]$ and $f_m(x,y)\in F[x,y]$, $m=0,1,2,\cdots$, satisfying

$$(x-y)f_m(x,y)=y^mf(x,y)-x^mf(y,y),$$

we need to prove that the linear transformation d determined by (2.1), with $g(y)=0$, is a left inner derivation of $F[x,y]$.

At first, by Theorem 2.1.3, d is a left derivation of $F[x,y]$.

Secondly, we have

$$\begin{aligned}
d(x^my^n) &= y^n(x-y)f_m(x,y) \\
&= y^n(y^mf(x,y)-x^mf(y,y)) \\
&= y^{m+n}f(x,y)-x^my^nf(y,y) \\
&= f(x,y)\dashv x^my^n - x^my^n\dashv f(x,y) \\
&= \widehat{\mathrm{ad}}f(x,y)(x^my^n).
\end{aligned}$$

Therefore $d=\widehat{\mathrm{ad}}f(x,y)$ is a left inner derivation. □

§ 2.2 Right derivations of the associative dialgebra $F[x,y]$.

Definition 2.2.1. *Let $(\mathcal{D},\dashv,\vdash)$ be an associative dialgebra, and let $d\in\mathrm{End}(\mathcal{D})$, if d satisfies*

$$d(a\vdash b)=d(a)\vdash b+a\vdash d(b),\quad \forall\, a,b\in\mathcal{D},$$

then d is called a right derivation of the associative dialgebra $(\mathcal{D}\dashv,\vdash)$. We denote the set of all right derivations of $\mathcal{D}$ by $\mathrm{RDer}(\mathcal{D})$.

Firstly, we define a new algebra $(F[x,y]^{\mathrm{op}},\vdash')$ as follows: as a vector space $F[x,y]^{\mathrm{op}}=F[x,y]$. For $f(x,y)$, $g(x,y)\in F[x,y]^{\mathrm{op}}$, we define $f(x,y)\vdash' g(x,y)=g(x,y)\vdash f(x,y)$, then $(F[x,y]^{\mathrm{op}},\vdash')$ is also an associative algebra.

We denote the set of all derivations of the associative algebra $(F[x,y]^{\mathrm{op}},\vdash')$ by $\mathrm{Der}(F[x,y]^{\mathrm{op}})$.

Lemma 2.2.2. *Define a linear map* $\eta : (F[x,y], \dashv) \to (F[x,y]^{\rm op}, \vdash')$ *by*

$$\eta(f(x,y)) = f(y,x),$$

then η *is an isomorphism of associative algebras.*

Proof. For any $f(x,y)$, $g(x,y) \in F[x,y]$, we have

$$\begin{aligned}\eta(f(x,y) \dashv g(x,y)) &= \eta(f(x,y)g(y,y)) \\ &= f(y,x)g(x,x).\end{aligned}$$

And

$$\begin{aligned}\eta(f(x,y)) \vdash' \eta(g(x,y)) &= f(y,x) \vdash' g(y,x) \\ &= g(y,x) \vdash f(y,x) = g(x,x)f(y,x).\end{aligned}$$

Thus η is a homomorphism from $(F[x,y], \dashv)$ to $(F[x,y], \vdash')$. Obviously, η is an isomorphism of vector spaces, so η is an isomorphism of associative algebras. □

Due to the above results, we have

Theorem 2.2.3. *If* $d \in \mathrm{Der}(F[x,y]^{\rm op})$, *then there are* $f_i(x,y) \in F[x,y]$, $i = 0,1,2,\cdots$, *and* $g(x) \in F[x]$, *such that*

$$d(x^m y^n) = (mx^{m-1}y^n + nx^{m+n-1})g(x) + (x-y)x^m f_n(x,y), \quad \forall\, m,n \in \mathbb{Z}_+. \tag{2.2}$$

Conversely, if $f_i(x,y) \in F[x,y]$, $i = 0,1,2,\cdots$, *and* $g(x) \in F[x]$, *then the linear transformation* d *of* $F[x,y]$ *given by (2.2) is an element of* $\mathrm{Der}(F[x,y]^{\rm op})$.

Theorem 2.2.4. $\mathrm{RDer}(F[x,y]) = \mathrm{Der}(F[x,y]^{\rm op})$.

Proof. For any linear transformation d of $F[x,y]$, and $a, b \in F[x,y]$, we have

$$d(a \vdash b) = d(a) \vdash b + a \vdash d(b)$$

holds if and only if

$$d(b \vdash' a) = b \vdash' d(a) + d(b) \vdash' a$$

holds. Thus $d \in \mathrm{RDer}(F[x,y])$ if and only if $d \in \mathrm{Der}(F[x,y]^{\rm op})$. Therefore, $\mathrm{RDer}(F[x,y]) = \mathrm{Der}(F[x,y]^{\rm op})$. □

Suppose that $(\mathcal{D}, \dashv, \vdash)$ is a dialgebra. For $z \in \mathcal{D}$, set $\widetilde{\mathrm{ad}}z(a) = z \vdash a - a \vdash z$, $\forall\, a \in \mathcal{D}$, then $\widetilde{\mathrm{ad}}z$ is a derivation of the associative algebra $(\mathcal{D}, \vdash)$, so $\widetilde{\mathrm{ad}}z$ is called a *right inner derivation* of the dialgebra $(\mathcal{D}, \dashv, \vdash)$. From the above discussion we have

Theorem 2.2.5. *If* d *is a right inner derivation of the dialgebra* $F[x,y]$, *then there exist* $f(x,y) \in F[x,y]$ *and* $f_i(x,y) \in F[x,y]$, $i = 0,1,2,\cdots$, *satisfying*

$$(x-y)f_n(x,y) = y^n f(x,x) - x^n f(x,y),$$

such that (2.2) hold with $g(x) = 0$. *Conversely, given* $f(x,y) \in F[x,y]$ *and* $f_i(x,y) \in F[x,y]$, $i = 0,1,2,\cdots$, *satisfying*

$$(x-y)f_n(x,y) = y^n f(x,x) - x^n f(x,y),$$

the linear transformation d of $F[x,y]$ determined by (2.2), with $g(x)=0$, is a right derivation of the dialgebra $F[x,y]$.

Proof. Suppose that $d=\widetilde{\mathrm{ad}}f(x,y)$ is a right derivation of $F[x,y]$, for some $f(x,y)\in F[x,y]$. Then

$$\begin{aligned} d(x^my^n) &= f(x,y)\vdash x^my^n - x^my^n\vdash f(x,y)\\ &= x^my^nf(x,x)-x^{m+n}f(x,y)\\ &= x^m(y^nf(x,x)-x^nf(x,y)). \end{aligned}$$

We can take $f_i(x,y)=\big(y^ig(x,x)-x^ig(x,y)\big)/(x-y)\in F[x,y]$, $i=0,1,2,\cdots$, and $g(x)=0$, then

$$(x-y)f_n(x,y)=y^nf(x,x)-x^nf(x,y),\quad n\in\mathbb{Z}_+,$$

and $d(x^my^n)=(x-y)x^mf_n(x,y)$, that is, (2.2) holds.

If there are $f(x,y)\in F[x,y]$ and $f_i(x,y)\in F[x,y]$, $i=0,1,2,\cdots$, such that

$$(x-y)f_n(x,y)=y^nf(x,x)-x^nf(x,y),$$

we have to prove that the linear transformation d of $F[x,y]$ determined by (2.2), with $g(x)=0$, is a right inner derivation of the associative dialgebra $F[x,y]$. At first, by Theorem 2.2.3 and 2.2.4, d is a right derivation. Secondly,

$$\begin{aligned} d(x^my^n) &= (x-y)x^mf_n(x,y)\\ &= x^m(y^nf(x,x)-x^nf(x,y))\\ &= x^my^nf(x,x)-x^{m+n}f(x,y)\\ &= f(x,y)\vdash x^my^n-x^my^n\vdash f(x,y)\\ &= \widetilde{\mathrm{ad}}f(x,y)(x^my^n). \end{aligned}$$

Therefore, $d=\widetilde{\mathrm{ad}}f(x,y)$ is a right inner derivation. □

§ 2.3 Derivations of the associative dialgebra $F[x,y]$.

Definition 2.3.1. *Let $(\mathcal{D},\dashv,\vdash)$ be an associative dialgebra, $d\in\mathrm{End}(\mathcal{D})$, if d satisfies*

$$\begin{aligned} &d(a\dashv b)=d(a)\dashv b+a\dashv d(b),\\ &d(a\vdash b)=d(a)\vdash b+a\vdash d(b),\quad \forall\, a,b\in\mathcal{D}, \end{aligned}$$

then d is called a derivation of the associative dialgebra $(\mathcal{D},\dashv,\vdash)$. We denote the set of all derivations of $(\mathcal{D},\dashv,\vdash)$ by $\mathrm{Der}(\mathcal{D})$. Obviously, $\mathrm{Der}(\mathcal{D})=\mathrm{LDer}(\mathcal{D})\cap\mathrm{RDer}(\mathcal{D})$.

By the definition of the dialgebra $F[x,y]$, we have

$$x^my^n=\underbrace{x\vdash x\vdash\cdots\vdash x}_{m}\vdash 1\dashv\underbrace{x\dashv x\dashv\cdots\dashv x}_{n}. \tag{2.3}$$

Suppose that $d\in\mathrm{Der}(F[x,y])$. Then from $\mathrm{Der}(F[x,y])=\mathrm{LDer}(F[x,y])\cap\mathrm{RDer}(F[x,y])$ and the above discussion about left derivations we know $(x-y)|d(1)$, hence we can set $d(1)=(x-y)f(x,y)$, where $f(x,y)\in F[x,y]$. By definition of derivation,

$$d(x\vdash 1)=d(x)=d(x)\vdash 1+x\vdash d(1).$$

Assume that $d(x) = g(x,y)$. Then $g(x,y) = g(x,x) + x(x-y)f(x,y)$. By (2.3) we have

$$\begin{aligned} d(x^m y^n) &= d(x \vdash \cdots \vdash x \vdash 1 \dashv x \dashv \cdots \dashv x) \\ &= d(x) \vdash x \vdash \cdots \vdash 1 \dashv x \dashv \cdots \dashv x + \cdots + x \vdash \cdots \vdash d(1) \dashv x \dashv \cdots \dashv x \\ &\quad + x \vdash \cdots \vdash 1 \dashv d(x) \dashv \cdots \dashv x + \cdots + x \vdash \cdots \vdash x \vdash 1 \dashv x \dashv \cdots \dashv d(x) \\ &= mx^{m-1}y^n g(x,x) + x^m y^n d(1) + nx^m y^{n-1} g(y,y). \end{aligned}$$

Therefore we have the following results.

Theorem 2.3.2. *If $d \in \mathrm{Der}(F[x,y])$, then there are $f(x) \in F[x]$ and $g(x,y) \in F[x,y]$, such that*

$$d(x^m y^n) = mx^{m-1}y^n f(x) + x^m y^n (x-y) g(x,y) + nx^m y^{n-1} f(y), \tag{2.4}$$

Conversely, for any $f(x) \in F[x]$ and $g(x,y) \in F[x,y]$, the linear transformation d of $F[x,y]$ determined by (2.4) is an element of $\mathrm{Der}(F[x,y])$.

Proof. Suppose that $d \in \mathrm{Der}\,(F[x,y])$. Then by the above discussions, we have

$$d(x^m y^n) = mx^{m-1}y^n f(x) + x^m y^n (x-y) g(x,y) + nx^m y^{n-1} f(y),$$

where $f(x) = d(x) \vdash 1$ and $g(x,y) = d(1)/(x-y) \in F[x,y]$.

Conversely, for any $f(x) \in F[x]$ and $g(x,y) \in F[x,y]$, we have to prove that the linear transformation d of $F[x,y]$ determined by (2.4) is a derivation of $F[x,y]$. For any $x^m y^n \in F[x,y]$, clearly, $(x-y) \mid x^{m-1}f(x) - y^{m-1}f(y)$, thus, $f_m(x,y) := x^m g(x,y) + \frac{m(x^{m-1}f(x) - y^{m-1}f(y))}{x-y} \in F[x,y]$, $m = 0, 1, 2, \cdots$, so

$$\begin{aligned} d(x^m y^n) &= mx^{m-1}y^n f(x) + x^m y^n (x-y) g(x,y) + nx^m y^{n-1} f(y) \\ &= (my^{m+n-1} + nx^m y^{n-1}) f(y) + (x-y) y^n f_m(x,y). \end{aligned}$$

By Theorem 2.1.3, $d \in \mathrm{LDer}(F[x,y])$. Similarly,

$$d(x^m y^n) = (mx^{m-1}y^n + nx^{m+n-1}) f(x) + (x-y) x^m g_n(x,y),$$

where $g_n(x,y) = y^n g(x,y) + \frac{n(y^{n-1}f(y) - x^{n-1}f(x))}{x-y} \in F[x,y]$, thus $d \in \mathrm{RDer}(F[x,y])$ and $d \in \mathrm{LDer}(F[x,y]) \cap \mathrm{RDer}(F[x,y]) = \mathrm{Der}(F[x,y])$, completing the proof. $\square$

Let $(\mathcal{D}, \dashv, \vdash)$ be an associative dialgebra over F. For $z \in \mathcal{D}$, set $\mathrm{ad}\, z(X) = X \dashv z - z \vdash X$ $(\forall\, X \in \mathcal{D})$, we can prove that $\mathrm{ad}\, z$ is a derivation of $(\mathcal{D}, \dashv, \vdash)$. For any $X, Y \in \mathcal{D}$, we have

$$\mathrm{ad}\, z(X \dashv Y) = (X \dashv Y) \dashv z - z \vdash (X \dashv Y),$$

and

$$\begin{aligned} \mathrm{ad}\, z(X) \dashv Y + X \dashv \mathrm{ad}\, z(Y) &= (X \dashv z - z \vdash X) \dashv Y + X \dashv (Y \dashv z - z \vdash Y) \\ &= X \dashv (z \dashv Y) - (z \vdash X) \dashv Y + X \dashv (Y \dashv z) - X \dashv (z \dashv Y) \\ &= X \dashv (Y \dashv z) - (z \vdash X) \dashv Y. \end{aligned}$$

Hence,

$$\operatorname{ad} z(X \dashv Y) = \operatorname{ad} z(X) \dashv Y + X \dashv \operatorname{ad} z(Y).$$

On the other hand,

$$\operatorname{ad} z(X \vdash Y) = (X \vdash Y) \dashv z - z \vdash (X \vdash Y),$$

and

$$\begin{aligned} \operatorname{ad} z(X) \vdash Y + X \vdash \operatorname{ad} z(Y) &= (X \dashv z - z \vdash X) \vdash Y + X \vdash (Y \dashv z - z \vdash Y) \\ &= (X \vdash z) \vdash Y - (z \vdash X) \vdash Y + X \vdash (Y \dashv z) - (X \vdash z) \vdash Y \\ &= X \vdash (Y \dashv z) - (z \vdash X) \vdash Y. \end{aligned}$$

Thus,

$$\operatorname{ad} z(X \vdash Y) = \operatorname{ad} z(X) \vdash Y + X \vdash \operatorname{ad} z(Y).$$

$\operatorname{ad} z$ is called an *inner derivation* of $\mathcal{D}$. We denote the set of all inner derivations of $F[x,y]$ by $\operatorname{Inder}(F[x,y])$. By the above definition, we have

$$\begin{aligned} \operatorname{ad}(x^m y^n)(x^i y^j) &= x^i y^j \dashv x^m y^n - x^m y^n \vdash x^i y^j \\ &= x^i y^j (y^{m+n} - x^{m+n}). \end{aligned}$$

Thus we have

Theorem 2.3.3. *If* $d \in \operatorname{Inder}(F[x,y])$*, then there exist* $h(x) \in F[x]$ *and* $g(x,y) \in F[x,y]$ *satisfying*

$$(x-y)g(x,y) = h(y) - h(x), \tag{2.5}$$

such that (2.4) hold with $f(x) = 0$*. Conversely, given* $h(x) \in F[x]$ *and* $g(x,y) \in F[x,y]$ *satisfying (2.5), the linear transformation* d *of* $F[x,y]$ *determined by (2.4), with* $f(x) = 0$*, is an element in* $\operatorname{Inder}(F[x,y])$.

Proof. Let $d = \operatorname{ad} h(x,y) \in \operatorname{Inder}(F[x,y])$, for some $h(x,y) \in F[x,y]$, then we have

$$\begin{aligned} d(x^m y^n) &= x^m y^n \dashv h(x,y) - h(x,y) \vdash x^m y^n \\ &= x^m y^n h(y,y) - x^m y^n h(x,x) \\ &= x^m y^n (h(y,y) - h(x,x)), \quad \forall\, m,n \in \mathbb{Z}_+. \end{aligned}$$

Take $g(x,y) = (h(y,y) - h(x,x))/(x-y)$, $h(x) = h(x,x)$ and $f(x) = 0$, then $g(x,y) \in F[x,y]$, $(x-y)g(x,y) = h(y) - h(x)$ and (2.4) holds.

Conversely, take $h(x) \in F[x]$ and $g(x,y) \in F[x,y]$, such that (2.5) hold. If d is the linear transformation of $F[x,y]$ determined by (2.4), with $f(x) = 0$, then by Theorem 2.3.2, d is a derivation, and

$$\begin{aligned} d(x^m y^n) &= x^m y^n (x-y) g(x,y) \\ &= x^m y^n (h(y) - h(x)) \\ &= x^m y^n h(y) - x^m y^n h(x) \\ &= x^m y^n \dashv h(x) - h(x) \vdash x^m y^n \\ &= \operatorname{ad} h(x)(x^m y^n). \end{aligned}$$

Thus $d = \operatorname{ad} h(x) \in \operatorname{Inder}(F[x,y])$. $\square$

§ 2.4 Automorphisms of the associative dialgebra $F[x,y]$.

Definition 2.4.1. *Suppose that* $(\mathcal{D}, \dashv, \vdash)$ *is an associative dialgebra,* $\sigma \in \mathrm{End}(\mathcal{D})$. *If* σ *satisfies the following conditions*

$$\begin{aligned}\sigma(a \dashv b) &= \sigma(a) \dashv \sigma(b),\\ \sigma(a \vdash b) &= \sigma(a) \vdash \sigma(b), \quad \forall\, a, b \in \mathcal{D},\end{aligned}$$

and σ *is bijective, then* σ *is called an automorphism of the dialgebra* $(\mathcal{D}, \dashv, \vdash)$. *We denote the set of all automorphisms of* $\mathcal{D}$ *by* $\mathrm{Aut}\,(\mathcal{D})$.

In this subsection, we always denote $\mathrm{Aut}(F[x,y])$ the automorphism group of the associative dialgebra $F[x,y]$.

By (2.3) we know $F[x,y]$ is generated by 1 and x as a dialgebra. Let $\sigma \in \mathrm{Aut}(F[x,y])$, then

$$\begin{aligned}\sigma(x) &= \sigma(x \dashv 1) = \sigma(x) \dashv \sigma(1),\\ \sigma(x) &= \sigma(1 \vdash x) = \sigma(1) \vdash \sigma(x).\end{aligned}$$

Suppose that $\sigma(x) = f(x,y)$ and $\sigma(1) = g(x,y)$. Then $g(x,x) = g(y,y) = 1$. By (2.3), we have

$$\sigma(x^m y^n) = f(x,x)^m g(x,y) f(y,y)^n.$$

For any $h(x,y) = \sum_{m,n} a_{m,n} x^m y^n \in F[x,y]$, we get

$$\begin{aligned}\sigma(h(x,y)) &= \sum_{m,n} a_{m,n} \sigma(x^m y^n)\\ &= \left(\sum_{m,n} a_{m,n} f(x,x)^m f(y,y)^n\right) g(x,y).\end{aligned}$$

Thus, $F[x,y] = \sigma(F[x,y]) \subseteq g(x,y)F[x,y]$, which implies that $\sigma(1) = g(x,y)$ is a non-zero element in F. From $g(x,x) = 1$ we get $\sigma(1) = 1$.

Now suppose that $f(x) = f(x,x) = \sum_{i=0}^{s} a_i x^i$ (where all $a_i \in F$ and $a_s \neq 0$). Then

$$\sigma(x^m y^n) = f(x)^m f(y)^n.$$

Assume that η is the inverse of σ, and $\eta(x) = g(x) = \sum_{j=0}^{t} b_j x^j$, where all $b_j \in F$ and $b_t \neq 0$. Then

$$\begin{aligned}x = \eta\sigma(x) = \eta(f(x)) &= \sum_{i=0}^{s} a_i \eta(x^i) = \sum_{i=0}^{s} a_i g(x)^i\\ &= a_0 + a_1(b_0 + b_1 x + \cdots + b_t x^t) + \cdots + a_s(b_0 + b_1 x + \cdots + b_t x^t)^s.\end{aligned}$$

Since the degree of the last polynomial of the above equation is 1, we have $st = 1$, that is, $s = t = 1$. Therefore, $f(x) = a + bx$, where $a, b \in F$ and $b \neq 0$.

By the above discussion we know that all automorphisms σ of the associative dialgebra $F[x,y]$ preserve the standard filtration, that is, $\sigma(F[x,y]_{(n)}) \subseteq F[x,y]_{(n)}$, $n = 0, 1, 2, \cdots$.

Now, let us determine the automorphism group $\mathrm{Aut}(F[x,y])$.

Theorem 2.4.2. *The automorphism group* $\mathrm{Aut}(F[x,y])$ *of the dialgebra* $F[x,y]$ *is*

$$\{\sigma \in \mathrm{End}\,(F[x,y]) \mid \sigma f(x,y) = f(a+bx, a+by),\ \forall\, f(x,y);\ a,b \in F,\ b \neq 0\}.$$

Proof. Set

$$A = \{\sigma \in \mathrm{End}\,(F[x,y]) \mid \sigma f(x,y) = f(a+bx, a+by),\ \forall\, f(x,y);\ a,b \in F,\ b \neq 0\}.$$

Then by the above discussion we know that $\mathrm{Aut}(F[x,y]) \subseteq A$. On the other hand, for any $\sigma \in A$, such that $\sigma(x) = a + bx$, it is easy to verify that σ is a homomorphism of dialgebras. We can define a linear transformation η of $F[x,y]$ by $\eta(x^m y^n) = (-ab^{-1} + b^{-1}x)^m(-ab^{-1} + b^{-1}y)^n$, for all $m, n \in \mathbb{Z}_+$. Then η is the inverse of σ, thus σ is a bijection and $A \subseteq \mathrm{Aut}(F[x,y])$. Therefore, $\mathrm{Aut}(F[x,y]) = A$. □

Corollary 2.4.3. $\mathrm{Aut}(F[x,y])$ *is a subgroup of the automorphism group of the polynomial associative algebra* $F[x,y]$.

3. Derivations and automorphisms of the Leibniz algebra $F[x,y]$

In this section we will consider $F[x,y]$ as a Leibniz algebra induced by its associative dialgebraic structure. Recall that its Leibniz brackets are given by

$$[x^m y^n, x^s y^t] = x^m y^n (y^{s+t} - x^{s+t}).$$

Our main purpose is to determine the derivation algebra and the automorphism group of $F[x,y]$. Since it is difficult to determine all derivations and all automorphisms of $F[x,y]$, we only discuss its homogeneous derivations and automorphisms preserving the standard filtration.

§ 3.1 Derivations of the Leibniz algebra $F[x,y]$.

Suppose that d is a derivation of the Leibniz algebra $F[x,y]$. Since $[1,1] = 0$, $d([1,1]) = 0$, thus $[f(x,y), d(1)] = f(x,y)[1, d(1)] = 0$, for any $f(x,y) \in F[x,y]$. Set $d(1) = g(x,y)$, then $g(x,y) \in Z^r(F[x,y])$. Hence by Lemma 1.2.4, there are $c \in F$ and $f(x,y) \in F[x,y]$, such that $d(1) = c + (y-x)f(x,y)$. But

$$d[x^m, x^n] = d(x^m \dashv x^n - x^n \vdash x^m) = d(x^m y^n) - d(x^{m+n}),$$

thus,

$$\begin{aligned} d(x^m y^n) &= d(x^{m+n}) + d[x^m, x^n] \\ &= d(x^{m+n}) + (y^n - x^n)d(x^m) + x^m[1, d(x^n)]. \end{aligned} \tag{3.1}$$

That is, $d(x^m y^n)$ can be determined by $d(x^{m+n})$, $d(x^m)$ and $d(x^n)$.

Lemma 3.1.1. *If* d *is a derivation of an associative dialgebra* $\mathcal{D}$*, then* d *is a derivation of* $\mathcal{D}$ *as a Leibniz algebra.*

By definition of derivation of Leibniz algebra we have

$$d[x^m, x^n] = [d(x^m), x^n] + [x^m, d(x^n)] = d(x^m y^n - x^{m+n}), \tag{3.2}$$

and

$$d[y^n, x^m] = [d(y^n), x^m] + [y^n, d(x^m)] = d(y^{m+n} - x^m y^n). \tag{3.3}$$

Adding (3.2) and (3.3) we get

$$d[x^m, x^n] + d[y^n, x^m] = d(y^{m+n}) - d(x^{m+n}). \tag{3.4}$$

Since $d[1, x^{m+n}] = d(y^{m+n}) - d(x^{m+n})$, substituting it into (3.4) we get

$$[d(x^m), x^n] + [x^m, d(x^n)] + [d(y^n), x^m] + [y^n, d(x^m)] = [d(1), x^{m+n}] + [1, d(x^{m+n})],$$

or

$$\begin{aligned}(y^n - x^n)d(x^m) + x^m[1, d(x^n)] + (y^m - x^m)d(y^n) + y^n[1, d(x^m)]\\ = (y^{m+n} - x^{m+n})d(1) + [1, d(x^{m+n})].\end{aligned}$$

Then

$$\begin{aligned}(y^n - x^n)d(x^m) + x^m[1, d(x^n)] + (y^m - x^m)\big(d(x^n) + (y^n - x^n)d(1)\\ + [1, d(x^n)]\big) + y^n[1, d(x^m)] = (y^{m+n} - x^{m+n})d(1) + [1, d(x^{m+n})].\end{aligned}$$

Therefore, we have

$$\begin{aligned}[1, d(x^{n+m})] = (y^m - x^m)d(x^n) + (y^n - x^n)d(x^m) + y^m[1, d(x^n)]\\ + y^n[1, d(x^m)] + (2x^{n+m} - x^m y^n - x^n y^m)d(1).\end{aligned} \tag{3.5}$$

Lemma 3.1.2. *For $f(x,y) \in F[x,y]$, we define a linear transformation $\widehat{f}(x,y)$ of $F[x,y]$ as follows:*

$$\begin{aligned}\widehat{f}(x,y):\ & F[x,y] \to F[x,y]\\ & g(x,y) \mapsto f(x,y)g(x,y), \quad \forall\, g(x,y) \in F[x,y].\end{aligned}$$

Then $\widehat{f}(x,y)$ is a derivation of the Leibniz algebra $F[x,y]$ if and only if $(x-y) \mid f(x,y)$.

Proof. Suppose that $\widehat{f}(x,y)$ is a derivation of the Leibniz algebra $F[x,y]$. Then

$$\widehat{f}(x,y)[1,y] = [\widehat{f}(x,y)(1), y] + [1, \widehat{f}(x,y)(y)].$$

Since

$$\widehat{f}(x,y)[1,y] = f(x,y)[1,y] = f(x,y)(y-x),$$

and

$$\begin{aligned}&[\widehat{f}(x,y)(1), y] + [1, \widehat{f}(x,y)(y)]\\ &\quad = [f(x,y), y] + [1, f(x,y)y]\\ &\quad = f(x,y)(y-x) + \big(f(y,y)y - f(x,x)x\big),\end{aligned}$$

we obtain that $f(x,x) = f(y,y) = 0$, so $(x-y) \mid f(x,y)$.

On the other hand, if $(x-y) \mid f(x,y)$, then due to Theorem 2.3.2, $\widehat{f}(x,y)$ is a derivation of the dialgebra $F[x,y]$. Thus it follows from Lemma 3.1.1 that $\widehat{f}(x,y)$ is a derivation of the Leibniz algebra $F[x,y]$. □

Definition 3.1.3. *Let $\mathcal{G}$ be a $\mathbb{Z}$-graded Leibniz algebra. If d is a derivation of $\mathcal{G}$ such that $d(\mathcal{G}_i) \subseteq \mathcal{G}_{i+j}$, for any $i \in \mathbb{Z}$, then d is called a homogeneous derivation of the Leibniz algebra $\mathcal{G}$ of degree j. We denote by $\mathrm{der}_j(\mathcal{G})$ the set of all homogeneous derivations of $\mathcal{G}$ with degree j.*

If a Leibniz algebra $\mathcal{G} = \bigoplus_{i\in\mathbb{Z}} \mathcal{G}_i$ is $\mathbb{Z}$-graded algebra and finite dimensional, then the derivation algebra $\mathrm{der}(\mathcal{G})$ is also $\mathbb{Z}$-graded (cf. [13, p119-120]). If $\mathcal{G}$ is not finite dimensional, then the derivation algebra $\mathrm{der}(\mathcal{G})$ is not necessarily $\mathbb{Z}$-graded (with respect to the original $\mathbb{Z}$-gradation of $\mathcal{G}$). But the space $\bigoplus_{i\in\mathbb{Z}} \mathrm{der}_i(\mathcal{G})$ generated by homogeneous derivations of $\mathcal{G}$ is a (Lie) subalgebra of $\mathrm{der}(\mathcal{G})$. We denote this subalgebra by $\mathrm{der}'(\mathcal{G})$ and call it the homogeneous derivation algebra of $\mathcal{G}$.

In the following we discuss the subalgebra $\mathrm{der}'(F[x,y])$.

Lemma 3.1.4.
$$\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \in \mathrm{der}_{-1}\left(F[x,y]\right).$$

Proof. By Lemma 3.1.1, a derivation of a dialgebra $\mathcal{D}$ is also a derivation of $\mathcal{D}$ as a Leibniz algebra. Thus if in Theorem 2.3.2 we put $f(x) = 1$ and $g(x,y) = 0$, then we have
$$d = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \in \mathrm{der}_{-1}\left(F[x,y]\right). \quad \square$$

In general, by direct checking we have

Lemma 3.1.5. *For any non-positive integer m,*
$$d_{m-1} := x^m \frac{\partial}{\partial x} + y^m \frac{\partial}{\partial y} \in \mathrm{der}_{m-1}(F[x,y]).$$

In particular, $d_{-1} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$.

Lemma 3.1.6. $\mathrm{der}_{-1}(F[x,y]) = \langle d_{-1}\rangle$*, where $\langle d_{-1}\rangle$ is the space generated by d_{-1}.*

Proof. We have proved that $\langle d_{-1}\rangle \subseteq \mathrm{der}_{-1}(F[x,y])$.

Now for any $\delta \in \mathrm{der}_{-1}(F[x,y])$, we have $\delta(1) = 0$ and $\delta(x) = r \in F$. Set $d = \delta - rd_{-1}$, then $d \in \mathrm{der}_{-1}(F[x,y])$, and $d(1) = d(x) = 0$. From (3.1), $d(y) = d(x) + (y-x)d(1) + [1, d(x)] = 0$. By (3.5), we get
$$[1, d(x^2)] = 2(y-x)d(x) + 2y[1, d(x)] + (2x^2 - 2xy)d(1) = 0.$$

But $d(x^2) \in F[x,y]_1$, so we can write $d(x^2) = b(y-x)$. On the other hand,
$$\begin{aligned}[1, d(x^3)] &= (y^2 - x^2)d(x) + (y-x)d(x^2) + y^2[1, d(x)] \\ &\quad + y[1, d(x^2)] + (2x^3 - x^2y - xy^2)d(1) \\ &= (y-x)d(x^2) \\ &= (y-x)(by - bx).\end{aligned}$$

Since $d(x^3) \in F[x,y]_2$, we can write $d(x^3) = f(x,y) \in F[x,y]_2$. As the right-hand side of the above equation is $f(y,y) - f(x,x)$, the coefficients of x and y in its second factor are the same, which implies that $b = 0$, that is, $d(x^2) = 0$ and $[1, d(x^3)] = 0$.

In the following by using the induction on k we prove that if d is a homogeneous derivation of the Leibniz algebra $F[x,y]$ with degree -1, then $d(x^k)=0$, for all $k\geq 0$.

Suppose that for $k=n$ (≥ 2) we have $d(x^n)=0$. Then for $k=n+1$, by (3.5) we get

$$\begin{aligned}[1,d(x^{n+1})] &= (y-x)d(x^n)+(y^n-x^n)d(x)+y[1,d(x^n)]+y^n[1,d(x)]\\ &\quad +(2x^{n+1}-xy^n-x^ny)d(1)=0.\end{aligned}$$

As $d(x^{n+1})\in F[x,y]_n$, we can set $d(x^{n+1})=\sum_{i=0}^n a_{n-i,i}x^{n-i}y^i$, where all $a_{i,j}\in F$ and $\sum_{i=0}^n a_{n-i,i}=0$. But $d(x^{n+2})\in F[x,y]_{n+1}$, again by (3.5) we have

$$\begin{aligned}[1,d(x^{n+2})] &= (y-x)d(x^{n+1})\\ &= (y-x)\left(\sum_{i=0}^n a_{n-i,i}x^{n-i}y^i\right).\end{aligned}$$

Since coefficients of monomials in the second factor of the right-hand side are all equal,

$$a_{n,0}=a_{n-1,1}=\cdots=a_{0,n}=0,$$

thus, $d(x^{n+1})=0$. By induction, we have $d(x^k)=0$, for any $k\geq 0$.

Now $d(x^my^n)=d(x^{m+n})+(y^n-x^n)d(x^m)+x^m[1,d(x^n)]$, thus $d(x^my^n)=0$, for all $m,n\geq 0$, that is, $d=0$. So $\delta=rd_{-1}$. Hence, $\mathrm{der}_{-1}(F[x,y])\subseteq\langle d_{-1}\rangle$. Consequently, $\mathrm{der}_{-1}(F[x,y])=\langle d_{-1}\rangle$. □

Lemma 3.1.7. *If d is a homogeneous derivation of the Leibniz algebra $F[x,y]$ with degree -2, then $d=0$.*

Proof. First, we prove by induction on k that if d is a such derivation, then $d(x^k)=0$, for all $k\geq 0$.

Since d is of degree -2, $d(1)=d(x)=d(y)=0$ and $d(x^2)=a\in F$. By (3.5) we have $[1,d(x^3)]=a(y-x)$, and $d(x^3)\in F[x,y]_1$, so we can put $d(x^3)=bx+(a-b)y$. By (3.5), we have

$$\begin{aligned}[1,d(x^4)] &= 2(y^2-x^2)d(x^2)+2y^2[1,d(x^2)]=2a(y^2-x^2)\\ &= (y-x)d(x^3)+(y^3-x^3)d(x)+y[1,d(x^3)]+y^3[1,d(x)]+(2x^4-x^2y-xy^2)d(1)\\ &= (y-x)(bx+(a-b)y+ay).\end{aligned}$$

Thus $a=b=0$, $d(x^2)=d(x^3)=0$ and $[1,d(x^4)]=0$.

Suppose that for $k=n$, $d(x^n)=0$ and $[1,d(x^{n+1})]=0$ hold. Then for $k=n+1$, by (3.5) we have $[1,d(x^{n+2})]=(y-x)d(x^{n+1})$, and

$$\begin{aligned}[1,d(x^{n+2})] &= (y^n-x^n)d(x^2)+(y^2-x^2)d(x^n)+y^n[1,d(x^2)]+y^2[1,d(x^n)]\\ &\quad +(2x^{n+2}-x^ny^2-x^2y^n)d(1)\\ &= 0.\end{aligned}$$

So $d(x^{n+1})=0$. Thus, by induction we have $d(x^k)=0$, for any $k\geq 0$. But $d(x^my^n)=d(x^{m+n})+(y^n-x^n)d(x^m)+x^m[1,d(x^n)]$, hence $d(x^my^n)=0$, for all $m,n\geq 0$. Consequently, $d=0$. □

Lemma 3.1.8. *For the Leibniz algebra* $F[x,y]$, $\mathrm{der}_{-3}(F[x,y]) = 0$.

Proof. At first, by induction on k we can prove that if $d \in \mathrm{der}_{-3}(F[x,y])$, then $d(x^k) = 0$, for any $k \geq 0$.

Since $\deg d = -3$, we have $d(1) = d(x) = d(x^2) = 0$ and $[1, d(x^3)] = 0$.

Suppose that for $k = n$ (≥ 2), $d(x^n) = 0$ and $[1, d(x^{n+1})] = 0$ hold. Then for $k = n+1$, by (3.5) we get $[1, d(x^{n+2})] = (y-x)d(x^{n+1})$ and

$$\begin{aligned}[1, d(x^{n+2})] &= (y^n - x^n)d(x^2) + (y^2 - x^2)d(x^n) + y^n[1, d(x^2)] + y^2[1, d(x^n)] \\ &\quad + (2x^{n+2} - x^n y^2 - x^2 y^n)d(1) \\ &= 0.\end{aligned}$$

That is, $d(x^{n+1}) = [1, d(x^{n+2})] = 0$. By induction on k we have $d(x^k) = 0$, for any $k \geq 0$. But $d(x^m y^n) = d(x^{m+n}) + (y^n - x^n)d(x^m) + x^m[1, d(x^n)]$, thus $d(x^m y^n) = 0$, for any $m, n \geq 0$. Consequently, $d = 0$. $\square$

Similar to the proof of Lemma 3.1.8, we have the following result.

Theorem 3.1.9. *For the Leibniz algebra* $F[x,y]$, $\mathrm{der}_m(F[x,y]) = 0$, $\forall\, m < -3$.

Now we discuss homogeneous derivations of the Leibniz algebra $F[x,y]$ with non-negative degrees.

Theorem 3.1.10. $\mathrm{der}_0(F[x,y]) = \langle d_0 \rangle$.

Proof. Note that $d_0 \in \mathrm{der}_0(F[x,y])$ and $d_0(f(x,y)) = sf(x,y)$, for any $f(x,y) \in F[x,y]_s$.

Now suppose that $\delta \in \mathrm{der}_0(F[x,y])$, and $\delta(x) = ay + bx$, then by Lemma 3.1.5,

$$\delta - bd_0 \in \mathrm{der}_0(F[x,y]).$$

Set $d = \delta - bd_0$, then $d(x) = ay$. Suppose $d(1) = a' \in F$, then

$$\begin{aligned}[1, d(x^2)] &= 2(y-x)d(x) + 2y[1, d(x)] + (2x^2 - 2xy)d(1) \\ &= (y-x)(-2a'x + 4ay).\end{aligned}$$

Since the coefficients of x and y in the second factor of the right-hand side of the above equation are equal, we get $a' = -2a$. Thus $d(1) = -2a$ and $d(x) = ay$. But $d(x^2) \in F[x,y]_2$, we can write $d(x^2) = a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2$, such that $a_{2,0} + a_{1,1} + a_{0,2} = 4a$. Now

$$\begin{aligned}[1, d(x^3)] &= (y^2 - x^2)d(x) + (y-x)d(x^2) + y^2[1, d(x)] + y[1, d(x^2)] \\ &\quad + (2x^3 - x^2 y - xy^2)d(1) \\ &= (y-x)\big((4a + a_{2,0})x^2 + (a + a_{1,1} + 4a + 2a)xy \\ &\quad + (a + a_{0,2} + a + 4a)y^2\big).\end{aligned}$$

Since the coefficients of x^2, xy, y^2 in the second factor of the right-hand side of the above equation are equal, $a_{2,0} = 3a, a_{1,1} = 0, a_{0,2} = a$, hence

$$d(x^2) = 3ax^2 + ay^2, [1,\ d(x^3)] = 7a(y^3 - x^3).$$

As $d(x^3) \in F[x,y]_3$, we can assume that $d(x^3) = \sum_{i=0}^{3} b_{3-i,i}x^{4-i}y^i$, where all $b_{i,j} \in F$ and $\sum_{i=0}^{3} b_{3-i,i} = 7a$. From (3.5) we have

$$\begin{aligned}[1, d(x^4)] &= (y^3 - x^3)d(x) + (y - x)d(x^3) + y[1, d(x^3)] + y^3[1, d(x)] \\ &\quad + (2x^4 - x^3y - xy^3)d(1) \\ &= (y - x)\big((4a + b_{3,0})x^3 + (b_{2,1} + 10a)x^2y + (b_{1,2} + 10a)xy^2 \\ &\quad + (b_{0,3} + 9a)y^3\big).\end{aligned}$$

As the coefficients of x^3, x^2y, xy^2, y^3 in the second factor of right-hand side of the above equation are equal, we have

$$4a + b_{3,0} = b_{2,1} + 10a = b_{1,2} + 10a = b_{0,3} + 9a.$$

That is, $b_{3,0} = 6a$, $b_{1,2} = b_{2,1} = 0$, $b_{0,3} = a$, thus

$$d(x^3) = 6ax^3 + ay^3,\ \ [1, d(x^4)] = 10a(y^4 - x^4).$$

As $d[xy, x] = [d(xy), x] + [xy, d(x)]$, and $d(x^my^n) = d(x^{m+n}) + (y^n - x^n)d(x^m) + x^m[1, d(x^n)]$, we have

$$\begin{aligned}d[xy, x] &= d(xy^2) - d(x^2y) = d(x^3) + (y^2 - x^2)d(x) + x[1, d(x^2)] \\ &\quad - d(x^3) - (y - x)d(x^2) - x^2[1, d(x)] \\ &= 5a(xy^2 - x^2y),\end{aligned}$$

and

$$\begin{aligned}[d(xy), x] + [xy, d(x)] &= (y - x)d(xy) + xy[1, d(x)] \\ &= (y - x)(2ax^2 + 2ay^2) + a(y - x)xy \\ &= -2ax^3 + (-a)(xy^2 - x^2y) + 2ay^3,\end{aligned}$$

which implies that $a = 0$, so $d(x) = d(x^2) = d(x^3) = 0$ and $[1, d(x^4)] = 0$.

By induction on k we can prove that if $d \in \mathrm{der}_0(F[x,y])$, then $d(x^k) = 0$, for any $k \in \mathbb{Z}_+$.

Suppose that for $k = n$, $d(x^k) = 0$. Then for $k = n + 1$, by (3.5) we have $[1, d(x^{n+1})] = 0$. Since $d(x^{n+1}) \in F[x,y]_{n+1}$, we can suppose that

$$d(x^{n+1}) = \sum_{i=0}^{n+1} c_{n+1-i,i}x^{n+1-i}y^i,$$

where all $c_{n+1-i,i} \in F$ and $\sum_{i=0}^{n+1} c_{n+1-i,i} = 0$. But

$$\begin{aligned}[1, d(x^{n+2})] &= (y^{n+1} - x^{n+1})d(x) + (y - x)d(x^{n+1}) + y[1, d(x^{n+1})] + y^{n+1}[1, d(x)] \\ &\quad + (2x^{n+2} - x^{n+1}y - xy^{n+1})d(1) \\ &= (y - x)d(x^{n+1}).\end{aligned}$$

As the coefficients of $x^n, x^{n-1}y, \cdots, y^n$ in the second factor of the right-hand side of the above equation are equal, we can get $d(x^{n+1}) = 0$. By induction we have $d(x^k) = 0$, for any $k \geq 0$.

Now $d(x^sy^t) = d(x^{s+t}) + (y^t - x^t)d(x^s) + x^s[1, d(x^t)]$, thus $d(x^sy^t) = 0$, that is $d = 0$, so $\delta = bd_0$. Hence $\mathrm{der}_0(F[x,y]) \subseteq \langle d_0\rangle$. Therefore, $\mathrm{der}_0(F[x,y]) = \langle d_0\rangle$. □

We determine general forms of derivations of positive degrees as follows.

Lemma 3.1.11. *For any positive integer* n, $\mathrm{der}_n(F[x,y]) = \langle \widehat{f}(x,y) \mid f(x,y) \in F[x,y]_n$*, and* $(x-y) \mid f(x,y)\rangle \oplus \langle d_n\rangle$.

Proof. At first, we want to prove that if $d \in \mathrm{der}_n(F[x,y])$, $d(1)=0$ and

$$d(x) = \sum_{i=1}^{n+1} a_{n+1-i,i}x^{n+1-i}y^i, \text{ where } a_{n+1-i,i} \in F, \text{ for all } i,$$

then $d=0$.

We have

$$\begin{aligned}
[1, d(x^2)] &= 2(y-x)d(x) + 2y[1,d(x)] + (2x^2-2xy)d(1)\\
&= 2(y-x)\left(\sum_{i=1}^{n+1} a_{n+1-i,i}x^{n+1-i}y^i\right) + 2\alpha y(y^{n+1}-x^{n+1})\\
&= 2(y-x)\left(\sum_{i=1}^{n+1} a_{n+1-i,i}x^{n+1-i}y^i + \alpha y\big(x^n + x^{n-1}y + \cdots + xy^{n-1} + y^n\big)\right)\\
&= 2(y-x)\sum_{i=1}^{n+1}(a_{n+1-i,i}+\alpha)x^{n+1-i}y^i,
\end{aligned}$$

where $\alpha = \sum_{i=1}^{n+1} a_{n+1-i,i}$. But the coefficients of $x^{n+1}, x^n y, \cdots, y^{n+1}$ in the second factor of $[1,d(x^2)]$ are equal, so $a_{n,1} = a_{n-1,2} = \cdots = a_{0,n+1} = 0$, thus $d(x)=0$ and $[1,d(x^2)]=0$.

Suppose that $d(x^k) = [1, d(x^{k+1})] = 0$ $(k \geq 1)$. We will prove that $d(x^{k+1}) = [1, d(x^{k+2})] = 0$. Let

$$d(x^{k+1}) = a_{n+k+1,0}x^{n+k+1} + a_{n+k,1}x^{n+k}y + \cdots + a_{0,n+k+1}y^{n+k+1},$$

where all $a_{n+k+1-i,i} \in F$. Then

$$\begin{aligned}
[1, d(x^{k+2})] &= (y^{k+1}-x^{k+1})d(x) + (y-x)d(x^{k+1}) + y^{k+1}[1,d(x)]\\
&\quad + y[1,d(x^{k+1})] + (2x^{k+2} - x^{k+1}y - xy^{k+1})d(1)\\
&= (y-x)d(x^{k+1})\\
&= (y-x)\big(a_{n+k+1,0}x^{n+k+1} + a_{n+k,1}x^{n+k}y + \cdots + a_{0,n+k+1}y^{n+k+1}\big).
\end{aligned}$$

Thus, since the coefficients of all monomials in the second factor of $[1,d(x^{k+2})]$ are equal we have $a_{n+k+1,0} = \cdots = a_{0,n+k+1}$. But $[1,d(x^{k+1})] = 0$ implies that $\sum_{i=0}^{n+k+1} a_{n+k+1-i,i} = 0$. Thus $a_{n+k+1,0} = \cdots = a_{0,n+k+1} = 0$, that is, $[1,d(x^{k+2})] = 0$ and $d(x^{k+1}) = 0$. By induction on m, we have $d(x^m) = 0$, for any $m \geq 0$. Now by (3.1), $d(x^s y^t) = d(x^{s+t}) + (y^t - x^t)d(x^s) + x^s[1, d(x^t)] = 0$, for any $s, t \in \mathbb{Z}_+$. Thus $d = 0$.

In general, suppose that $\delta \in \mathrm{der}_n(F[x,y])$. Then $\delta(1) \in F[x,y]_n$. Let $\delta(1) = f(x,y)$, then $(x-y) \mid f(x,y)$. By Lemma 3.1.2, $\widehat{f}(x,y) \in \mathrm{der}_n(F[x,y])$, and $\delta(1) = \widehat{f}(x,y)(1)$. Set $d' = \delta - \widehat{f}(x,y) \in \mathrm{der}_n(F[x,y])$, then $d'(1) = 0$. Suppose that $d'(x) = a_{n+1,0}x^{n+1} + a_{n,1}x^n y + \cdots + a_{0,n+1}y^{n+1}$, where all $a_{n+1-i,i} \in F$. Then

$$(d' - a_{n+1,0}d_n)(x) = d'(x) - a_{n+1,0}x^{n+1} = a_{n,1}x^n y + \cdots + a_{0,n+1}y^{n+1},$$

and $(d' - a_{n+1,0}d_n)(1) = 0$. Set $d = d' - a_{n+1,0}d_n$, then $d \in \mathrm{der}_n(F[x,y])$, $d(1) = 0$ and $d(x) = b_n x^n y + \cdots + b_0 y^{n+1}$. Now by above discussion, we have $d = 0$. Thus $\delta = \widehat{f}(x,y) + a_{n+1,0}d_n$. Consequently, $\mathrm{der}_n(F[x,y]) = \langle \widehat{f}(x,y) \mid f \in F[x,y]_n$, and $(x-y) \mid f(x,y)\rangle + \langle d_n\rangle$. If $\widehat{f}(x,y) \in \langle d_n\rangle$, then $0 = \widehat{f}(x,y)(1) = f(x,y)$, thus $\widehat{f}(x,y) = 0$ and

$$\mathrm{der}_n(F[x,y]) = \langle \widehat{f}(x,y) \mid f \in F[x,y]_n, \text{ and } (x-y) \mid f(x,y)\rangle \oplus \langle d_n\rangle. \quad \square$$

We collect all these results into the following theorem.

Theorem 3.1.12. *For the $\mathbb{Z}$-graded Leibniz algebra $F[x,y]$, its homogeneous derivation algebra*

$$\mathrm{der}'(F[x,y]) = \{\widehat{f}(x,y) \mid f(x,y) \in (y-x)F[x,y]\} \oplus \langle d_m \mid m \geq -1\rangle.$$

Corollary 3.1.13. $\mathrm{Der}(F[x,y]) = \mathrm{der}'(F[x,y])$*, that is, the derivation algebra of $F[x,y]$ as an associative dialgebra is the same as the homogeneous derivation algebra of $F[x,y]$ as a $\mathbb{Z}$-graded Leibniz algebra.*

Set

$$\mathcal{D}_1(F[x,y]) = \{\widehat{f}(x,y) \mid f(x,y) \in (y-x)F[x,y]\},$$

and

$$\mathcal{D}_2(F[x,y]) = \langle d_m \mid m \geq -1\rangle.$$

Thus, by Theorem 3.1.12, $\mathrm{der}'(F[x,y]) = \mathcal{D}_1(F[x,y]) \oplus \mathcal{D}_2(F[x,y])$. In the following theorem we point out the structure of the Lie algebra $\mathrm{der}'(F[x,y])$.

Theorem 3.1.14. *For the homogeneous derivation algebra $\mathrm{der}'(F[x,y])$ of the $\mathbb{Z}$-graded Leibniz algebra $F[x,y]$, we have*

(1) $\mathcal{D}_1(F[x,y])$ is an abelian ideal of $\mathrm{der}'(F[x,y])$;

(2) $\mathcal{D}_2(F[x,y])$ is a subalgebra of $\mathrm{der}'(F[x,y])$ and is isomorphic to Witt algebra $W(1)$;

(3) For any $f(x,y) \in (y-x)F[x,y]$ and $m \geq -1$,

$$[d_m, \widehat{f}(x,y)] = \widehat{g}(x,y),$$

where $g(x,y) = d_m(f(x,y)) \in (y-x)F[x,y]$.

Proof. Since $F[x,y]$ is a commutative algebra as a polynomial algebra (with respect to the ordinary multiplication), we have that $[\widehat{f}(x,y), \widehat{g}(x,y)] = 0$, and $\mathcal{D}_1(F[x,y])$ is an abelian subalgebra.

It is easy to verify that

$$[d_m, d_n] = \begin{cases} (n-m)d_{m+n}, & \text{if } m+n \geq -1; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, (2) is true.

For any $f(x,y) \in (y-x)F[x,y]$, $h(x,y) \in F[x,y]$ and integer $m \geq -1$, one has

$$\begin{aligned}
[d_m, \widehat{f}(x,y)]h(x,y) &= d_m\widehat{f}(x,y)(h(x,y)) - \widehat{f}(x,y)d_m(h(x,y)) \\
&= d_m(f(x,y)h(x,y)) - f(x,y)\left(x^{m+1}\frac{\partial h(x,y)}{\partial x} + y^{m+1}\frac{\partial h(x,y)}{\partial y}\right) \\
&= x^{m+1}\frac{\partial f(x,y)}{\partial x}h(x,y) + y^{m+1}\frac{\partial f(x,y)}{\partial y}h(x,y) \\
&= \widehat{g}(x,y)h(x,y),
\end{aligned}$$

where $g(x,y) = x^{m+1}\frac{\partial f(x,y)}{\partial x} + y^{m+1}\frac{\partial f(x,y)}{\partial y} = d_m(f(x,y))$. Since $[d_m, \widehat{f}(x,y)]$ is also a derivation, by Lemma 3.1.2, $g(x,y) \in (y-x)F[x,y]$. Thus (3) holds, and $\mathcal{D}_1(F[x,y])$ is an ideal of $\mathrm{der}'(F[x,y])$. So (1) is true. □

§ **3.2 Automorphisms of the Leibniz algebra $F[x,y]$.**

In this subsection, we discuss automorphisms of the Leibniz algebra $F[x,y]$.

Definition 3.2.1. *Let $\mathcal{G}$ be a Leibniz algebra, $\sigma \in \mathrm{End}(\mathcal{G})$. If*

$$\sigma[a,b] = [\sigma(a), \sigma(b)], \quad \forall, a, b \in \mathcal{G},$$

and σ is bijective, then σ is called an automorphism of the Leibniz algebra $\mathcal{G}$. We denote the automorphism group of the Leibniz algebra $\mathcal{G}$ by $\mathrm{Aut}(\mathcal{G})$.

Suppose that σ is an automorphism of the Leibniz algebra $F[x,y]$, that is, $\sigma \in \mathrm{Aut}(F[x,y])$. Then

$$\sigma[x^m, x^n] = \sigma(x^m y^n - x^{m+n}). \tag{3.6}$$

Thus $\sigma(x^m y^n) = \sigma(x^{m+n}) + \sigma(x^m)[1, \sigma(x^n)]$. That is, $\sigma(x^m y^n)$ can be determined by $\sigma(x^{m+n})$, $\sigma(x^m)$ and $\sigma(x^n)$. Further,

$$\sigma[y^n, x^m] = \sigma(y^{m+n} - x^m y^n). \tag{3.7}$$

By adding (3.6) and (3.7) we get

$$\sigma[x^m, x^n] + \sigma[y^n, x^m] = \sigma(y^{m+n}) - \sigma(x^{m+n}).$$

Since $\sigma[1, x^{m+n}] = \sigma(y^{m+n} - x^{m+n})$,

$$\sigma[x^m, x^n] + \sigma[y^n, x^m] = \sigma[1, x^{m+n}]. \tag{3.8}$$

$$\begin{aligned}
\sigma(y^n) &= \sigma(x^n)[1, \sigma(1)] + \sigma(1)[1, \sigma(x^n)] + \sigma(x^n) \\
&= \sigma(x^n) + \sigma(1)[1, \sigma(x^n)].
\end{aligned}$$

Substituting this expression into (3.8) we have

$$\sigma(1)[1, \sigma(x^{m+n})] = \sigma(x^m)[1, \sigma(x^n)] + \sigma(x^n)[1, \sigma(x^m)] + \sigma(1)[1, \sigma(x^m)][1, \sigma(x^n)]. \tag{3.9}$$

Suppose that $\sigma(1) = f(x,y)$, then $f(y,y) = f(x,x)$ follows from $[1, \sigma(1)] = 0$. Thus there are $c \in F$ and $g(x,y) \in F[x,y]$, such that $f(x,y) = c + (y-x)g(x,y)$.

Next, we will prove that $c \neq 0$.

By Lemma 1.2.4, the right annihilator $Z^r(F[x,y])$ of the Leibniz algebra $F[x,y]$ is $\{a+(y-x)h(x,y) \mid a \in F,\ h(x,y) \in F[x,y]\}$. On the other hand, by Lemma 1.2.2, $F[x,y]^{(1)} = (y-x)F[x,y]$.

Obviously, for any $\sigma \in \mathrm{Aut}(F[x,y])$, $F[x,y]^{(1)}$ and $Z^r(F[x,y])$ are all σ-invariant. Hence, $(y-x)F[x,y]$ is σ-invariant.

Now $1 \notin (y-x)F[x,y]$ implies that $\sigma(1) \notin (y-x)F[x,y]$, thus $c \neq 0$.

By (3.9), we have

$$\sigma(1)[1,\sigma(x^2)] = 2\sigma(x)[1,\sigma(x)] + \sigma(1)[1,\sigma(x)]^2, \tag{3.10}$$

or

$$\sigma(1)\big([1,\sigma(x^2)] - [1,\sigma(x)]^2\big) = 2\sigma(x)[1,\sigma(x)]. \tag{3.11}$$

Suppose that σ preserves the standard filtration $\{F[x,y]_{(n)}\}_{n\geq 0}$ of $F[x,y]$. Then $\sigma(1) = c$, $\sigma(x) \in F[x,y]_{(1)}$. So we can write $\sigma(x) = a+bx+ey$, for some $a,b,e \in F$. Then

$$\begin{aligned} 0 \neq c[1,\sigma(x^2)] &= 2\sigma(x)[1,\sigma(x)] + c[1,\sigma(x)]^2 \\ &= (b+e)(y-x)(2a+2bx+2ey+c(b+e)(y-x)). \end{aligned}$$

As the coefficients of x^2 and y^2 in $[1,\sigma(x^2)]$ are equal, $(1-c)b = (1+c)e$. Hence, if $c=1$, then $e=0$; if $c=-1$, then $b=0$. So in the following we first consider two special cases: $c=1$ or $c=-1$.

Theorem 3.2.2. *Let σ be an automorphism of the Leibniz algebra $F[x,y]$ which preserves the standard filtration and $\sigma(1)=1$, then*

$$\sigma(x^m y^n) = (a+bx)^m(a+by)^n, \quad \forall\, m,n \in \mathbb{Z}_+, \tag{3.12}$$

where $a,b \in F$ and $b \neq 0$.

Conversely, for any $a,b \in F$ and $b \neq 0$, the linear transformation σ of $F[x,y]$ defined by (3.12) is an automorphism of the Leibniz algebra $F[x,y]$.

Proof. If $\sigma \in \mathrm{End}(F[x,y])$ defined by (3.12), then it is easy to see that

$$\sigma f(x,y) = f(a+bx, a+by), \quad \forall\, f(x,y) \in F[x,y].$$

Obviously,

$$\begin{aligned} \sigma[f(x,y),g(x,y)] &= f(a+bx,a+by)\big(g(a+by,a+by) - g(a+bx,a+bx)\big) \\ &= [\sigma f(x,y), \sigma g(x,y)]. \end{aligned}$$

Thus σ is a homomorphism. On the other hand, if we define $\eta \in \mathrm{End}(F[x,y])$ by

$$\eta f(x,y) = f(-ab^{-1} + b^{-1}x, -ab^{-1} + b^{-1}y), \quad \forall\, f(x,y) \in F[x,y].$$

then η is the inverse of σ. Hence, $\sigma \in \mathrm{Aut}(F[x,y])$.

If σ is any automorphism of $F[x,y]$ preserving the standard filtration and $\sigma(1) = 1$, $\sigma(x) = a+bx$, for some $a,b \in F$, $b \neq 0$, then by (3.9), $[1,\sigma(x^2)] = b(y-x)(2a+bx+$

$by)$. Suppose that $\sigma(x^2)=\sum_{k=0}^{2}\sum_{i=0}^{k}a_{i,k-i}x^iy^{k-i}$ and the coefficients satisfying $a_{1,0}+a_{0,1}=2ab$, $a_{2,0}+a_{1,1}+a_{0,2}=b^2$. Then by (3.9), we have

$$\begin{aligned}[1,\sigma(x^3)] &= \sigma(x)[1,\sigma(x^2)]+\sigma(x^2)[1,\sigma(x)]+[1,\sigma(x)][1,\sigma(x^2)]\\ &= b(y-x)((2a^2+a_{00})+(ab+a_{1,0})x+(3ab+a_{0,1})y+a_{2,0}x^2\\ &\quad +(b^2+a_{1,1})xy+(b^2+a_{0,2})y^2).\end{aligned}$$

Since the coefficients of x,y and of x^2,xy,y^2 in the second factor of the right-hand side of the above equation are equal respectively, we have

$$\sigma(x^2)=c+2abx+b^2x^2,\ [1,\sigma(x^3)]=b(y-x)(2a^2+c+3ab(x+y)+b^2(x^2+xy+y^2)).$$

Now suppose that

$$\sigma(x^3)=\sum_{k=0}^{3}\sum_{i=0}^{k}b_{i,k-i}x^iy^{k-i},$$

and the coefficients satisfy

$$b_{1,0}+b_{0,1}=b(2a^2+c),\ \sum_{i=0}^{2}b_{i,2-i}=3ab^2,\ \sum_{i=0}^{3}b_{i,3-i}=b^3.$$

Then

$$\begin{aligned}[1,\sigma(x^4)] &= 2\sigma(x^2)[1,\sigma(x^2)]+[1,\sigma(x^2)]^2\\ &= b(y-x)\big(4ac+(4a^2b+2bc)(x+y)+4ab^2(x^2+xy+y^2)\\ &\quad +b^3(x^3+x^2y+xy^2+y^3)\big)\\ &= \sigma(x)[1,\sigma(x^3)]+\sigma(x^3)[1,\sigma(x)]+[1,\sigma(x)][1,\sigma(x^3)]\\ &= b(y-x)\big(2a^3+ac+b_{0,0}+(3a^2b+b_{1,0})x+(5a^2b+bc+b_{0,1})y\\ &\quad +(ab^2+b_{2,0})x^2+(4ab^2+b_{1,1})xy+(4ab^2+b_{0,2})y^2\\ &\quad +b_{3,0}x^3+(b_{2,1}+b^3)x^2y+(b_{1,2}+b^3)xy^2+(b_{0,3}+b^3)y^3\big).\end{aligned}$$

Thus $c=a^2, b_{0,0}=a^3$, which implies that $\sigma(x^2)=(a+bx)^2$ and $\sigma(x^3)=(a+bx)^3$.

In the following we prove by induction that $\sigma(x^k)=(a+bx)^k$, for all $k\geq 0$.

Suppose that for $k=n$, $\sigma(x^k)=(a+by)^k$ holds. Then for $k=n+1$, by (3.9) we have $[1,\sigma(x^{n+1})]=(a+by)^{n+1}-(a+bx)^{n+1}$. Thus

$$\begin{aligned}[1,\sigma(x^{n+2})] &= \sigma(x^2)[1,\sigma(x^n)]+\sigma(x^n)[1,\sigma(x^2)]+[1,\sigma(x^2)][1,\sigma(x^n)]\\ &= (a+by)^{n+2}-(a+bx)^{n+2}\\ &= \sigma(x)[1,\sigma(x^{n+1})]+\sigma(x^{n+1})[1,\sigma(x)]+[1,\sigma(x)][1,\sigma(x^{n+1})].\end{aligned}$$

Hence

$$\sigma(x^{n+1})=(a+bx)^{n+1}.$$

By induction, for any $k\geq 0$, $\sigma(x^k)=(a+bx)^k$ holds.

As $\sigma(x^my^n)=\sigma(x^{m+n})+\sigma(x^m)[1,\sigma(x^n)]$, for any $m,n\in\mathbb{Z}_+$, we get

$$\sigma(x^my^n)=(a+bx)^m(a+by)^n,$$

which completes the proof. $\square$

Theorem 3.2.3. *Let σ be an automorphism of the Leibniz algebra $F[x,y]$ which preserves the standard filtration and $\sigma(1) = -1$, then there are $a, b \in F$, and $b \neq 0$, such that*

$$\sigma f(x,y) = -f(a + by, a + bx), \quad \forall\, f(x,y) \in F[x,y]. \tag{3.13}$$

Conversely, for any $a, b \in F$ and $b \neq 0$, the linear transformation σ of $F[x,y]$ defined by (3.13) is an automorphism of the Leibniz algebra $F[x,y]$.

Proof. For any $a, b \in F$ and $b \neq 0$, if σ is the transformation of $F[x,y]$ defined by (3.13), then similar to the proof of Theorem 3.2.2, we obtain that σ is an automorphism of the Leibniz algebra $F[x,y]$, and clearly, σ preserves the standard filtration.

On the other hand, it is easy to verify that the linear transformation γ of $F[x,y]$ defined by

$$\gamma:\ f(x,y) \mapsto -f(y,x), \quad \forall\, f(x,y) \in F[x,y]$$

is an automorphism of $F[x,y]$, and $\gamma(1) = -1$. Obviously, γ preserves the standard filtration.

Now consider any automorphism σ of $F[x,y]$ which preserves the filtration and $\sigma(1) = -1$. Set $\sigma' = \gamma^{-1}\sigma$, then σ' is also an automorphism preserving the filtration, and $\sigma'(1) = \gamma^{-1}\sigma(1) = \gamma^{-1}(-1) = 1$. By Theorem 3.2.2, there are $a, b \in F$, $b \neq 0$, such that $\sigma' f(x,y) = f(a + bx, a + by)$, for any $f(x,y) \in F[x,y]$. Hence,

$$\begin{aligned}\sigma f(x,y) &= \gamma\sigma' f(x,y) = \gamma f(a + bx, a + by) \\ &= -f(a + by, a + bx), \quad \forall\, f(x,y) \in F[x,y],\end{aligned}$$

which is the result we want to get. □

In the following, we will prove that if σ is any automorphism of the Leibniz algebra $F[x,y]$ preserving the standard filtration, and $\sigma(1) = a \in F$, then $a = \pm 1$, or equivalently, if $a \neq -1$ then $a = 1$.

So suppose that $\sigma(1) = a \neq 1$, and $\sigma(x) = b + cx + ey$. As $a[1, \sigma(x^2)] = 2\sigma(x)[1, \sigma(x)] + a[1, \sigma(x)]^2$, we have

$$\begin{aligned}a[1, \sigma(x^2)] &= 2(b + cx + ey)(c + e)(y - x) + a(c + e)^2(y - x)^2 \\ &= (c + e)(y - x)\big(2b + (2c - ae - ac)x + (2e + ac + ae)y\big).\end{aligned}$$

Thus $e = \frac{1-a}{1+a}c$ and

$$\sigma(x) = b + cx + \frac{1-a}{1+a}cy,\ a[1, \sigma(x^2)] = \frac{4bc}{1+a}(y - x) + \frac{4c^2}{(1+a)^2}(y^2 - x^2).$$

In this case $c \neq 0$, otherwise,

$$\sigma(y - x) = \sigma[1, x] = \sigma(1)[1, \sigma(x)] = 0,$$

and σ is injective, a contradiction. Hence, $c \neq 0$.

Suppose that $\sigma(x^2) = a_{0,0} + a_{1,0}x + a_{0,1}y + a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2$, where all $a_{i,j} \in F$ and

$$a_{1,0} + a_{0,1} = \frac{4bc}{a(1+a)}, a_{2,0} + a_{1,1} + a_{0,2} = \frac{4c^2}{a(1+a)^2}.$$

By (3.9) we have

$$
\begin{aligned}
a[1,\sigma(x^3)] &= \frac{2c}{a(1+a)}(y-x)\Big(2b+\frac{2c}{1+a}x+\frac{2c}{1+a}y\Big)\Big(b+cx+\frac{1-a}{1+a}cy \\
&\quad +\frac{2ac}{1+a}(y-x)\Big)+\frac{2c}{1+a}(y-x)\sigma(x^2) \\
&= \frac{2c}{1+a}(y-x)\Big(\frac{2b^2}{a}+\frac{2bc(2-a)}{a(1+a)}x+\frac{2bc(2+a)}{a(1+a)}y+\frac{2c^2(1-a)}{a(1+a)^2}x^2 \\
&\quad +\frac{4c^2}{a(1+a)^2}xy+\frac{2c^2}{a(1+a)}y^2+a_{0,0}+a_{1,0}x+a_{0,1}y+a_{2,0}x^2+a_{1,1}xy+a_{0,2}y^2\Big).
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
&\frac{2bc(2-a)}{a(1+a)}+a_{1,0}=\frac{2bc(2+a)}{a(1+a)}+a_{0,1}, \\
&\frac{2c^2(1-a)}{a(1+a)^2}+a_{2,0}=\frac{4c^2}{a(1+a)^2}+a_{1,1}=\frac{2c^2}{a(1+a)}+a_{0,2}.
\end{aligned}
$$

Hence,

$$
a_{1,0}=\frac{2bc}{a},\ a_{0,1}=\frac{2bc(1-a)}{a(1+a)},\ a_{2,0}=\frac{2c^2}{a(1+a)},\ a_{1,1}=0, a_{0,2}=\frac{2c^2(1-a)}{a(1+a)^2}.
$$

Thus

$$
\sigma(x^2)=a_{0,0}+\frac{2bc}{a}x+\frac{2bc(1-a)}{a(1+a)}y+\frac{2c^2}{a(1+a)}x^2+\frac{2c^2(1-a)}{a(1+a)^2}y^2.
$$

On the other hand, $\sigma[xy,x]=[\sigma(xy),\sigma(x)]=\sigma(xy)[1,\sigma(x)]$, and

$$
\begin{aligned}
\sigma[xy,x] &= \sigma(xy^2-x^2y) \\
&= \sigma(x^3)+\sigma[x,x^2]-\sigma(x^3)-\sigma[x^2,x] \\
&= \sigma(x)[1,\sigma(x^2)]-\sigma(x^2)[1,\sigma(x)] \\
&= \frac{\sigma x}{a}(2\sigma(x)[1,\sigma(x)]+a[1,\sigma(x)]^2)-\sigma(x^2)[1,\sigma(x)] \\
&= [1,\sigma(x)]\Big(\frac{2(\sigma(x))^2}{a}+\sigma(x)[1,\sigma(x)]-\sigma(x^2)\Big).
\end{aligned}
$$

But

$$
\sigma[xy,x]=\sigma(xy)[1,\sigma(x)]=[1,\sigma(x)](\sigma(x^2)+\sigma(x)[1,\sigma(x)]),
$$

and $[1,\sigma(x)]\neq 0$, hence

$$
\frac{2(\sigma(x))^2}{a}+\sigma(x)[1,\sigma(x)]-\sigma(x^2)=\sigma(x^2)+\sigma(x)[1,\sigma(x)],
$$

which implies that $(\sigma(x))^2=a\sigma(x^2)$. By comparing the coefficients of x^2 in two sides of the equation we have $a=1$. Hence, we have proved the following

Theorem 3.2.4. *If σ is an automorphism of the Leibniz algebra $F[x,y]$ and preserves the standard filtration $\{F[x,y]_{(n)}\}_{n\in\mathbb{Z}_+}$, then $\sigma(1)=\pm 1$.*

Thus, we can describe all automorphisms of the Leibniz algebra $F[x,y]$ which preserve the standard filtration.

Theorem 3.2.5. *Let $\mathrm{Aut}'(F[x,y])$ be the set of automorphisms of the Leibniz algebra $F[x,y]$ which preserve the standard filtration $\{F[x,y]_{(n)}\}_{n\in\mathbb{Z}_+}$ of $F[x,y]$, then $\mathrm{Aut}'(F[x,y])$ is a subgroup of $\mathrm{Aut}(F[x,y])$, and $\mathrm{Aut}'(F[x,y])$ is isomorphic to the following matrix multiplication group*

$$G=\left\{\begin{pmatrix}1 & a & a\\ 0 & b & 0\\ 0 & 0 & b\end{pmatrix}\,\middle|\, a,b\in F,\ b\neq 0\right\}\cup\left\{\begin{pmatrix}-1 & a & a\\ 0 & 0 & b\\ 0 & b & 0\end{pmatrix}\,\middle|\, a,b\in F,\ b\neq 0\right\}.$$

Proof. Clearly, $\mathrm{Aut}'(F[x,y])$ is a subgroup of the automorphism group $\mathrm{Aut}(F[x,y])$, and by Theorem 3.2.4, for any $\sigma\in\mathrm{Aut}'(F[x,y])$, $\sigma(1)=\pm 1$. But by Theorem 3.2.2 and 3.2.3, such an automorphism σ can be determined uniquely by its images $\sigma(1)$, $\sigma(x)$ and $\sigma(y)$. If we identify σ with the matrix of linear transformation $\sigma|_{F[x,y]_{(1)}}$ with respect to the basis $1,x,y$, we get the result. $\square$

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